Homework

1. Use the method we applied to S_3 and A_4 to classify the irreps of the non-abelian order 21 group.

2. Consider the rep of $\mathbb{Z}_n$ on $\mathbb{R}^2$ defined by $1 \mapsto \mathrm{Rot}_{2\pi/n}$. Prove that for $n > 2$, it is irreducible over $\mathbb{R}$. Find $\mathrm{End}_{\mathbb{Z}_n} \mathbb{R}^2$, prove that it is a 2-dim'l alg, and identify it.

   Then regard $1 \mapsto \mathrm{Rot}_{2\pi/n}$ as a rep of $\mathbb{Z}_n$ on $\mathbb{C}^2$. Decompose it into irreps and find $\mathrm{End}_{\mathbb{Z}_n} \mathbb{C}^2$.

3. Consider the perm rep Perm of S_n on $\mathbb{C}^n$.

   (a) Prove that Perm is the direct sum $\pi_1 \oplus \pi_{1-n}$ of two irreps. Prove that this is also true for $\mathrm{Perm}|_{A_n}$.

   (b) Prove that $\Lambda^2 \mathrm{Perm} \cong \pi_{1-n} \oplus \Lambda^2 \pi_{1-n}$, and that $\Lambda^2 \pi_{1-n}$ is irr. (This is a notational challenge. Try for $n=6$ first.)

   (c) Difficult: prove $\mathcal{S}^2 \mathrm{Perm} = \pi_1 \oplus \pi_{1-n} \oplus \mathcal{S}^2 \pi_{1-n}$, and decompose $\mathcal{S}^2 \pi_{1-n}$ into irreps.

④ Suppose $(V, \langle , \rangle)$ is a complex vec space with a PDH form. Assume that $T: \mathbb{C}^n \to V$ is a linear bijection sth $\langle Tz, Tz' \rangle = z^H z'$ for all $z, z' \in \mathbb{C}^n$. Prove that $Ad_T(A) := TAT^{-1}$ defines an isomorphism from U_n to $U(V, \langle , \rangle)$.

⑤ Let A, A' be $n \times n$ matrices.

(a) If MAM^{-1} & $M'A'M'^{-1}$ are diagonal (upper Δular), the $(M \otimes M')(A \otimes A')(M \otimes M')^{-1}$ is diag (upper Δ).

(b) If $Av = \lambda v$, $A'v' = \lambda' v'$, then $(A \otimes A')(v \otimes v') = \lambda\lambda'(v \otimes v')$.

(c) Find the evals and det of $A \otimes A'$.

⑥ Compute the char tables (w/ ring str) of S_3, A_4, S_4.

⑦ Let π, π' be reps of G on $\mathbb{C}^n$, $\mathbb{C}^{n'}$. Prove $\pi' \otimes \pi \cong \pi' \otimes \pi$.

If π'' is a third rep, on $\mathbb{C}^{n''}$, prove $(\pi \otimes \pi') \otimes \pi'' \cong \pi \otimes (\pi' \otimes \pi'')$.

If $\pi_1, \pi_2, \pi_1', \pi_2'$ are reps, prove

$$(\pi_1 \oplus \pi_2) \otimes (\pi_1' \oplus \pi_2') \cong \pi_1 \otimes \pi_1' \oplus \pi_1 \otimes \pi_2' \oplus \pi_2 \otimes \pi_1' \oplus \pi_2 \otimes \pi_2'.$$

Do 7 in two ways: first, show that the reps have the same character, and second, find explicit isomorphisms.

8 Prove (a) $S^2(\pi_1 \oplus \pi_2) \cong S^2\pi_1 \oplus (\pi_1 \otimes \pi_2) \oplus S^2\pi_2$,

(b) $\Lambda^2(\pi_1 \oplus \pi_2) \cong \Lambda^2\pi_1 \oplus (\pi_1 \otimes \pi_2) \oplus \Lambda^2\pi_2$. (Do in same two ways as 7.)

9 Find the char table of A_5, with its ring structure.

10 Let X a set w/ a left G-action: so $g \in G, x \in X \Rightarrow gx \in X$.

Prove that $(\lambda(g)f)(x) = f(g^{-1}x)$ defines a rep λ of G on L^2X, and that $\langle f, f' \rangle = \frac{1}{|X|}\sum_{x \in X} \overline{f(x)} f'(x)$ is λ-iPDH.

