# NO $n$-POINT SET IS $\sigma$-COMPACT

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ABSTRACT. Let $n$ be an integer greater than 1. We prove that there exist no $F_\sigma$-subsets of the plane that intersect every line in precisely $n$ points.

Let $n \geq 2$ be some fixed integer. A subset of the plane $\mathbb{R}^2$ is called an *$n$-point set* if every line in the plane meets the set in precisely $n$ points. The question whether $n$-point sets can be Borel sets is a long standing open problem, see e.g. Mauldin [6] for details. Sierpinski [7, p. 447] has given a simple example of a closed set that meets every line in $\aleph_0$ points. It was shown by Baston and Bostock [1] and by Bouhjar, Dijkstra, and van Mill [2] that 2-point sets respectively 3-point sets cannot be $F_\sigma$ in the plane. Both papers use a method suggested by Larman [5] for the case $n = 2$ which consists of proving on the one hand that 2-point sets cannot contain arcs and on the other hand that 2-point sets that are $F_\sigma$ must contain arcs. Observe that to prove the result that is the subject of this note Larman's program cannot be followed because it was shown in [2] that $n$-point sets can contain arcs whenever $n \geq 4$.

**Theorem.** *Let $n \geq 2$. No $n$-point set is an $F_\sigma$-subset of the plane.*

The three authors of this note each, independently of each other, found a proof for this theorem. We decided to publish the shortest proof jointly.

*Proof.* Assume that $A$ is an $n$-point set that is an $F_\sigma$-subset of the plane. Let $xy$ be an *arbitrary* rectangular coordinate system for the plane and let $\lambda$ be the Lebesgue measure on $\mathbb{R}$. According to [2, Proposition 3.2] there exists a nondegenerate interval $[a, b]$ on the $x$-axis and continuous functions $f_1 < f_2 < \cdots < f_n$ from $[a, b]$ into $\mathbb{R}$ such that $A$ contains the graph of each $f_i$. Consider an $f_i$ and its graph $G_i$. Since $A$ is an $n$-point set each horizontal line intersects $G_i$ in at most $n$ points. So every fibre of $f_i$ has cardinality at most $n$. Consequently, according to Banach [4,

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Exercise 17.34], the variation of $f_i$ is bounded by $n(M-m)$, where $m$ and $M$ are the minimum and maximum values of $f_i$. According to Lebesgue [4, Theorem 17.17] the derivative of a function of bounded variation such as $f_i$ exists almost everywhere. Select a Borel set $B \subset [a,b]$ such that $\lambda(B) = b-a$ and every $f_i$ is differentiable at every point of $B$. By the Whitney Extension Theorem for $C^1$ functions [3, Theorem 3.1.16] there exists a set $C \subset B$ such that $\lambda(C) > 0$ and continuously differentiable functions $g_i : [a,b] \to \mathbb{R}$ with $g_i|C = f_i|C$ for $1 \leq i \leq n$. The functions $g_i$ satisfy the premises of Theorem 7 in [6] so we may conclude that the set $A$ is bounded or intersects some line in $n+1$ points. Either way, the result is inconsistent with $A$ being an $n$-point set. □

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