Math 1720 Final Review Problems

Note that the final IS COMPREHENSIVE for the entire semester's material but these review problems only cover a sample of problems from sections 8.4, 8.7 and §9. You should also review the earlier sections. For problems on this material I suggest looking again at the earlier review problems, midterms, quizzes, and other problems in the text. In the exam there will be a little more focus on the most recent material, but not too much.

Problems

1. Compute, or show that it does not exist,

$$\int_0^{11} \frac{4}{\sqrt[3]{x-10}}.$$

Solution.
Note that the interval $0 \leq x \leq 11$ includes an asymptote of the integrand at $x = 10$. For both 4 and $\sqrt[3]{x-10}$ are continuous, and as $x \to 10$, we have $4 \to 4 \neq 0$, and $\sqrt[3]{x-10} \to \sqrt[3]{10-10} = 0$. Note that also for $x > 10$, $x-10 > 0$, so $\sqrt[3]{x-10} > 0$, so as $x \to 10^+$, we have $\sqrt[3]{x-10} \to 0^+$. And the numerator $\to 4 > 0$. So as $x \to 10^+$, the integrand $\to +\infty$. Similarly, as $x \to 10^-$, $\sqrt[3]{x-10} \to 0^-$, and the integrand $\to -\infty$.

The integrand is, however, continuous over the intervals $[0, 10)$ and $(10, 11]$. So we compute the integral by breaking it into these two sub-intervals:

$$\int_0^{10} \frac{4}{\sqrt[3]{x-10}} dx + \int_{10}^{11} \frac{4}{\sqrt[3]{x-10}} dx,$$

given that each of these two integrals exists itself, or both are $+\infty$, or both are $-\infty$.

For the first one:

$$\int_0^{10} \frac{4}{\sqrt[3]{x-10}} dx,$$

and as discussed above, the integrand $\to -\infty$ as $x \to 10^-$, and is continuous over $[0, 10)$, so we compute this integral as the limit:

$$= \lim_{b \to 10^-} \int_0^b \frac{4}{\sqrt[3]{x-10}} dx$$

$$= \lim_{b \to 10^-} 4 \int_0^b (x-10)^{-1/3} dx$$

And since $((x-10)^{2/3})' = (2/3)(x-10)^{-1/3}(x-10)' = (2/3)(x-10)^{-1/3}$, we have

$$= 4 \lim_{b \to 10^-} (3/2)(x-10)^{2/3}\Big|_0^b$$

$$= 6 \lim_{b \to 10^-} [(b-10)^{2/3} - (-10)^{2/3}]$$

$$= 6 \lim_{b \to 10^-} [(b-10)^{2/3} - \sqrt[3]{100}]$$

$$= 6[(\lim_{b\to 10^-}(b-10)^{2/3}) - \lim_{b\to 10^-}\sqrt[3]{100}]$$

$$= 6[(\lim_{b\to 10^-}(b-10)^{2/3}) - \sqrt[3]{100}]$$

And since the function $f(x) = (x-10)^{2/3}$ is continuous, we just plug $x = 10$ in to compute the remaining limit, so this is

$$= 6[(10-10)^{2/3} - \sqrt[3]{100}] = 6[0^{2/3} - \sqrt[3]{100}]$$

$$= 6[0 - \sqrt[3]{100}] = -6\sqrt[3]{100}.$$

For the second one:

$$\int_{10}^{11} \frac{4}{\sqrt[3]{x-10}}dx$$

For similar reasons as with the first one, we compute this interal as a limit:

$$= \lim_{b\to 10^+}\int_b^{11} \frac{4}{\sqrt[3]{x-10}}dx$$

$$= \lim_{b\to 10^+} 4\int_b^{11}(x-10)^{-1/3}dx$$

$$= 4\lim_{b\to 10^+}(3/2)(x-10)^{2/3}\Big|_b^{11}$$

$$= 6\lim_{b\to 10^+}[(11-10)^{2/3} - (b-10)^{2/3}]$$

$$= 6[1 - \lim_{b\to 10^+}(b-10)^{2/3}]$$

And again since $(x-10)^{2/3}$ is continuous, this is

$$= 6[1 - (10-10)^{2/3}] = 6[1-0] = 6.$$

So both sides existed, so the overall integral exists and is their sum:

$$= -6\sqrt[3]{100} + 6$$

$$= 6(1 - \sqrt[3]{100}).$$

2. Compute, or show that it does not exist,

$$\int_0^{11} \frac{4}{3x-10}dx$$

Solution.
The integrand $f(x) = 4/(3x-10)$ is continuous over $[0, 10/3)$ and over $(10/3, 11]$, but has an asymptote at $x = 10/3$: As $x \to (10/3)^-$, the numerator $\to 4$ and the denominator $3x - 10 \to 0^-$, so $f(x) \to -\infty$. And as $x \to (10/3)^+$, the numerator $\to 4$ and the denominator $3x - 10 \to 0^+$, so $f(x) \to +\infty$. So like in the previous problem, we compute the integral as a sum of two integrals, each of which has a single asymptote at one of its end points $x = 10/3$:

$$= \int_0^{10/3} f(x)dx + \int_{10/3}^{11} f(x)dx$$

And since $f(x)$ is continuous over $[0, 10/3)$ and $f(x) \to -\infty$ as $x \to 10/3^-$, we compute the the first as

$$\int_0^{10/3} f(x)dx = \lim_{b \to 10/3^-} \int_0^b f(x)dx$$

$$= \lim_{b \to 10/3^-} 4 \int_0^b \frac{1}{3x - 10} dx$$

And since $(\ln(|3x - 10|))' = \frac{1}{3x-10}(3x - 10)' = \frac{3}{3x-10}$, we have

$$= 4 \lim_{b \to 10/3^-} \frac{1}{3} \ln(|3x - 10|)\Big|_0^b$$

$$= (4/3) \lim_{b \to 10/3^-} [\ln(|3b - 10|) - \ln(| - 10|)]$$

$$= (4/3) \left[[\lim_{b \to 10/3^-} \ln(|3b - 10|)] - \lim_{b \to 10/3^-} \ln(10) \right]$$

$$= (4/3) \left[[\lim_{b \to 10/3^-} \ln(|3b - 10|)] - \ln(10) \right]$$

Now as $b \to 10/3^-$, $|3b - 10| \to 0^+$, so $\ln(|3b - 10|) \to -\infty$. So this is

$$(4/3)[-\infty - \ln(10)] = -\infty.$$

Now consider the second integral in the sum:

$$\int_{10/3}^{11} f(x)dx$$

Note that $f(x)$ has odd symmetry about the line $x = 10/3$ (i.e. $f(10/3 - x) = -f(10/3 + x)$ for all $x \neq 0$). So the integral (a)

$$\int_0^{10/3} f(x)dx$$

is odd-symmetric with the integral (b)

$$\int_{10/3}^{20/3} f(x)dx.$$

I.e., the value of (a) is the negative of the value of (b). We calclulated the value of (a) as $-\infty$ above. So (b) has value $+\infty$, which means

$$\lim_{b \to 10/3^+} \int_b^{20/3} f(x)dx = \infty.$$

But $10/3 < 20/3 < 11$, and $f(x) > 0$ for all $x > 10/3$, so

$$A = \int_{20/3}^{11} f(x)dx \geq 0,$$

and for any $b > 10/3$,

$$\int_b^{20/3} f(x)dx + \int_{20/3}^{11} f(x)dx = \int_b^{11} f(x)dx,$$

so

$$\int_b^{20/3} f(x)dx + A = \int_b^{11} f(x)dx,$$

where $A \geq 0$, so

$$\int_b^{20/3} f(x)dx \leq \int_b^{11} f(x)dx.$$

As $b \to 10/3^+$, the integral $\int_b^{20/3}$ goes $\to \infty$ (as shown just above), and therefore since $\int_b^{20/3} \leq \int_b^{11}$, we also have that $\int_b^{11} \to \infty$. Therefore

$$\int_{10/3}^{11} f(x)dx = \infty.$$

(You could alternatively compute this integral directly, much like the first one above, and find its value to be $+\infty$ in that way.)

