Math 1720 Homework 3, due Friday Feb 10
Explain all answers and show all calculations.

Problem 7.1:18*:
Find the inverse of $f(x) = x^2 + 4$ over each of the intervals $x \geq 0$, and $x \leq 0$.
Verify the formulas $f(f^{-1}(x)) = x$ (x in the domain of the inverse) and $f^{-1}(f(x)) = x$ ($x \geq 0$ or $x \leq 0$ accordingly) for each of the intervals.
Find the domain and range of the inverse for each of the intervals.

Solution.
(Note that graphing $y = x^2 + 4$, it's just the regular parabola for $y = x^2$, shifted upward vertically by 4 units. So by the HLT, it is 1-1 over $(-\infty, 0]$ and over $[0, \infty)$ (and in fact not 1-1 over any larger interval, also by the HLT, using the graph). So f does indeed have an inverse over each of these intervals.)

For any numbers x, y we have:

$$y = x^2 + 4$$

iff (since $x^2 \geq 0$):

$$y = x^2 + 4, \quad y \geq 4$$

iff

$$y - 4 = x^2, \quad y - 4 \geq 0$$

iff

$$\sqrt{y-4} = \sqrt{x^2}, \quad y - 4 \geq 0$$

iff

$$\sqrt{y-4} = |x|, \quad y - 4 \geq 0.$$

So for the interval $D = [0, \infty)$, we have $x \geq 0$ so $x = |x|$ so $y = x^2 + 4$ iff

$$\sqrt{y-4} = x, \quad y \geq 4.$$

Let g be the inverse of f over D. From the line above, the formula for g is

$$g(x) = \sqrt{x-4}.$$

The range of g is $D = [0, \infty)$. The domain of g is the range of f over D. Since $f(x) = x^2+4$, its graph is just the regular parabola ($y = x^2$) shifted up vertically by 4 units. So its range over $0 \leq x < \infty$ is $4 \leq y < \infty$, i.e. the interval $[4, \infty)$. Thus the domain of g is $[4, \infty)$.

For the interval $D_2 = (-\infty, 0]$, we have $x \leq 0$ so $-x = |x|$ so $y = x^2 + 4$ iff

$$\sqrt{y-4} = -x, \quad y \geq 4$$

iff

$$-\sqrt{y-4} = x, \quad y \geq 4.$$

Let h be the inverse of f over D_2. From the line above, the formula for h is

$$h(x) = -\sqrt{x-4}.$$

The range of h is $D_2 = (-\infty, 0]$. The domain of g is the range of f over D_2, which as in the previous case, is $[4, \infty)$.

(b) To verify the cancellation formulas: For $x \geq 0$, the inverse is g. We have

$$g(f(x)) = \sqrt{f(x) - 4} = \sqrt{x^2 + 4 - 4} = \sqrt{x^2} = |x| = x,$$

the last equation since $x \geq 0$. And for x in the domain of g, i.e. $x \geq 4$,

$$f(g(x)) = g(x)^2 + 4 = \sqrt{x-4}^2 + 4 = (x - 4) + 4 = x.$$

For $x \leq 0$, the inverse is h. We have

$$h(f(x)) = -\sqrt{f(x) - 4} = -\sqrt{x^2 + 4 - 4} = -\sqrt{x^2} = -|x| = x,$$

the last equation since $x \leq 0$. And for x in the domain of h, i.e. $x \geq 4$,

$$f(h(x)) = h(x)^2 + 4 = (-\sqrt{x-4})^2 + 4 = (x - 4) + 4 = x.$$

A. Find the longest intervals over which

$$f(x) = x^2 + 2x - 8$$

is 1-1 (one-to-one). Find the range of f over each of these intervals. Find the inverse of f over each of these intervals, and also find the inverse's domain and range.

Solution.
Since

$$f'(x) = 2x + 2,$$

we have

$$f'(x) = 0$$

iff

$$2x + 2 = 0$$

iff

$$x = -1.$$

And f' is continuous and is a linear function with positive slope, so $f'(x) > 0$ for $x > -1$ and $f'(x) < 0$ for $x < -1$. Also f is continuous.

So f is increasing over the interval $[-1, \infty)$. And f is decreasing over the interval $(-\infty, -1]$.

And since f is also continuous, we have that f is 1-1 over these intervals, and these are the largest such intervals.

(You could also have sketched the graph and used the H.L.T. for this part.)

So we know that f has an inverse over the interval $(-\infty, -1]$, and also over the interval $[-1, \infty)$.

Ranges: We know f has a local and absolute min at $x = -1$ (since f is continuous and using where f is inc/decreasing). So this is $f(-1) = (-1)^2 +$

$2(-1)+8=-9$. And over $[-1,\infty)$, f is increasing, $\lim_{x\to\infty} f(x)=\infty$, and f is continuous, so the range of f over this interval is $[-9,\infty)$.

Similarly, the range of f over the interval $(-\infty,-1]$ is also $[-9,\infty)$.

Inverses: for any numbers x, y we have

$$y = x^2 + 2x - 8$$

iff

$$0 = x^2 + 2x - 8 - y$$

iff (by the quad formula)

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-8-y)}}{2(1)}$$

iff

$$x = \frac{-2 \pm \sqrt{4 + 32 + 4y}}{2}$$

iff

$$x = \frac{-2 \pm \sqrt{4(9+y)}}{2}$$

iff

$$x = \frac{-2 \pm 2\sqrt{9+y}}{2}$$

iff

$$x = -1 \pm \sqrt{9+y}.$$

Now for the interval $[-1,\infty)$, i.e. $x \geq -1$, we therefore have $y = x^2 + 2x - 8$ iff

$$x = -1 + \sqrt{9+y}$$

(since $\sqrt{9+y} \geq 0$). Now let g be the inverse of f over $[-1,\infty)$. Then the range of g is $[-1,\infty)$ and the domain of g is f's range over this interval, i.e. $[-9,\infty)$. So (using the calculation above), the formula for g is

$$g(x) = -1 + \sqrt{9+x}, \quad x \geq -9.$$

Similarly, for the interval $(-\infty,-1]$, i.e. $x \leq -1$, we have $y = x^2 + 2x - 8$ iff

$$x = -1 - \sqrt{9+y}.$$

Now let h be the inverse for f over $(-\infty,-1]$. Then the range of h is $(-\infty,-1]$, and the domain of h is f's range over this interval, i.e. $[-9,\infty)$. So (using the calculation above), the formula for h is

$$h(x) = -1 - \sqrt{9+x}, \quad x \geq -9.$$

I have found the domain, range of the two inverses already above.