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THE BAIRE ORDER OF THE FUNCTIONS CONTINUOUS ALMOST EVERYWHERE. II

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ABSTRACT. Let S be a complete and separable metric space and μ a σ-finite, complete Borel measure on S with $\mu(S) > 0$. Let Φ be the family of all real-valued functions defined on S whose set of points of discontinuity is of μ-measure 0. Let $B_\alpha(\Phi)$ be the functions of Baire's class α generated by Φ. It is shown that $B_1(\Phi) = B_2(\Phi)$ if and only if μ is a purely atomic measure whose set of atoms forms a scattered subset of S and that if $B_1(\Phi) \neq B_2(\Phi)$, then the Baire order of Φ is ω_1; in other words, if $0 \leq \alpha < \omega_1$, then $B_\alpha(\Phi) \neq B_{\alpha+1}(\Phi)$. This answers a generalized version of a problem raised by Sierpinski and Felsztyn. An example is given of a normal space with Borel order 2 and Baire order ω_1.

Sierpinski and Felsztyn in the first volume of Fundamenta Mathematicae raised the following problem:

(*) Is there a function of Baire's class 2 on the unit interval which is not the pointwise limit of a sequence of functions each continuous almost everywhere [5]?

There is a discussion of this problem in the appendix of the 1937 edition of the first volume. This problem was solved by Zalcwasser and Kantorovitch. Also, see [4].

In Theorem 4 of [4], the author shows that for each countable ordinal α, there is a function of Baire's class $\alpha + 1$ which is not in the α class generated by the functions continuous almost everywhere. Therefore, the answer to (*) and to a generalized version of (*) is yes.

This paper contains a number of generalizations of results contained in [4].

Definitions and notation. If X is a topological space and μ is a complete Borel measure on X, A is a subset of X, and B is a subset of A, then

(a) $\Phi(A, \mu)$ will denote the family of all real-valued functions defined on A whose set of points of discontinuity is of μ-measure zero, and

(b) $\Phi(A, B)$ will denote the family of all real-valued functions defined

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on A which are continuous at each point of B.

If X is a set and Φ is a family of real-valued functions defined on X, then $B_0(\Phi)$ will denote Φ and for each ordinal α, $\alpha > 0$, $B_\alpha(\Phi)$ will denote the family of all pointwise limits of sequences from $\bigcup_{\gamma<\alpha} B_\gamma(\Phi)$. Of course, $B_{\omega_1}(\Phi) = \bigcup_{\alpha<\omega_1} B_\alpha(\Phi)$ and thus, $B_{\omega_1}(\Phi) = B_{\omega_1+1}(\Phi)$. The first ordinal α for which $B_\alpha(\Phi) = B_{\alpha+1}(\Phi)$ will be called the Baire order of Φ.

The unit interval will be denoted by I.

Recall that a subset M of a topological space is said to be scattered if there is no subset of M which is dense in itself. Also, in this paper the Borel sets form the σ-algebra generated by the open sets and a measure μ is regular means $\mu(E) = \sup\{\mu(F)\colon F = \overline{F} \leq E\} = \inf\{\mu(U)\colon U$ is open and $E \leq U\}$, for each μ-measurable set E.

Theorem 1. *Suppose μ is a finite, positive complete Borel measure on I and $\mu(I) > 0$. If μ is not a purely atomic measure whose set of atoms forms a scattered set, then the Baire order of $\Phi(I, \mu)$ is ω_1.*

Proof. Let M be the set of all atoms of the measure μ. Either (1) the countable set M contains a dense in itself subset K, or (2) $\mu(I - M) > 0$. If the first case holds, then $\overline{K}$ is a perfect subset of I such that if an open set U meets $\overline{K}$, then $\mu(\overline{K} \cap U) > 0$. If the second case holds, then there is a perfect set lying in $I - M$ such that if an open set meets P, then $\mu(P \cap U) > 0$.

It is easy to check that one may now proceed exactly as in [4], and conclude that the Baire order of $\Phi(I, \mu)$ is ω_1.

Theorem 2. *Let K be a subset of a metric space S and let D and A be G_δ subsets of S containing K with $K \subseteq D \subseteq A$. Then*

(a) *if $\alpha > 0$, each function in $B_\alpha(\Phi(D, K))$ has an extension to a function in $B_\alpha(\Phi(A, K))$,*

(b) *the Baire order of $\Phi(D, K)$ is no more than the Baire order of $\Phi(A, K)$,*

(c) *if the Baire order of $\Phi(D, K)$ is > 0, then $\Phi(A, K)$ and $\Phi(D, K)$ have the same order.*

Proof. (a) If $f \in B_\alpha(\Phi(D, K))$ and $\alpha > 0$, then by Theorem 3 of [2], there is a function g of Baire's class α (in other words, $g \in B_\alpha(\Phi(D, D))$ such that $M = \{x | f(x) \neq g(x)\}$, is a subset of an F_σ set, W, with respect to D and W does not intersect K.