So one side is $-\infty$, and the other side $+\infty$. So the overall integral does not exist (we can't add $-\infty$ with $+\infty$).

2.5. Compute, or show that it does not exist,

$$\int_0^1 \frac{x}{\sqrt{1-x^4}} dx$$

(Hint: at some point, make a substitution to make the integrand involve a form related to trig subs.) Use symmetry to deduce what $\int_{-1}^1$ is, with the same integrand.

Solution.
Note $\sqrt{1-x^4}$ is continuous over $0 \leq x \leq 1$, and is > 0 for $0 \leq x < 1$. The numerator x is continuous everywhere. So $f(x) = x/\sqrt{1-x^4}$ is continuous over $[0, 1)$. As $x \to 1^-$, the numerator $x \to 1^-$, and the denominator $\sqrt{1-x^4} \to 0^+$, since $x^4 \to 1^-$, so $1 - x^4 \to 0^+$, so $\sqrt{1-x^4} \to 0^+$. So as $x \to 1^-$, $f(x) \to 1/0^+ = +\infty$. So we compute the integral as a limit:

$$= \lim_{b \to 1^-} \int_0^b \frac{x}{\sqrt{1-x^4}} dx$$

Now the integrand invovles the form $1 - x^4$, which is similar to $1 - u^2$, a trig sub form. We would like to arrange that $u^2 = x^4$, so to do this we sub $u = x^2$. This gives $du = 2xdx$, so $xdx = du/2$, and note that x is in the numerator of $f(x)$. Making the sub:

$$= \lim_{b \to 1^-} \int_{x=0}^b \frac{du/2}{\sqrt{1-u^2}}.$$

Now when $x = 0$, $u = x^2 = 0$, and when $x = b$, $u = x^2 = b^2$:

$$= \lim_{b \to 1^-} \frac{1}{2} \int_{u=0}^{b^2} \frac{1}{\sqrt{1-u^2}} du$$

$$= \frac{1}{2} \lim_{b \to 1^-} \arcsin(u)\Big|_0^{b^2}$$

$$= \frac{1}{2} \lim_{b \to 1^-} \left[\arcsin(b^2) - \arcsin(0)\right]$$

Now $\arcsin(0) = \alpha$ where $\sin(\alpha) = 0$ and $-\pi/2 \le \alpha \le \pi/2$, so this is $\alpha = 0$:

$$= \frac{1}{2} \lim_{b \to 1^-} \left[\arcsin(b^2) - 0\right]$$

$$= \frac{1}{2} \lim_{b \to 1^-} \arcsin(b^2)$$

And as $b \to 1^-$, $b^2 \to 1^-$, and arcsin is continuous over its domain $-1 \le x \le 1$, so

$$= \frac{1}{2} \arcsin(1),$$

and $\arcsin(1) = \alpha$ where $\sin(\alpha) = 1$ and $-\pi/2 \le \alpha \le \pi/2$, so this is $\alpha = \pi/2$:

$$= \frac{1}{2}\pi/2$$

$$= \pi/4.$$

3.(a) Find

$$\int_4^5 \frac{x^3 - x - 3}{(x-3)x^3} dx$$

Solution.
The integrand is rational, and there is no straightforward sub to make, so we use partial fractions. The degree of the numerator is less than that of the denominator, so we can go straight to partial fraction form. The denominator has roots $x = 0$ with multiplicity 3; and $x = 3$ with multiplicity 1. So the partial fraction form is:

$$\frac{x^3 - x - 3}{(x-3)x^3} = \frac{A}{(x-3)} + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3}.$$

Multiplying through by the denominator:

$$x^3 - x - 3 = Ax^3 + Bx^2(x-3) + Cx(x-3) + D(x-3)$$

Evaluating at the zeros:
Plugging $x = 0$ gives:

$$-3 = A(0) + B(0) + C(0) + D(-3)$$

So $D = 1$.
Plugging $x = 3$ gives:

$$27 - 3 - 3 = A(27) + B(0) + C(0) + D(0)$$

$$21 = 27A$$

$$7 = 9A$$

$$A = 7/9.$$

Now there are no more zeros to plug in, so we match some coefficients:

$$x^3 - x - 3 = Ax^3 + Bx^2(x-3) + Cx(x-3) + D(x-3)$$

$$x^3 - x - 3 = (A+B)x^3 + (-3B+C)x^2 + (-3C+D)x - 3D$$

So matching the coefficient of x^3:

$$1 = A + B = (7/9) + B$$

(since we already found $A = 7/9$ earlier). So $B = 2/9$. And matching the x coefficient:

$$-1 = -3C + D$$

$$-1 = -3C + 1$$

(since we found $D = 1$). So $-2 = -3C$ and $C = 2/3$.

So the integral is

$$\int_4^5 \frac{A}{(x-3)} + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3} dx$$

$$= \int_4^5 \frac{A}{(x-3)} dx + \int_4^5 \frac{B}{x} dx + \int_4^5 \frac{C}{x^2} dx + \int_4^5 \frac{D}{x^3} dx$$

$$= A\ln(|x-3|) + B\ln(|x|) - Cx^{-1} - \frac{1}{2}Dx^{-2}|_4^5$$

$$= A(\ln(|5-3|) - \ln(|4-3|)) + B(\ln(|5|) - \ln(|4|)) - C(\frac{1}{5} - \frac{1}{4}) - \frac{1}{2}D(\frac{1}{5^2} - \frac{1}{4^2})$$

$$= A(\ln(2) - \ln(1)) + B\ln(5/4) - C(-1/20) - \frac{1}{2}D(\frac{4^2 - 5^2}{5^2 4^2})$$

$$= A\ln(2) + B\ln(5/4) + C/20 - \frac{1}{2}D(\frac{-9}{5^2 4^2})$$

$$= \frac{7}{9}\ln(2) + \frac{2}{9}\ln(5/4) + \frac{1}{30} + \frac{9}{2(5^2)4^2}$$