11 Prove that the left and right reg reps λ and ρ of G on L^2G are equiv, and find an explicit isomorphism.

Prove also that they commute: $\lambda(g) \circ \rho(h) = \rho(h) \circ \lambda(g)$.

12 Let X in 10 be "homogeneous": a single G-orbit.

Fix $x_0 \in X$, set $S = \mathrm{Stab}_G x_0$. Prove that $\chi_{(\lambda, L^2X)}(g)$ is

$\boxed{\text{~~the number of elts of Class(g) in S: $|S \cap \mathrm{Class}(g)|$~~}}$

the number of cosets xS sth $gxS = xS$. Equate this number to $|C_G(g)|\,|S \cap Cl_G(g)| / |S|$.

(13) Prove that $\big((\lambda \times \rho)(g,h)f\big)(x) = f(g^{-1}xh)$ defines a rep $\lambda \times \rho$ of $G \times G$ on L^2G. Let $\Delta : G \to G \times G$ be the homo $\Delta(g) = (g,g)$. The adjoint rep Ad of G on L^2G is $(\lambda \times \rho) \circ \Delta$. Prove that

$$(Ad(g)f)(x) = f(g^{-1}xg).$$

Prove that $\{f \in L^2G : Ad(g)f = f \ \forall g\} = L^2_{cl}G$.

(See 18) (14) Take H a sbgrp of G, (ψ, W) a rep of H. Let π be $\mathrm{Ind}_H^G(\psi, W)$. So π acts on

$$\{f : G \to W : f(gh^{-1}) = \psi(h)f(g) \ \forall\, h \in H, g \in G\},$$

by $(\pi(g)f)(x) = f(g^{-1}x)$. Regard χ_ψ as a fnctn on G, via $\chi_\psi(g) = 0$ for $g \notin H$. Prove $\chi_\pi(g) = \frac{1}{|H|} \sum_{x \in G} \chi_\psi(xgx^{-1})$

$= \frac{|C_G(g)|}{|H|} \sum_{y \in H \cap Cl_G(g)} \chi_\psi(y)$. Match with F4H Ex 3.19.

15 Take H a sbgrp of G, W a rep of H, V a rep of G.

Define a map from $\mathrm{Hom}_H(V,W)$ to $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W)$

by $(\xi : V \xrightarrow{H} W) \mapsto (\Xi : V \to \mathrm{Ind}_H^G W)$, where

$$(\Xi(v))(g) = \xi(g^{-1}v).$$

Prove that $\Xi(v) : G \to W$ is in fact in $\mathrm{Ind}_H^G W$,

and that $\xi \mapsto \Xi$ is a linear bijection. (Frobenius reciprocity)

16 Let (π_1, V_1) be a rep of G_1, (π_2, V_2) a rep of G_2.

(a) Prove that $(\pi_1 \boxtimes \pi_2)(g_1, g_2) = \pi_1(g_1) \otimes \pi_2(g_2)$ defines

a rep of $G_1 \times G_2$ on $V_1 \boxtimes V_2$.

(b) Prove that π_1, π_2 irr $\Leftrightarrow$ $\pi_1 \boxtimes \pi_2$ irr.

(c) (Hard!) Prove that if π an irrep of $G_1 \times G_2$, then

$\pi \cong \pi_1 \boxtimes \pi_2$ for some irreps π_1, π_2.

17 Jacobsen Density (Hard!): if π an irrep of G on V, then

the algebra generated by the image $\pi(G)$ is all of $\mathrm{End}\, V$.

(18) (This is a "prequel" to (14).) Let $\{x_1 H, \cdots, x_q H\} = G/H$.

Prove that restriction from G to $\{x_1, \cdots, x_q\}$ is a bijection $\mathrm{Ind}_H^G W$ to the space of all fnctns from $\{x_1, \cdots, x_q\}$ to W:

$$\mathrm{Res}\big|_{\{x_1, \cdots, x_q\}} : \mathrm{Ind}_H^G W \xrightarrow{\text{bij}} \{f : \{x_1, \cdots, x_q\} \to W\}.$$

Conclude that $\dim \mathrm{Ind}_H^G W = |G/H| \dim W$.