Let

$$\hat{f}(x) = \begin{cases} f(x), & x \in D, \\ g(x), & x \in A - D. \end{cases}$$

The set of all x such that $\hat{f}(x) \neq \hat{g}(x)$ is M. Let $W = \bigcup_{n=1}^{\infty} F_n$, where for each n, F_n is closed with respect to D and let $\hat{F}_n$ be the closure of F_n in A. Then $M \subset \hat{W} = \bigcup_{n=1} \hat{F}_n$ and $\hat{W}$ is an F_σ set with respect to A which does not meet K. Thus, by Theorem 3 of [2], $\hat{f} \in B_\alpha(\Phi(A, K))$.

(b) It may be shown by transfinite induction, that for all α, $0 \leq \alpha$, if $f \in B_\alpha(\Phi(A, K))$, then the restriction of f to D is in the family $B_\alpha(\Phi(D, K))$. From this we see that if f is exactly of class $B_\alpha(\Phi(D, K))$ ($f \in B_\alpha(\Phi(D, K)) - \bigcup_{\gamma<\alpha} B_\gamma(\Phi(D, K))$), then no extension of f to A can be of lower class with respect to $\Phi(A, K)$. Thus, the Baire order of $\Phi(D, K)$ is no more than the Baire order of $\Phi(A, K)$.

(c) Suppose the Baire order of $\Phi(A, K)$ is greater than γ, the Baire order of $\Phi(D, K)$. Let f be a function of exactly class $B_{\gamma+1}(\Phi(A, K))$ and let h be the restriction of f to D. Then $h \in B_{\gamma+1}(\Phi(D, K))$ and therefore $h \in B_\gamma(\Phi(D, K))$. Since $\gamma > 0$, by part (a), there is an extension $\hat{h}$ of h to A which is in $B_\gamma(\Phi(A, K))$. Let $M = \{x | \hat{h}(x) \neq f(x)\}$. The set M is a subset of $A - D$. But, $A - D$ is an F_σ set with respect to A which does not meet K. It follows from Theorem 3 of [2], that $f \in B_\gamma(\Phi(A, K))$. This contradiction completes the argument for part (c).

Theorem 3. *Let A and D be G_δ subsets of a metric space S with $D \subseteq A$. Let μ be a finite regular complete Borel measure defined on A. If $\mu(A - D) = 0$, then*

(a) if $\alpha > 0$, each function in $B_\alpha(\Phi(D, \mu))$ has an extension to a function in $B_\alpha(\Phi(A, \mu))$,

(b) the Baire order of $\Phi(D, \mu)$ is no more than the Baire order of $\Phi(A, \mu)$, and

(c) if the Baire order of $\Phi(D, \mu)$ is > 0, then $\Phi(A, \mu)$ and $\Phi(D, \mu)$ have the same order.

The proof of this theorem follows the corresponding proofs of Theorem 2.

Theorem 4. *Let R be the set of all rational numbers in I, let B be a G_δ subset of I containing R. Then the Baire order of $\Phi(B, R)$ is ω_1.*

Proof. Let μ be a finite, complete Borel measure on I such that μ is purely atomic and R is the set of all atoms of μ. Then, the family $\Phi(I, R)$ is $\Phi(I, \mu)$. It is easy to see that the Baire order of $\Phi(B, R)$ is not 0. Therefore, by Theorem 2 (c), the Baire order of $\Phi(B, R)$ is ω_1.

Theorem 5. *Let K be a countable dense in itself subset of a complete and separable metric space S and let A be a G_δ subset of S containing K. Then the Baire order of $\Phi(A, K)$ is ω_1.*

Proof. Let ϕ be a homeomorphism of K with the set of all rational numbers in the unit interval I [1, p. 287]. Let $\hat{\phi}$ be an extension of ϕ defined on a G_δ set B containing K to a G_δ set, $\hat{\phi}(B)$, in I such that $\hat{\phi}$ is a homeomorphism of B and $\hat{\phi}(B)$ [1, p. 429].

It follows easily by transfinite induction that $f \in B_\alpha(\Phi(A \cap B, K))$ if and only if $f \circ \hat{\phi}^{-1} \in B_\alpha(\Phi(\hat{\phi}(A \cap B), R))$. Therefore, the order of the family $\Phi(A \cap B, K)$ is ω_1 by Theorem 3. Thus, the Baire order of the family $\Phi(A, K)$ is ω_1 by Theorem 2 (c).