$$= \frac{7}{9}\ln(2) + \frac{2}{9}\ln(5/4) + \frac{107}{2400}$$

(b) Find

$$\int_2^3 \frac{x}{x^3 - 1} dx$$

Solution.
The "possible rational zeros" theorem tells us that the possible rational zeros of the denominator $x^3 - 1$ are $\pm\frac{1}{1}$, i.e. ± 1. Trying $x = 1$, we see that $1^3 - 1 = 0$, so $x = 1$ is a zero of $x^3 - 1$. Factoring with synthetic division leads to:

$$x^3 - 1 = (x-1)(x^2 + x + 1)$$

And $x^2 + x + 1$ is irreducible since the "$b^2 - 4ac$" term of the quadratic formula is $1^2 - 4(1)(1) = -3 < 0$. (So the zeros of $x^2 + x + 1$ are non-real, so it can't be factored with real linear factors.) So for $x^3 - 1$, we have one real linear factor $x - 1$, with multiplicity 1, and one irreducible quadratic factor $x^2 + x + 1$, also with multiplicity 1 (i.e. $(x^2+x+1)^1$ is what appears, not $(x^2+x+1)^2$ or any higher power). Therefore the partial fractions form is

$$\frac{x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

So multiplying through:

$$x = A(x^2+x+1) + (Bx+C)(x-1)$$

So plugging in the zero $x=1$:

$$1 = A(3) + (B(1)+C)(0)$$

So $A = 1/3$. Now matching coefficients:

$$0x^2 + 1x + 0 = (A+B)x^2 + (A-B+C)x + (A-C)$$

Matching the constant term (i.e. the coefficient of 1):

$$0 = A - C$$

So $C = 1/3$ since $A = 1/3$. And matching the coefficients of x^2:

$$0 = A + B$$

So $B = -1/3$ since $A = 1/3$.

So the integral is

$$\int_2^3 \frac{1/3}{x-1} + \frac{(-1/3)x + (1/3)}{x^2+x+1} dx$$

$$= \frac{1}{3}\int_2^3 \frac{1}{x-1} + \frac{-x+1}{x^2+x+1} dx$$

Note that $(x^2+x+1)' = 2x+1$, so if we separate $-x-\frac{1}{2}$ from the numerator $-x+1$ in the second term, we can easily integrate part of the second term:

$$= \frac{1}{3}\left[\int_2^3 \frac{1}{x-1} dx + \int_2^3 \frac{-x+(-\frac{1}{2}+\frac{3}{2})}{x^2+x+1} dx\right]$$

$$= \frac{1}{3}\left[\ln(|x-1|)\Big|_2^3 + \int_2^3 \frac{-x-\frac{1}{2}}{x^2+x+1} dx + \int_2^3 \frac{3/2}{x^2+x+1} dx\right]$$

$$= \frac{1}{3}\left[\ln(|3-1|) - \ln(|2-1|) - \frac{1}{2}\int_2^3 \frac{2x+1}{x^2+x+1} dx + \frac{3}{2}\int_2^3 \frac{1}{x^2+x+1} dx\right]$$

$$= \frac{1}{3}\left[\ln(2) - \ln(1) - \frac{1}{2}(\ln(|x^2+x+1|)\Big|_2^3 + \frac{3}{2}\int_2^3 \frac{1}{x^2+x+1} dx\right]$$

$$= \frac{1}{3}\left[\ln(2) - 0 - \frac{1}{2}\left(\ln(|3^2+3+1|) - \ln(|2^2+2+1|)\right) + \frac{3}{2}\int_2^3 \frac{1}{(x+\frac{1}{2})^2 - (\frac{1}{2})^2 + 1} dx\right]$$

(in the last term I've just completed the square, which I'm doing so as to convert the x^2+x+1 to a form like u^2+a^2, with which we can do a trig sub); subbing now $u = x + \frac{1}{2}$ in the last term, which gives $du = dx$:

$$= \frac{1}{3}\left[\ln(2) - \frac{1}{2}(\ln(13) - \ln(7)) + \frac{3}{2}\int_{x=2}^{3} \frac{1}{u^2+\frac{3}{4}} du\right]$$

And when $x = 2$, $u = x + \frac{1}{2} = \frac{5}{2}$, and when $x = 3$, $u = x + \frac{1}{2} = \frac{7}{2}$; and converting the remaining integral to the form $c/(v^2+1)$ with a constant c (in order to antidifferentiate to $c\arctan(v)$):

$$= \frac{1}{3}\left[\frac{1}{2}(2\ln(2) - \ln(13) + \ln(7)) + \frac{3}{2}\int_{u=\frac{5}{2}}^{\frac{7}{2}} \frac{1}{\frac{3}{4}(\frac{4}{3}u^2+1)} du\right]$$

$$= \frac{1}{3}\left[\frac{1}{2}(\ln(2^2) - \ln(13) + \ln(7)) + \frac{3}{2}\frac{4}{3}\int_{5/2}^{7/2} \frac{1}{(\frac{2}{\sqrt{3}}u)^2+1} du\right]$$

Subbing $v = \frac{2}{\sqrt{3}}u$, which gives $dv = \frac{2}{\sqrt{3}}du$:

$$= \frac{1}{3}\left[\frac{1}{2}\ln((4/13)(7)) + 2\int_{u=5/2}^{7/2} \frac{1}{v^2+1}\frac{\sqrt{3}}{2} dv\right]$$

$$= \frac{1}{3}\left[\frac{1}{2}\ln(28/13) + 2\frac{\sqrt{3}}{2}\int_{v=5/\sqrt{3}}^{7/\sqrt{3}} \frac{1}{v^2+1} dv\right]$$

$$= \frac{1}{3}\left[\ln(\sqrt{28/13}) + \sqrt{3}\arctan(v)\Big|_{5/\sqrt{3}}^{7/\sqrt{3}}\right]$$

$$= \frac{1}{3}\left[\ln(\sqrt{28/13}) + \sqrt{3}(\arctan(7/\sqrt{3}) - \arctan(5/\sqrt{3}))\right]$$

4. Let $\{a_n\}_{n=5}^{\infty}$ be the sequence with terms

$$a_n = \int_0^{2n} 4^{-x+10} dx.$$

(a) Determine whether $\lim_{n\to\infty} a_n$ exists, and if so, its value.

(b) How can you write this as an improper integral?

Solution.

