The exam will have two parts: theory and computation. All answers must be proven. You will see that this practice exam is quite difficult! The actual exam will be similarly difficult, but keep in mind that you are not expected to do anywhere near all the problems. The best strategy is to begin by looking through all the problems and marking those you plan to work on first. I will count the exam as worth 150 points, but the total points possible will be higher than that. I will decide on the point cutoffs for each letter grade after grading the exams.

Part I: Theory.

1. Let G be a group. For any $g \in G$, write $\mathrm{Cl}_G\, g$ for the conjugacy class of g.
   (a) Find the normalizer $N_G\big(\langle \mathrm{Cl}_G\, g\rangle\big)$ of the subgroup generated by $\mathrm{Cl}_G\, g$.
   (b) Prove that a subgroup $H \leq G$ is normal if and only if it is a union of G-conjugacy classes.
   (c) List the conjugacy classes in A_5: for each one, give an element and the size of its class.
   (d) Prove that A_5 is simple, *i.e.,* has no normal subgroups. Hint: use Part (b).
   (e) For each prime p dividing the order of A_5, give an example of a p-Sylow subgroup S_p of A_5, find the number n_p of such subgroups, and compute the size of the normalizer $N_{A_5}S_p$.

2. Fix a prime p. Suppose that P is a group of order p^α and $Q_1 < P$ is a proper subgroup of order p^γ. Denote the set of all subgroups of P conjugate to Q_1 under the adjoint action of P by

$$\mathcal{Q} := \big\{Q_1, Q_2, \ldots, Q_r\big\}.$$

   (a) Prove that the size $|N_P(Q_1)|$ of the normalizer of Q_1 is p^β for some $\beta \geq \gamma$.
   (b) Prove that $r = p^{\alpha-\beta}$.
   (c) Consider the adjoint action of Q_1 on $\mathcal{Q}$. Prove that if $\beta < \alpha$, then the number of size 1 Q_1-orbits in $\mathcal{Q}$ is a positive multiple of p.
   (d) Prove that if $\big\{\mathrm{Ad}_g(Q_1)\big\}$ is a size 1 Q_1-orbit in $\mathcal{Q}$, then $\mathrm{Ad}_{g^{-1}}(Q_1)$ normalizes Q_1.
   (e) Prove that $\beta > \gamma$.

3. Assume that G is a finite group with exactly three conjugacy classes.
   (a) If G is abelian, prove that $G \cong \mathbb{Z}_3$.
   (b) Prove that S_3 is an example of such a group, and write its class equation.
   (c) Prove that if G is non-abelian, then $|G| = 6$. Hint: use the form of the class equation in which 1 is written as a sum of reciprocals of integers.
   (d) If G is an arbitrary non-abelian group of order 6, use the Sylow theorems to prove that it has an order 2 subgroup $\langle b\rangle$ and a normal order 3 subgroup $\langle c\rangle$ such that $bc = c^{-1}b$. Conclude that $G \cong D_6 \cong S_3$.

4. Let G be a finite group. Here is a fact: if $H < G$ is a proper subgroup, then there is a G-conjugacy class $\mathrm{Cl}_G\, g$ which does not intersect H.
   (a) Suppose that $g_0 = e, g_1, \ldots, g_r$ are representatives of all of the conjugacy classes of G. Prove that if these representatives all commute with each other, then G is abelian. Hint: apply the above fact to the centralizers $C_G(g_i)$.
   (b) The goal of Parts (b) and (c) is to prove the above fact by contradiction. Suppose that $H < G$ and the g_i in Part (a) are all in H (but do not assume that they commute). Prove that for each i, $H \cap \mathrm{Cl}_G\, g_i$ is union of one or more H-conjugacy classes, say represented by

$$h_{i,1} := g_i,\ h_{i,2},\ \ldots,\ h_{i,n_i}.$$

   (c) Write the class equation for both G and H under the assumptions of Part (b) and derive a contradiction. Hint: use again the hint in 3c.

Part II: Computation.

5.

(a) Prove that groups of order $2^2 \cdot 3 \cdot 11$ are not simple.

(b) Suppose that $|G| = 3 \cdot 7 \cdot 11$. Prove that $n_7 = n_{11} = 1$ and G has a central element of order 11. Compute the number of elements of G of orders 7, 11, and 77.

(c) Find a non-abelian subgroup of S_7 of order 21.

(d) Prove that S_7 does not contain an abelian subgroup of order 21.

6. Let G be a group containg two elements c and q such that $c \neq e$ and $q^2 \neq e$, and

$$c^3 = e, \qquad q^4 = e, \qquad qcq^{-1} = c^{-1}.$$

(a) Prove that the following twelve element set H is a subgroup of G:

$$H := \{c^i q^j : 0 \leq i < 3,\ 0 \leq j < 4\}.$$

Hint: first prove that qc is in this set.

(b) Prove that $\langle c \rangle$ is normal in H. Is the quotient group $H/\langle c \rangle$ cyclic? Explain.

(c) Draw the subgroup lattice of H. Indicate the size of each subgroup.

(d) Which subgroups of H are normal? What is its center?

(e) Show that S_7 contains such a subgroup, but S_6 does not.

7. Consider the group $A = \mathbb{Z}_2 \times \mathbb{Z}_4$, written in additive vector form:

$$A := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \in \mathbb{Z}_2,\ y \in \mathbb{Z}_4 \right\}.$$

(a) For each order two element t in A, explain whether or not the quotient $A/\langle t \rangle$ is cyclic.

(b) List all pairs of elements (t, f) of A such that t is order two, f is order four, and t and f generate A.

(c) Prove that $|\operatorname{Aut}(A)| \leq 8$. Hint: use your answer to Part (b).

(d) Prove that S_6 contains a subgroup C isomorphic to A but S_5 does not. Prove that the image of the normalizer $N_{S_6}(C)$ in $\operatorname{Aut}(C)$ is of size two. Hint: what is the size of a 2-Sylow subgroup of S_6?

(e) Consider matrices of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where a and b are in $\mathbb{Z}_2$, c and d are in $\mathbb{Z}_4$, and c is even. Check that each such matrix defines a homomorphism from A to itself by the usual rule for multiplying vectors by matrices. Which matrices give automorphisms? Use this to prove that $\operatorname{Aut}(A) \cong D_8$.

8. Consider the action of $\mathrm{GL}_2\,\mathbb{Z}_3$ on the nine element set $\mathbb{Z}_3^2$. This set contains four lines, of slopes 0, ± 1, and ∞ (I will draw a picture of this on the board if asked to). Denote the set of these lines by

$$\mathcal{L} := \{L_0, L_1, L_{-1}, L_\infty\}.$$

(a) Prove that $\mathrm{GL}_2\,\mathbb{Z}_3$ acts on the set $\mathcal{L}$, giving a homomorphism $\phi : \mathrm{GL}_2\,\mathbb{Z}^3 \to S_{\mathcal{L}} \cong S_4$.

(b) Prove that ϕ is surjective. Hint: compute explicitly the actions of the three matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(c) Use ϕ to prove that the quotient of $\mathrm{SL}_2\,\mathbb{Z}_3$ by its center is isomorphic to A_4.

(d) What is the size of a 2-Sylow subgroup of $\mathrm{GL}_2\,\mathbb{Z}_3$? Is it abelian? Dihedral?