Theorem 6. *Let M be a subset of a complete and separable metric space. If M contains a perfect set, then the Baire order of $\Phi(S, M)$ is ω_1. If M is countable, then* (1) *the Baire order of $\Phi(S, M)$ is ≤ 1, if M is scattered and* (2) *the Baire order of $\Phi(S, M)$ is ω_1, if M is not scattered.*

Proof. Suppose M contains a perfect set K. Since $\Phi(K, K)$ is the space of all real valued continuous functions defined on K, it follows that the Baire order of $\Phi(K, K)$ is ω_1. Also, for each α, $0 \leq \alpha$, each function in $B_\alpha(\Phi(K, K))$ has an extension to a function in $B_\alpha(\Phi(S, S))$ [1, p. 434] and thus to a function in $B_\alpha\Phi(S, M)$. It follows that if $f \in B_\alpha(\Phi(K, K))$ but to none of the preceding classes, then any extension of f to a function in $B_\alpha(\Phi(S, M))$ cannot belong to any class $B_\gamma(\Phi(S, M))$, $\gamma < \alpha$.

Therefore, the order of $\Phi(S, M)$ is ω_1.

Now, suppose M is countable.

Case 1. The set M is scattered. In this case, Theorem 2 of [3] states that the Baire order of $\Phi(S, M)$ is ≤ 1.

Case 2. The set M is not scattered. Let K be the dense in itself kernel of M.

If M is K, then by Theorem 5 the Baire order of $\Phi(S, M) = \Phi(S, K)$ is ω_1.

If K is a proper subset of M, then the set $M - K$ is scattered. Therefore $M - K$ is an F_σ set [1, p. 258]. Then $S - (M - K)$ is a G_δ set containing K and the Baire order of $\Phi(S - (M - K), K)$ is ω_1 by Theorem 5.

If f is of exactly class $B_{\alpha+1}(\Phi(S - (M - k), K))$, $\alpha > 0$, then there is a function g of Baire's class $\alpha + 1$ on $S - (M - K)$ such that the set $M = \{x | f(x) \neq g(x)\}$ is a subset of a set W which is an F_σ set with respect to

$S-(M-K)$. Let $\hat{g}$ be an extension of g to S of Baire's class $\alpha+1$. Then obviously, $g \in B_{\alpha+1}(\Phi(S, M))$. Assume $g \in B_\alpha(\Phi(S, M))$. Then there is a function h in Baire's class α on S such that the set $M_1 = \{x | \hat{g}(x) \neq h(x)\}$ is a subset of an F_σ set W_1 in S such that W_1 does not intersect K [2, Theorem 3]. But, then l, the restriction of h to $S-(M-K)$, is a function of Baire's α on $S-(M-K)$ and the set of all x such that $l(x) \neq f(x)$ is a subset of $W_1 \cap (S-(M-K))$, which is an F_σ set in $S-(M-K)$ which does not meet K. Therefore, by Theorem 3 of [2], f is in $B_\alpha(\Phi(S-(M-K), K))$. This contradiction proves that the order of $\Phi(S, M)$ is ω_1.

Questions. Is there a subset M of I such that the Baire order of $\Phi(I, M)$ is 2? For each ordinal α, $2 \leq \alpha < \omega_1$, is there a subset M of I such that the Baire order of $\Phi(I, M)$ is α?

Theorem 6. *Let μ be a finite regular Borel measure defined on the space N consisting of all irrational numbers between* 0 *and* 1. *If μ has no atoms and $\mu(N) > 0$, then the order of $\phi(I, \mu)$ is ω_1.*

Proof. Let $\hat{\mu}$ be the unique extension of μ to a complete Borel measure defined on I such that $\hat{\mu}(I-N) = 0$. Then $\hat{\mu}(I) > 0$ and $\hat{\mu}$ has no atoms. Therefore the Baire order of $\Phi(I, \mu)$ is ω_1. Therefore, by Theorem 2, the Baire order of $\Phi(N, \mu)$ is ω_1.

Theorem 7. *Let μ be a σ-finite regular Borel measure defined on a complete and separable metric space S with $\mu(S) > 0$. Then* (1) *the order of $\Phi(S, \mu)$ is ≤ 1 if and only if μ is purely atomic and the set of atoms of μ forms a scattered set, and* (2) *the order of $\Phi(S, \mu)$ is ω_1, if μ does not meet the conditions described in* 1.