(a) Evaluating, with a substitution of $u = -x+10$, which gives $du = -dx$, one gets

$$a_n = \int_{x=0}^{2n} 4^u(-du)$$

$$= \int_{10}^{-2n+10} 4^u(-du)$$

$$= -\frac{1}{\ln(4)}4^u\Big|_{10}^{-2n+10}$$

$$= \frac{1}{\ln(4)}(4^{10} - 4^{10-2n})$$

So letting f be the function

$$f(x) = \frac{1}{\ln(4)}(4^{10} - 4^{10-2x})$$

we have $a_n = f(n)$ for each n. But

$$\lim_{x\to\infty} \frac{1}{\ln(4)}(4^{10} - 4^{10-2x})$$

$$= \frac{1}{\ln(4)}\left[(\lim_{x\to\infty} 4^{10}) - \lim_{x\to\infty} 4^{10}4^{-2x}\right]$$

$$= \frac{1}{\ln(4)}\left[4^{10} - 4^{10}\lim_{x\to\infty}(\frac{1}{16})^x\right]$$

and as $x \to \infty$, $(1/16)^x \to 0$, since $(1/16) < 1$

$$= \frac{1}{\ln(4)}\left[4^{10} - 4^{10}(0)\right]$$

$$= \frac{1}{\ln(4)}4^{10}.$$

Therefore, since $a_n = f(n)$ and $\lim_{x\to\infty} f(x)$ exists, we know $\lim_{n\to\infty} a_n$ exists and

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \frac{1}{\ln(4)}4^{10}.$$

(b) The limit

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \lim_{b\to\infty} f(b),$$

as discussed above. But

$$\lim_{b\to\infty} f(b) = \lim_{b\to\infty} \int_0^{2b} 4^{-x+10}dx$$

by the same integration calcluation made earlier (although we were assuming that n was an integer there, since it was the index to a_n, the same calculation works for any real number b in place of n). Now as $b \to \infty$, we have $2b \to \infty$, and vice versa. So this limit is

$$\lim_{b'\to\infty} \int_0^{b'} 4^{-x+10}dx$$

which is the improper integral

$$\int_0^{\infty} 4^{-x+10}dx.$$

(Remark: there are also other possible solutions here.)

5. Find

$$\int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+1} dx$$

Solution.
We must split the integral into parts so that each part involves at most one limit. The denominator $e^{2x}+1 > 1$ for all x, since $e^{2x} > 0$, and e^x and e^{2x+1} are both continuous, so the integrand is continuous over $(-\infty, \infty)$. So we need only make one split, and we choose $x = 0$ for this, for convenience.

So the integral is

$$\int_{-\infty}^{0} \frac{e^x}{e^{2x}+1} dx + \int_{0}^{\infty} \frac{e^x}{e^{2x}+1} dx,$$

given both sides exist, or both are $+\infty$, or both are $-\infty$.

Left side:

$$\int_{-\infty}^{0} \frac{e^x}{e^{2x}+1} dx$$

$$= \lim_{b \to -\infty} \int_{b}^{0} \frac{e^x}{e^{2x}+1} dx$$

Note the denominator is $(e^x)^2 + 1 = u^2 + 1$ if $u = e^x$, and we can integrate $\int \frac{1}{u^2+1} du$, and e^x is the numerator. So sub $u = e^x$, which gives $du = e^x dx$:

$$= \lim_{b \to -\infty} \int_{x=b}^{0} \frac{du}{u^2+1}$$

$$= \lim_{b \to -\infty} \int_{u=e^b}^{e^0} \frac{1}{u^2+1} du$$

$$= \lim_{b \to -\infty} \arctan(u)\Big|_{e^b}^{1}$$

$$= \lim_{b \to -\infty} (\arctan(1) - \arctan(e^b))$$

Now $\arctan(1) = \alpha$ where $\tan(\alpha) = 1$ and $-\pi/2 < \alpha < \pi/2$, which is $\alpha = \pi/4$.

And as $b \to -\infty$, $e^b \to 0$, and arctan is continuous over its domain $(-\infty, \infty)$, and $\lim_{b \to -\infty} e^b = 0$, which is in the domain of arctan, so $\lim_{b \to -\infty} \arctan(e^b) = \arctan(\lim_{b \to -\infty} e^b) = \arctan(0) = 0$ (the last equality since $\arctan(0) = \alpha$ where $\tan(\alpha) = 0$ and $-\pi/2 < \alpha < \pi/2$, which is $\alpha = 0$).

So

$$\lim_{b \to -\infty} (\arctan(1) - \arctan(e^b))$$

$$= \pi/4 - 0 = \pi/4.$$

Right side (method 1):

$$\int_{0}^{\infty} \frac{e^x}{e^{2x+1}} dx$$

As in the left side,

$$= \lim_{b \to \infty} (\arctan(e^b) - \arctan(1))$$

And $\arctan(1) = \pi/4$ as discussed above, and as $b \to \infty$, $e^b \to \infty$, and as $x \to \infty$, $\arctan(x) \to \pi/2$, since arctan has a horizontal asymptote at $y = \pi/2$

as $x \to \infty$, since the graph of $y = \tan(x)$, over the interval $-\pi/2 < x < \pi/2$, reflects to the graph of arctan, and $\tan(x) \to \infty$ as $x \to \pi/2^-$. So

$$= \pi/2 - \pi/4 = \pi/4.$$

Right side (method 2): Note that

$$f(x) = \frac{e^x}{e^{2x}+1} = \frac{e^x/e^x}{e^{2x}/e^x + 1/e^x} = \frac{1}{e^x + e^{-x}},$$

and from the expression on the right we see f is even symmetric about $x = 0$ (i.e. $f(x) = f(-x)$ where $f(x) = 1/(e^x + e^{-x})$). Therefore

$$\int_{-\infty}^{0} f(x)dx = \int_{0}^{\infty} f(x)dx$$

and therefore

$$\pi/4 = \int_{0}^{\infty} f(x)dx$$

since we already computed the left side was $\pi/4$.

So both sides of the integral exist, so the overall integral exists, and equals their sum:

$$\int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+1}dx = \pi/4 + \pi/4 = \pi/2.$$

6. Find

$$\int_{10}^{\infty} \frac{1}{x\ln(x^3)}dx$$

Solution.

$$= \lim_{b\to\infty} \int_{10}^{b} \frac{1}{x\ln(x^3)}dx$$

Since $\ln(x^3) = 3\ln(x)$ this is

$$= \frac{1}{3} \lim_{b\to\infty} \int_{10}^{b} \frac{1}{x}\frac{1}{\ln(x)}dx$$

Now subbing $u = \ln(x)$ gives $du = dx/x$, and $1/x$ multiplies the $1/\ln(x)$ term:

$$= \frac{1}{3} \lim_{b\to\infty} \int_{u=\ln(10)}^{\ln(b)} \frac{1}{u}du$$

$$= \frac{1}{3} \lim_{b\to\infty} \ln(|u|)\Big|_{\ln(10)}^{\ln(b)}$$

$$= \frac{1}{3} \lim_{b\to\infty} [\ln(|\ln(b)|) - \ln(|\ln(10)|)]$$

Now as $b \to \infty$, $\ln(b) \to \infty$, so $|\ln(b)| \to \infty$, and as $x \to \infty$, $\ln(x) \to \infty$, so as $b \to \infty$, we have $\ln(|\ln(b)|) \to \infty$. And $\ln(10) > 0$ so $\ln(|\ln(10)|) = \ln(\ln(10))$ is just a constant. So we have

$$= \frac{1}{3}(\infty - \ln(\ln(10))) = \infty.$$

So the integral does not exist (it diverges) but it diverges to $= +\infty$.

