Vector fields on the line

This talk will be a gentle introduction to some aspects of the theory of representations of Lie algebras by means of an example: the Lie algebra $\mathrm{Vec}(\mathbb{R})$ of vector fields on the line.

Vector fields have a natural algebraic operation, the *Lie bracket,* i.e., the commutator

$$[X, Y] = XY - YX.$$

Under this operation, $\mathrm{Vec}(\mathbb{R})$ contains an important 3-dimensional subalgebra:

$$\left\{ f(x) \tfrac{d}{dx} : \ f \text{ is a polynomial of degree } \leq 2 \right\}.$$

This subalgebra is isomorphic to $\mathfrak{sl}_2$, the traceless 2×2 matrices. $\mathfrak{sl}_2$ is an important Lie algebra in its own right: it is the smallest semisimple Lie algebra.

$\mathrm{Vec}(\mathbb{R})$ acts naturally on the space $\mathrm{Diff}(\mathbb{R})$ of all differential operators on the line, and under this action $\mathrm{Diff}(\mathbb{R})$ can be written as a direct sum of subspaces invariant under $\mathfrak{sl}_2$. In this talk we will describe this decomposition, which leads to some beautiful combinatorics discovered as recently as the 1990's in work of Cohen, Manin, and Zagier. Along the way we will see the finite dimensional representations of $\mathfrak{sl}_2$ and, time permitting, their Clebsch-Gordan rules.

Since the objects involved are quite concrete, no prior knowledge of Lie algebras will be assumed: only basic calculus and linear algebra.