Proof. Part (1) of the conclusion is Theorem 3 of [3].

Let $\{K_n\}_{n=1}^\infty$ be a sequence of disjoint Borel sets of finite μ-measure filling up S. Let $\mu_n(A) = \mu(A \cap K_n)$, for each n and each μ-measurable set A. Let

$$\nu = \sum_{n=1}^{\infty} \frac{2^{-n}}{\mu_n(K_n)+1} \mu_n.$$

Then ν is a finite regular Borel measure on S and a subset E of S is of μ-measure 0 if and only if $\nu(E) = 0$.

Let $\nu = \nu_d + \nu_s$, where ν_d is purely atomic and ν_s has no atoms. Let M be the set of atoms of ν_d. Of course, M is the set of atoms of μ. It follows from part (1) of the conclusion that either M is not scattered or $\nu(S-M) > 0$.

Case 1. Suppose M is not scattered. Let K be the dense in itself kernel of M and let A be a G_δ set containing K such that $\nu(A) = \nu(K)$. Then $\nu(A - K) = 0$ and $\Phi(A, \nu) = \Phi(A, K)$. Therefore, by Theorem 5, the order of $\Phi(A, \nu)$ is ω_1 and by Theorem 3 the order of $\Phi(S, \nu) = \Phi(S, \mu)$ is ω_1.

Case 2. Suppose $\nu(S - M) > 0$.

Let J be a perfect set lying in $S - M$ such that if an open set U meets J, then $\nu(J \cap U) > 0$. Let $\{y_n\}_{n=1}^{\infty}$ be a dense subset of J and for each n, let $\{\delta_{np}\}_{p=1}^{\infty}$ be a decreasing sequence of positive numbers converging to zero such that $\nu\left(\overline{B(y_n, \delta_{np})} - B(y_n, \delta_{np})\right) = 0$, where $B(y_n, \delta_{np})$ is the ball with center y_n and radius δ_{np}. Let Q be the union of all the sets $\overline{B(y_n, \delta_{np})} - B(y_n, \delta_{np})$. It follows that $Q \cap J$ is an F_σ subset of J with $\nu(Q) = 0$ such that $J - Q$ is 0-dimensional.

Let $W = J - Q$. Then W is a dense in itself 0-dimensional G_δ set lying in J. By Theorem 3, the Baire order of $\Phi(J, \nu)$ is the same as the order of $\Phi(W, \nu)$.

Let ϕ be a homeomorphism of W onto N, the set of all irrational numbers between 0 and 1 [1, p. 441], and for each ν-measurable set E lying in W, let $\lambda(\phi(E)) = \nu(E)$. It follows that λ is a complete Borel measure on N and a function f is in the class $B_\alpha(\Phi(N, \lambda))$ if and only if $f \circ \phi$ is in the class $B_\alpha(\Phi(W, \lambda))$. By Theorem 5, the Baire order of $\Phi(N, \lambda)$ is ω_1. Thus, the order of $\Phi(J, \nu)$ is ω_1.

Finally, if $h \in B_\alpha(\Phi(S, \nu))$, then the restriction of h to J is in $B_\alpha(\Phi(J, \nu))$. Also, if $\alpha > 0$ and $f \in B_\alpha(\Phi(J, \nu))$, then there is a function g of Baire's class α defined on J such that the set M of all x such that $g(x) \neq f(x)$ is a subset of an F_σ set T with respect to J.

Let $\hat{g}$ be an extension of Baire's class α to all of S [1, p. 434], let $\hat{f}(x) = f(x)$, $x \in J$, and $\hat{f}(x) = g(x)$, $x \in S - J$. Then the set of all x such that $\hat{f}(x) \neq \hat{g}(x)$ is a subset of T. Since T is an F_σ set with respect to J, T is an F_σ set in S of ν-measure zero. Therefore, by Theorem 3 of [3], $\hat{f} \in B_\alpha(\Phi(S, \nu))$.

From the above considerations, it follows that the order of $\Phi(S, \nu)$, which is $\Phi(S, \mu)$, is ω_1.

Theorem 8. *There is a hereditarily paracompact space which has Borel order* 2 *and Baire order* ω_1.

Proof. Let X be the unit interval and let a subset W of X be open if and only if $W = U \cap V$ where U is open and V is any subset of $X - R$, where R is the rationals. The space X is hereditarily paracompact [6].

S. Willard in [7] shows that every Borel subset of X is a $G_{\delta\sigma}$ set in X. If $f \in C(X)$, then f is continuous in the usual topology at each point of R. Thus, by Theorem 4, X has Baire sets of arbitrarily high class.

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