7.(a) Given the recurrence $a_1 = 3$, $a_{n+1} = \frac{a_n - 1}{a_n}$, find the first eight terms of the sequence. Does $\lim_{n\to\infty} a_n$ exist? Explain.
Solution.

$$a_1 = 3$$

Using the recurrence formula with $n = 1$ gives

$$a_{1+1} = \frac{a_1 - 1}{a_1}$$

$$a_2 = \frac{3-1}{3} = \frac{2}{3}.$$

Now using the recurrence with $n = 2$ gives

$$a_3 = \frac{a_2 - 1}{a_2} = \frac{(2/3) - 1}{2/3} = \frac{-1/3}{2/3} = -\frac{1}{2}$$

With $n = 3$ gives:

$$a_4 = \frac{-\frac{1}{2} - 1}{-\frac{1}{2}} = \frac{-3/2}{-\frac{1}{2}} = 3$$

With $n = 4$ gives:

$$a_5 = \frac{3-1}{3} = \frac{2}{3}.$$

With $n = 5$ gives:

$$a_6 = \frac{(2/3) - 1}{2/3} = -\frac{1}{2}.$$

Notice that we are now just repeating the terms we started with: $a_4 = a_1 = 3$ and $a_5 = a_2 = 2/3$ and $a_6 = a_3 = -\frac{1}{2}$. Since a_{n+1} depends only on a_n, this means that if $n \geq 4$ and $a_n = a_{n-3}$ (as is the case with $a_4 = a_1$) then

$$a_{n+1} = \frac{a_n - 1}{a_n} = \frac{a_{n-3} - 1}{a_{n-3}} = a_{n-2}$$

(where the first equality is the recurrence formula, the second is because $a_n = a_{n-3}$, and the third is the recurrence formula when "n" is replaced by "$n-3$", which is correct for $n \geq 4$, since the recurrence holds for all $n \geq 1$.) So we deduce that

$$a_{n+1} = a_{n-2} = a_{(n+1)-3}.$$

Then from the above line we can likewise deduce that

$$a_{n+2} = a_{(n+2)-3},$$

and so on. So for all $n \geq 4$ we have $a_n = a_{n-3}$, so the sequence just keeps repeating the same 3 terms:

$$3, 2/3, -\frac{1}{2}, 3, 2/3, -\frac{1}{2}, 3, 2/3, \ldots$$

So the first 8 terms are as displayed here.

Since the sequence just keeps repeating 3 distinct terms, it never converges on any particular value, so the limit does not exist.

(b) Find a recurrence, and a general formula for a_n, for the sequence

$$4, 7, 12, 19, 28, 39, \dots,$$

assuming that the first index is $n = 1$. (Hint: for the general formula, try subtracting 3 from every term.)

Solution.

Note that

$$\begin{aligned} a_1 &= 4, \\ a_2 &= 7 = 4 + 3 = a_1 + 3, \\ a_3 &= 12 = 7 + 5 = a_2 + 5, \\ a_4 &= 19 = 12 + 7 = a_3 + 7, \\ a_5 &= 28 = 19 + 9 = a_4 + 9, \\ a_6 &= 39 = 28 + 11 = a_5 + 11. \end{aligned}$$

Note this is

$$\begin{aligned} a_1 &= 4, \\ a_{1+1} &= a_1 + (2(1) + 1) \\ a_{2+1} &= a_2 + (2(2) + 1) \\ &\dots \\ a_{5+1} &= a_5 + (2(5) + 1) \end{aligned}$$

So we have a recurrence

$$\begin{aligned} a_1 &= 4 \\ a_{n+1} &= a_n + (2n + 1). \end{aligned}$$

(There are also other possible recurrences you can use.)

For the explicit formula, following the hint:

$$a_1 - 3 = 1$$

$$\begin{aligned} a_2 - 3 &= 4 \\ a_3 - 3 &= 9 \\ a_4 - 3 &= 16 \\ a_5 - 3 &= 25 \\ a_6 - 3 &= 36 \end{aligned}$$

Notice that the numbers on the right are just the first 6 perfect squares. So we have:

$$\begin{aligned} a_1 &= 1^2 + 3 \\ a_2 &= 2^2 + 3 \\ a_3 &= 3^2 + 3 \end{aligned}$$

and so on,...

$$a_6 = 6^2 + 3$$

So we have a formula

$$a_n = n^2 + 3.$$

(Checking this agrees with the recurrence:

$$\begin{aligned} a_{n+1} &= (n+1)^2 + 3 \\ &= n^2 + 2n + 1 + 3 \\ &= (n^2 + 3) + 2n + 1 = a_n + (2n+1), \end{aligned}$$

which agrees.)

(c) Find a formula for a_n, for the sequence

$$4, 5 + \frac{1}{2}, 4 + \frac{3}{4}, 5 + \frac{1}{8}, 4 + \frac{15}{16}, \ldots$$

Based on your formula, does the limit of the sequence exist? Explain.
Solution.
Notice the sequence is

$$5 - \frac{1}{1}, 5 + \frac{1}{2}, 5 - \frac{1}{4}, 5 + \frac{1}{8}, 5 - \frac{1}{16}, \ldots$$

$$5 - \frac{(-1)^0}{2^0}, 5 - \frac{(-1)^1}{2^1}, 5 - \frac{(-1)^2}{2^2}, 5 - \frac{(-1)^3}{2^3}, 5 - \frac{(-1)^4}{2^4}, \ldots$$

So starting with index $n = 0$ we have

$$a_n = 5 - \frac{(-1)^n}{2^n}$$

or

$$a_n = 5 - (\frac{-1}{2})^n$$

Note that $(-1)^n = \cos(\pi n)$, so

$$a_n = 5 - \frac{\cos(\pi n)}{2^n}.$$

Let

$$f(x) = 5 - \frac{\cos(\pi x)}{2^x}.$$

Then f is a function with domain $(-\infty, \infty)$, $a_n = f(n)$ for integers $n \geq 0$ and

$$\lim_{x\to\infty} f(x) = 5 - \lim_{x\to\infty} \frac{\cos(\pi x)}{2^x}.$$

We can use the Squeeze Theorem to compute the remaining limit. For all x, we have $-1 \leq \cos(\pi x) \leq 1$, and $2^x > 0$, so

$$-\frac{1}{2^x} \leq \frac{\cos(\pi x)}{2^x} \leq \frac{1}{2^x},$$

and

$$\lim_{x\to\infty} -\frac{1}{2^x} = 0 = \lim_{x\to\infty} \frac{1}{2^x} = 0$$

So by the Squeeze Theorem,

$$\lim_{x\to\infty} \frac{\cos(\pi x)}{2^x} = 0.$$

So

$$\lim_{x\to\infty} f(x) = 5.$$

Therefore (since this limit exists, and $a_n = f(n)$), the limit $\lim_{n\to\infty} a_n$ also exists. (And it equals

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = 5.$$

though you weren't actually asked for its value.)

8.(a) If the sequence $a_0, a_1, a_2, \ldots$ is geometric and $a_1 = 5$ and $a_3 = 15$, what can you say about the ratio r of the sequence? What is the value of a_{101}?
Solution.
Since the sequence is geometric, letting r be the ratio of the sequence (a non-zero constant), we have a constant $a \neq 0$ such that

$$a_n = ar^n$$

for all $n \geq 0$. Since

$$5 = a_1 = ar^1 = ar$$

and

$$15 = a_3 = ar^3,$$

dividing the 2nd equation here by the first,

$$15/5 = ar^3/(ar)$$

$$3 = r^2$$

So $r = \pm\sqrt{3}$. Since we're not given any further information, it could be either way: if $r = \sqrt{3}$ then $5 = a_1 = ar = a\sqrt{3}$, so $a = 5/\sqrt{3}$. The sequence

$$a_n = (5/\sqrt{3})\sqrt{3}^n, \quad n \geq 0$$

is geometric, has first index $n = 0$, and satisfies the requirements that $a_1 = 5$ and $a_3 = 15$.