(19) Check: $\mathrm{Ind}_G^G V = V$, $\mathrm{Ind}_e^G \mathbb{C} = L^2 G$, $\mathrm{Ind}_H^G \mathbb{C} = L^2(G/H)$,

$\mathrm{Ind}_{A_3}^{S_3}$: $\mathbb{C}_1 \mapsto \pi_1 \oplus \pi_{-1}$, $\mathbb{C}_{\zeta_3^{\pm 1}} \mapsto \pi_2$,

$\mathrm{Ind}_V^{A_4}$: $\mathbb{C}_{1,1} \mapsto \pi_1 \oplus \pi_{\zeta_3} \oplus \pi_{\zeta_3^{-1}}$, $\mathbb{C}_{\text{non-triv}} \mapsto \pi_3$ (in all 3 cases),

$\mathrm{Ind}_{S_{n-1}}^{S_n} \mathbb{C}_1 \mapsto \mathrm{Perm}$, $\mathrm{Ind}_{S_{n-1}}^{S_n} \mathbb{C}_{-1} \mapsto ?$,

$\mathrm{Ind}_{A_4}^{A_5}$: $\pi_1 \mapsto \mathrm{Perm}$, $\pi_{\zeta_3^{\pm}} \mapsto ?$, $\mathrm{Ind}_{D_5}^{A_5} \mathbb{C}_{\pm 1} \mapsto ?$

(20) For $K < H < G$, $\mathrm{Ind}_H^G \circ \mathrm{Ind}_K^H = \mathrm{Ind}_K^G$.

(21) Prove: $\exp(MAM^{-1}) = M(\exp A)M^{-1}$, $\exp\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix}$,

$\exp(A+B) = e^A e^B$ if $AB = BA$, $\exp\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix} = e^\lambda \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and

exp (any non-diagable matrix) = a non-diagable matrix.

22 Prove: $\mathbb{R}$, S^1, and S^3 have no sbgrps of finite index

(Here the π_n are S^3 rep's) 23 Prove: for $n' \leq n$, $\pi_n \otimes \pi_{n'} \cong \pi_{n-n'+1} \oplus \pi_{n-n'+3} \oplus \cdots \oplus \pi_{n+n'-3} \oplus \pi_{n+n'-1}$.

24 Consider $\mathrm{Ind}_{S^1}^{S^3} e^{ik\theta}$. By HW 15, "Frobenius reciprocity",

$$\mathrm{Ind}_{S^1}^{S^3} e^{ik\theta} \cong \bigoplus_{n=1}^{\infty} m_n \pi_n,$$

where the multiplicity m_n of π_n is the same as the multiplicity of $e^{ik\theta}$ in π_n restricted to S^1. Deduce that $m_n = 1$ if $n - |k|$ is non-negative and even, and $m_n = 0$ o.w.

(This exercise is not intended to be done rigorously if you only know what we have covered, as for G and H Lie grps, Ind_H^G (Rep of H) is an ∞ dim'l topological vector space, a ~~the~~ space of sections of a vector bundle over G/H. The allowed sections depend on the type of induction. Here we would want L^2 sections.)

25 $(z + j\zeta) \mapsto \begin{pmatrix} z & -\bar{\zeta} \\ \zeta & \bar{z} \end{pmatrix}$ is an iso: $\mathbb{H} = \{z + j\zeta : z, \zeta \in \mathbb{C}\}$ to ~~SU_2~~ a subalgebra of $\mathrm{Mat}_2\,\mathbb{C}$. Deduce $S\mathbb{H} \cong SU_2$.

(26) If G is an abelian closed subgroup of $GL_n\mathbb{C}$, its $[,]$ is 0.

(27) If π is any complex-linear rep of $sl_2\mathbb{C}$ on $\mathbb{C}^m$, then $\pi: sl_2\mathbb{C} \to sl_m\mathbb{C}$.

(28) Let π be a $(\mu+1)$-dim'l irrep of $sl_2\mathbb{C}$. Let v_μ be a μ-wgt vector. For $0 \le k \le \mu$, let $v_{\mu-2k} = \frac{1}{k!} f^k v_\mu$. Find the matrices of h, e, f in the basis $\{v_\mu, \cdots, v_{-\mu}\}$.

(29) Let $\mathfrak{g} \subset \mathfrak{gl}_n\mathbb{C}$ be any Lie alg. Define $ad(X)Y = [X, Y]$ for X, Y in $\mathfrak{g}$. Prove that ad is a rep of $\mathfrak{g}$ on itself. It is called the "adjoint rep".

(30) Prove that the adjoint rep of $sl_2\mathbb{C}$ is irreducible.