If $r = -\sqrt{3}$, then $5 = a_1 = ar = a(-\sqrt{3})$, so $a = -5/\sqrt{3}$, so

$$a_n = (-5/\sqrt{3})(-\sqrt{3})^n, \quad n \geq 0$$

also gives a geometric sequence, has first index $n = 0$, and satisfies the requirements.

So $r = \pm\sqrt{3}$.

We have

$$a_{101} = ar^{101},$$

and

$$5 = a_1 = ar.$$

So

$$a_{101}/5 = a_{101}/a_1 = ar^{101}/(ar) = r^{100} = \sqrt{3}^{100} = 3^{50}.$$

So

$$a_{101}/5 = 3^{50}$$

$$a_{101} = 5(3^{50}).$$

(Alternatively if you use either of the possible formulas for a_n derived above, and compute a_{101} from these, both give $a_{101} = 5(3^{50})$, so this is the only possible value for a_{101}.)

(b) If the sequence $a_0, a_1, a_2, \ldots$ is such that $a_1 = 5$ and $a_3 = 15$, what can you say about a_{101}?

Solution.

All we know is that a_{101} is some number: given any number x, the sequence

$$x, 5, x, 15, x, x, x, x, x, x, x, x, x, \ldots$$

(starting with index $n = 0$, and $a_n = x$ for all $n \geq 4$) has first index $n = 0$, and $a_1 = 5$, $a_3 = 15$, and $a_{101} = x$. So the fact that $a_1 = 5$ and $a_3 = 15$ and the indexing starts at $n = 0$ does not restrict the possible values of a_{101}. (In part (a), the sequence was assumed to be *geometric*, which puts a strong restriction on the possibilities for the sequence. But in part (b) there is no such assumption.)

9. Suppose $\{c_n\}_{n=3}^{\infty}$ is a sequence such that $\lim_{n\to\infty} c_n = 6$, and $\{d_n\}_{n=3}^{\infty}$ has limit 3. In problems (a*), (a) and (b), compute the limit of the sequence, if possible, where in (a*) the sequence has the terms e_n shown, in (a), the terms a_n and in (b), the terms b_n. Do part (c).

(a*) The sequence with terms

$$e_n = nd_n^3 - \frac{n^2 d_n^3 - d_n}{n+1}, \quad n \geq 3$$

(a) The sequence with terms

$$a_n = nd_n - \frac{(n^2 - 5n + 1)c_n}{2n}, \quad n \geq 3$$

(b) The sequence with terms

$$b_n = n^{\frac{1}{n^3}}, \quad n \geq 1$$

(c) Determine whether

$$\sum_{n=3}^{\infty} d_n/c_n$$

converges.

Solution.

(a*) We have

$$e_n = nd_n^3 - \frac{n^2 d_n^3 - d_n}{n+1},$$

so now we can't use the "subtraction" limit law, since one can show that $\lim_{n\to\infty} nd_n^3 = \infty(3^3) = \infty$ and $\lim_{n\to\infty} \frac{n^2 d_n^3 - d_n}{n+1} = \infty$, so we'd get $\infty - \infty$, which is not valid (this is an indeterminate form). So we need to simplify first:

$$\lim_{n\to\infty} e_n = \lim_{n\to\infty} nd_n^3 - \frac{n^2 d_n^3 - d_n}{n+1}$$

$$= \lim_{n\to\infty} \frac{n(n+1)d_n^3 - n^2 d_n^3 + d_n}{n+1}$$

$$= \lim_{n\to\infty} \frac{nd_n^3 + d_n}{n+1}$$

$$= \lim_{n\to\infty} \frac{n}{n+1} d_n^3 + \frac{d_n}{n+1}$$

And applying limit laws (which will apply as long as the various limits end up existing):

$$= (\lim_{n\to\infty} \frac{n}{n+1}) \lim_{n\to\infty} d_n^3 + \lim_{n\to\infty} \frac{d_n}{n+1}$$

Applying more limit laws:

$$= 1(\lim_{n\to\infty} d_n)^3 + \lim_{n\to\infty} \frac{d_n}{n+1}$$

$$= (3)^3 + \lim_{n\to\infty} \frac{d_n}{n+1}$$

And as $n \to \infty$, $d_n \to 3$ and $n+1 \to \infty$, so the remaining limit has form $3/\infty$ which is a determinate form, and gives a result of 0, so $\lim_{n\to\infty}(d_n/(n+1))$ exists and $= 0$:

$$= 27 + 0$$

(Since the various limits existed at the end, the earlier limits also exist, so all the limit laws do in fact apply.) (Not to say that $\lim_{n\to\infty} n+1$ exists, but just that $\lim_{n\to\infty} d_n/(n+1) = 0$ exists and $\lim_{n\to\infty} n/(n+1) = 1$ exists and $\lim_{n\to\infty} d_n^3 = 27$ exists.)

So

$$\lim_{n\to\infty} e_n = 27.$$

(a) It is not possible to compute the limit. For first note that

$$a_n = nd_n - \frac{(n^2 - 5n + 1)c_n}{2n}$$

$$= \frac{2n^2 d_n - c_n n^2 + (5n-1)cn}{2n}$$

$$= \frac{n^2(2d_n - c_n)}{2n} + \frac{(5n-1)c_n}{2n}$$

$$= n\frac{(2d_n - c_n)}{2} - \frac{c_n}{2n} + \frac{5c_n}{2}.$$

So

$$\lim_{n\to\infty} a_n$$

$$= \lim_{n\to\infty} n\frac{(2d_n - c_n)}{2} - \frac{c_n}{2n} + \frac{5c_n}{2}$$

And using some limit laws:

$$= (\lim_{n\to\infty} n\frac{(2d_n - c_n)}{2}) - (\lim_{n\to\infty} c_n)(\lim_{n\to\infty} \frac{1}{2n}) + \frac{5}{2}\lim_{n\to\infty} c_n$$

$$= (\lim_{n\to\infty} n\frac{(2d_n - c_n)}{2}) - 6(\lim_{n\to\infty} \frac{1}{2n}) + \frac{5}{2}(6)$$

Now the function $f(x) = 1/(2x)$ satisfies $1/(2n) = f(n)$ for all $n \geq 3$, and $\lim_{x\to\infty} f(x) = 0$, so $\lim_{n\to\infty}(1/(2n)) = 0$ also. So we have the overall limit is

$$\lim_{n\to\infty} n\frac{2d_n - c_n}{2} - 6(0) + 15$$

But note that $\lim_{n\to\infty}(2d_n - c_n)/2 = \lim_{n\to\infty} d_n - \frac{1}{2}\lim_{n\to\infty} c_n$ (from some limit laws), which $= 3 - \frac{1}{2}(6) = 0$. And $\lim_{n\to\infty} n = \infty$. So the remaining limit has form $\infty \cdot 0$, which is indeterminate. So we would need to know more information about the sequences c_n and d_n in order to compute the limit.

(E.g., it's possible that $d_n = 3$ and $c_n = 6$ for all n (since $\lim_{n\to\infty} 3 = 3$ and $\lim_{n\to\infty} 6 = 6$); then $2d_n - c_n = 0$ for all n, in which case the overall limit would be $(\lim_{n\to\infty} n(2d_n - c_n)/2) + 15 = 0 + 15 = 15$. But on the other hand, it's also possible that $d_n = 3$ and $c_n = 6 + (1/n)$ for all n, in which case the overall limit would be $(\lim_{n\to\infty} n(2d_n - c_n)/2) + 15 = (\lim_{n\to\infty} n(-1/(2n))) + 15 = (\lim_{n\to\infty} -\frac{1}{2}) + 15 = 14.5$. And with other sequences c_n, d_n (which converge to 6 and 3 respectively), you could also have any other limit, or the limit might not exist.)

(b) Here we compute $\lim_{n\to\infty} n^{1/n^3}$. Let

$$f(x) = x^{1/x^3}.$$

Then for $n \geq 1$, $b_n = f(n)$. So if $\lim_{x\to\infty} f(x)$ exists, then it equals the limit we need to find.

So consider

$$\lim_{x\to\infty} x^{1/x^3}.$$

As $x \to \infty$, the base $x \to \infty$, and the exponent $1/x^3 \to 0$, so this limit has form ∞^0, indeterminate. So,

$$\lim_{x\to\infty} f(x)$$

$$= \lim_{x\to\infty} e^{\ln(x)/x^3}$$

$$= e^L$$

where

$$L = \lim_{x\to\infty} \frac{\ln(x)}{x^3}.$$

As $x \to \infty$, $\ln(x) \to \infty$ and $x^3 \to \infty$, so this limit has form ∞/∞, indeterminate, but L'Hopital's rule is valid for it. So

$$= \lim_{x\to\infty} \frac{(\ln(x))'}{(x^3)'}$$

$$= \lim_{x\to\infty} \frac{1/x}{3x^2}$$

$$= \lim_{x\to\infty} \frac{1}{3x^3} = 0.$$

So $L = 0$, so $(\lim_{x\to\infty} f(x)) = e^L = e^0 = 1$. So this limit exists, so the original limit $\lim_{n\to\infty} b_n$ also exists and

$$\lim_{n\to\infty} b_n = \lim_{x\to\infty} f(x) = 1.$$

(c)

$$\sum_{n=3}^{\infty} d_n/c_n$$

Since the terms $d_n \to 3$ and $c_n \to 6$, we have

$$\lim_{n\to\infty} \frac{d_n}{c_n} = \frac{\lim_{n\to\infty} d_n}{\lim_{n\to\infty} c_n} = \frac{3}{6} = \frac{1}{2}.$$

So in the series, for large values of n, the term d_n/c_n is very close to $1/2$, so it ends up adding term after term, with the terms very close to $\frac{1}{2}$.

This implies that the series diverges: for large enough values of n, $d_n/c_n > \frac{1}{4}$ (since $d_n/c_n \to \frac{1}{2}$). Let's say that N is some fixed integer such that for all $n > N$, we have $d_n/c_n > 1/4$. Then the series is

$$\sum_{n=3}^{\infty} \frac{d_n}{c_n} = \frac{d_3}{c_3} + \frac{d_4}{c_4} + \ldots + \frac{d_N}{c_N} + \frac{d_{N+1}}{c_{N+1}} + \frac{d_{N+2}}{c_{N+2}} + \ldots$$

When $M > N$, the partial sum $(\mathfrak{BS})_M$ has the form

$$(\mathfrak{BS})_M = \sum_{n=3}^{M} \frac{d_n}{c_n} = \frac{d_3}{c_3} + \frac{d_4}{c_4} + \ldots + \frac{d_N}{c_N} + \frac{d_{N+1}}{c_{N+1}} + \frac{d_{N+2}}{c_{N+2}} + \ldots + \frac{d_M}{c_M}$$

(recall that $(\mathfrak{BS})_M$ is the partial sum given by making the upper bound equal M; this is different to S_M, the notation in the book; S_M is the sum of the first M terms).

$$= A + \frac{d_{N+1}}{c_{N+1}} + \frac{d_{N+2}}{c_{N+2}} + \ldots + \frac{d_M}{c_M}$$

where $A = \sum_{n=3}^{N} \frac{d_n}{c_n}$. But since $n = N+1 > N$, our choice of N implies $\frac{d_{N+1}}{c_{N+1}} > \frac{1}{4}$. Similarly $\frac{d_{N+2}}{c_{N+2}} > \frac{1}{4}$, and so on, through $\frac{d_M}{c_M} > \frac{1}{4}$. So

$$(\mathfrak{BS})_M > A + \frac{1}{4} + \frac{1}{4} + \ldots + \frac{1}{4}$$

where there is a "$\frac{1}{4}$" term for each $n = N+1, \ldots, M$. So there are $M - N$ terms "$\frac{1}{4}$". So

$$(\mathfrak{BS})_M > A + (M-N)\frac{1}{4}$$

Note that the term A is just the sum of the terms $\frac{d_3}{c_3} + \ldots + \frac{d_N}{c_N}$, and N was a fixed constant, independent of the upper index M. So

$$(\mathfrak{BS})_M > (A - \frac{N}{4}) + \frac{M}{4},$$

and the term $A - \frac{N}{4} = B$ is a constant independent of M.

$$(\mathfrak{BS})_M > B + \frac{M}{4}.$$

But then as $M \to \infty$, $M/4 \to \infty$, and B is a constant, so $B + (M/4) \to \infty$ also. Since $(\mathfrak{BS})_M > B + M/4$, we get $(\mathfrak{BS})_M \to \infty$ also:

$$\lim_{M\to\infty} (\mathfrak{BS})_M = \infty,$$

and

$$\sum_{n=3}^{\infty} \frac{d_n}{c_n} = \lim_{M\to\infty} (\mathfrak{BS})_M = \infty.$$

So the series diverges (to $+\infty$).

(Remark: A similar argument shows that if $\lim_{n\to\infty} a_n = L \neq 0$, then $\sum_{n\to\infty} a_n$ diverges (to $+\infty$ if $L > 0$, or to $-\infty$ if $L < 0$). In fact, *if* the series $\sum_{n=k}^{\infty} a_n$ converges, *then* the limit of the terms a_n is 0: $\lim_{n\to\infty} a_n = 0$. But given that $\lim_{n\to\infty} a_n = 0$, you *cannot* conclude that $\sum_{n=k}^{\infty} a_n$ converges; e.g. the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, to $+\infty$, even though the terms $\frac{1}{n} \to 0$ as $n \to \infty$.)

10.(a) Find the first four partial sums for the series

$$\sum_{n=0}^{\infty} \frac{1}{7 \cdot 3^n}$$

(b) Evaluate the sum

$$\sum_{n=0}^{1{,}000{,}000} \frac{1}{7 \cdot 3^n},$$

simplifying fully.

(c) Does the series in (a) converge? If so, find the value it converges to.

(d) Repeat (c) for

$$\sum_{n=0}^{\infty} (-7/4)^n (3/4)^n$$

(e) Repeat (c) for

$$\sum_{n=0}^{\infty} 500(-7/4)^n (3/4)^{2n}$$

Solution.

(a) The first four partial sums are: First:

$$\sum_{n=0}^{0} \frac{1}{7 \cdot 3^n}$$

$$= \frac{1}{7 \cdot 3^0} = \frac{1}{7}.$$

(Note: The above partial sum is $(\mathfrak{BS})_0$ (the 0-bounded partial sum, i.e. the partial sum with upper bound 0; the somewhat difficult to read symbols "$\mathfrak{BS}$" are "B" and "S" for "bounded sum"), or in the other terminology, is S_1 (the 1st partial sum), i.e. the sum of first term. Since the series is summing starting from index $n = 0$, summing just one term means we get $\sum_{n=0}^{0}$, which has upper bound 0, which gives $(\mathfrak{BS})_0$. So when the series starts summing from index $n = 0$, we have $S_1 = (\mathfrak{BS})_0$.)

Second: (S_2 = sum of first two terms, which since our series starts summing at $n = 0$, this is $\sum_{n=0}^{1}$, which has upper bound 1, i.e. giving $(\mathfrak{BS})_1$.)

$$\sum_{n=0}^{1} \frac{1}{7 \cdot 3^n}$$

$$= \frac{1}{7 \cdot 3^0} + \frac{1}{7 \cdot 3^1}$$
$$= \frac{1}{7} + \frac{1}{21}.$$

Third: ($S_3 = (\mathfrak{BS})_2$ for this series):

$$\sum_{n=0}^{2} \frac{1}{7 \cdot 3^n}$$

$$= \frac{1}{7 \cdot 3^0} + \frac{1}{7 \cdot 3^1} + \frac{1}{7 \cdot 3^2}$$
$$= \frac{1}{7} + \frac{1}{21} + \frac{1}{63}$$

Fourth: ($S_4 = (\mathfrak{BS})_3$ for this series):

$$\sum_{n=0}^{3} \frac{1}{7 \cdot 3^n}$$

$$= \frac{1}{7 \cdot 3^0} + \frac{1}{7 \cdot 3^1} + \frac{1}{7 \cdot 3^2} + \frac{1}{7 \cdot 3^3}$$
$$= \frac{1}{7} + \frac{1}{21} + \frac{1}{63} + \frac{1}{189}$$

(Remark: in general for a series summing from lower bound $n = 0$, we have "sum of first n terms" $= S_n =$ "sum with upper bound $n - 1$" $= (\mathfrak{BS})_{n-1}$. If the series instead starts summing from lower bound $n = 1$, we just have $S_n = (\mathfrak{BS})_n$.)

(b)

$$\sum_{n=0}^{1,000,000} \frac{1}{7 \cdot 3^n},$$

$$= \frac{1}{7(3^0)} + \frac{1}{7(3^1)} + \ldots + \frac{1}{7(3^{1,000,000})},$$

but we need to simplify fully, so we need another way. We use the fact that for a geometric sequence with first term a and ratio r, the sum of the first M terms $a + ar + ar^2 + \ldots + ar^{M-1}$ is

$$\sum_{n=0}^{M-1} ar^n = a\frac{1-r^M}{1-r}$$

Our series is

$$\sum_{n=0}^{\infty} a_n$$

where

$$a_n = \frac{1}{7(3^n)}, \quad n \geq 0.$$

We can write this like

$$a_n = \frac{1}{7}\frac{1}{3^n} = \frac{1}{7}(\frac{1}{3})^n = ar^n, \quad n \geq 0.$$

where $a = 1/7$ and $r = 1/3$. This is in the correct form for a geometric sequence, and our sequence has first index $n = 0$. The first term is then $a_0 = (1/7)$, and ratio $r = (1/3)$. So then

$$\sum_{n=0}^{1,000,000} a_n$$

is the sum of the first $M = 1,000,001$ terms is

$$\sum_{n=0}^{1,000,000} a_n = a\frac{1-r^{1,000,001}}{1-r} = \frac{1}{7}(\frac{1-(1/3)^{1,000,001}}{2/3})$$

$$= \frac{3(1-(1/3)^{1,000,001})}{14}.$$

(Note that since $(1/3)^{1,000,001}$ is very close to 0, this sum is almost $3/14$.)

(c) Yes, since it is a geometric series with ratio $r = 1/3$, so the ratio r satisfies $|r| < 1$. So the series conveges, to

$$a\frac{1}{1-r},$$

where a is the first term.

$$= \frac{1}{7}\frac{1}{1-(1/3)}$$

$$= \frac{1}{7}\frac{3}{2} = \frac{3}{14}.$$

(The calculations from part (b) can be used to verify the formula $a/(1-r)$ here:

$$\sum_{n=0}^{\infty} a_n$$

$$= \lim_{M\to\infty} \sum_{n=0}^{M} a_n$$

$$= \lim_{M\to\infty} \sum_{n=0}^{M} ar^n$$

$$= \lim_{M\to\infty} \sum_{n=0}^{M} \frac{1}{7}(\frac{1}{3})^n$$

and the sum is the sum of the first $M+1$ terms, starting at first term $a = 1/7$, and ratio $r = 1/3$, so this is

$$= \lim_{M\to\infty} a\frac{1-r^{M+2}}{1-r}$$

$$= \lim_{M\to\infty} \frac{1}{7}\frac{1-(\frac{1}{3})^{M+2}}{1-\frac{1}{3}}$$

$$= \lim_{M\to\infty} \frac{3(1-(\frac{1}{3})^{M+2})}{14}$$

And as $M \to \infty$, the term $(1/3)^{M+2} \to 0$, resulting in

$$= \frac{3(1-0)}{14} = \frac{3}{14}.$$

(d)

$$\sum_{n=0}^{\infty} (-7/4)^n (3/4)^n$$

$$= \sum_{n=0}^{\infty} (\frac{-21}{16})^n = \sum_{n=0}^{\infty} r^n,$$

where $r = -21/16$. This is a geometric series with ratio $r = -21/16$. Since the ratio r satifies $|r| \geq 1$, the series does not converge.

(e)

$$\sum_{n=0}^{\infty} 500(-7/4)^n (3/4)^{2n}$$

$$= \sum_{n=0}^{\infty} 500(\frac{-7}{4}\frac{9}{16})^n$$

$$= \sum_{n=0}^{\infty} 500(\frac{-63}{64})^n$$

$$= \sum_{n=0}^{\infty} ar^n$$

where $a = 500$ and $r = -63/64$. This is a geometric series, with ratio $r = -63/64$, so the ratio r satisfies $|r| < 1$, so the series converges; note the first term is $a = 500$ (the first index is when $n = 0$) so the series converges to

$$a\frac{1}{1-r} = 500(\frac{1}{1-(-63/64)}) = \frac{500}{127/64} = \frac{500(64)}{127}.$$