Exam 2 will be formatted like Exam 1:

- Notes, books, and calculators will not be allowed.
- For full credit, show enough work to explain your method.
- The problems will mainly be computational.

Here is a list of topics, with the relevant sections. For preparation, redo some of the problems from Homeworks 6, 7, 8, and 9, and if you have time, do some of the additional practice problems given here.

Pseudoinverses

These will not be tested on this exam, as we will cover them more completely when we study the singular value decomposition.

Determinants

- Know the cofactor formula for the determinant.
- Know the behaviour of the determinant under row and column operations.
- Know that the determinant distributes over matrix multiplication.
- Know that the determinant is the product of the pivots.
- Section 5.3, on determinants as volumes, will not be tested.
- *Problems:* Section 5.1: 2, 3, 5, 8; Section 5.2: 2, 4, 10, 12, 13.

Eigenvectors and eigenvalues

- Be able to compute the characteristic polynomial and the eigenvalues.
- The trace and determinant are the sum and product of the eigenvalues.
- Be able to compute eigenvectors and bases of eigenspaces.
- *Problems:* Section 6.1: 9, 11, 18, 21, 23, 25, 27, 29, 30, 31.

Diagonalization

- A is diagonalizable if and only if for every eigenvalue λ, the dimension of $\mathrm{Null}(A-\lambda)$ is the multiplicity of λ as a root of $\mathrm{char}_A(t)$.
- For A diagonalizable, be able to find X and Λ such that $A = X\Lambda X^{-1}$.
- Be able to use $A = X\Lambda X^{-1}$ to compute A^k.
- If $A = YBY^{-1}$, then A and B have the same eigenvalues.
- *Problems:* Section 6.2: 4, 5, 6, 11, 12, 16, 17, 25, 28.

Symmetric matrices

Given a symmetric matrix S:

- Be able to diagonalize it by an orthogonal matrix: $S = Q\Lambda Q^T$.
- Know how to compute the projection form of the spectral theorem:

$$S = \lambda_1 P_{V_1} + \lambda_2 P_{V_2} + \cdots + \lambda_r P_{V_r},$$

  where $\lambda_1, \ldots, \lambda_r$ are the *distinct* eigenvalues, and V_s is the λ_s-eigenspace.
- *Problems:* Section 6.3: 14, 15, 16, 17, 18.

Positive definite and positive semidefinite matrices

Let S be a symmetric matrix.

- Definition: S is PDS (PSDS) if all eigenvalues are positive (nonnegative).
- Test: S is PDS (PSDS) iff $x^T Sx$ is positive (nonnegative) for all $x \neq 0$.
- Test: S is PDS iff all upper left subdeterminants are positive.
- If S is PDS, then all diagonal subdeterminants are positive.
- Test: S is PDS iff it has an LU-decomposition with positive pivots.
- $A^T A$ is always PSDS, and it is PDS iff A's columns are independent.
- For S PDS, be able to compute its Cholesky decomposition $C^T C$.
- For S PSDS, be able to compute its PSDS square root.
- For S 2×2 PDS, be able to graph $x^T Sx = \gamma$ for any $\gamma > 0$.
- *Problems:* Section 6.3: 4, 24, 26, 29, 30, 34, 35, 37, 40, 41, 42.

Orthogonal motions of $\mathbb{R}^2$ and $\mathbb{R}^3$

- Know Propositions 6.3 and 6.4 in the supplemental notes.
- For S 2×2 symmetric, be able to find all four orthogonal matrices diagonalizing it, and to describe geometrically the way they act on $\mathbb{R}^2$.

Here is a copy of a previous exam which you may use for practice. This year's exam will cover the same material. You will be allowed to do the problems in any order. For full credit, you will have to show all work and explain all answers.

1. Suppose that A is a 2×2 matrix such that

- $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of eigenvalue 2,
- $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is an eigenvector of eigenvalue 1.

(a) Find X and Λ such that $A = X\Lambda X^{-1}$.

(b) Find A.

(c) Find an explicit formula for A^k.

2. Suppose that B is 3×3, $\text{trace}(B) = 6$, $\det(B) = 6$, and $\text{Col}_1(B) = e_1$.

(a) Find an eigenvector of B.

(b) Find all eigenvalues of B.

3. Consider $\begin{pmatrix} 4 & -8 \\ -8 & 25 \end{pmatrix}$ and $\begin{pmatrix} 4 & 12 \\ 12 & 25 \end{pmatrix}$.

(a) Is either of these matrices positive definite symmetric? Explain.

(b) Factor both matrices into products of the form LDL^T, where L is lower triangular with 1's on the diagonal and D is diagonal.

(c) If the matrix is PDS, factor it as C^TC, where C is upper triangular.

4. Find the Cholesky decomposition C^TC of $\begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 6 \end{pmatrix}$.

5. Let $S := \begin{pmatrix} 3 & 1 & -2 \\ 1 & 3 & 2 \\ -2 & 2 & 0 \end{pmatrix}$. Given: $\mathrm{char}_S(t) = (t+2)(t-4)^2$.

(a) Find orthonormal bases of S's eigenspaces.

(b) Find an orthogonal diagonalization $M\Lambda M^T$ of S.

(c) Find the matrices projecting orthogonally to S's eigenspaces.

6. Consider $E := \frac{1}{4}\begin{pmatrix} 7 & 3\sqrt{3} \\ 3\sqrt{3} & 13 \end{pmatrix}$. Given: $E = Q\Lambda Q^T$ for

$$Q = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 4 & \\ & 1 \end{pmatrix}.$$

(a) Graph $x^T E x = 4$. Be as precise as possible.

(b) Describe the action of Q on the plane $\mathbb{R}^2$. Find any angles involved.

(c) Find an orthogonal matrix M of determinant -1 such that $M\Lambda M^T$ is also equal to E, and describe its action on $\mathbb{R}^2$.

7. (a) Find the PDS square root of the matrix E in Problem 6.

(b) Find the PDS square root of $\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix}$.

8. (Extra credit) One of these matrices is a rotation and one is not:

$$K := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L := \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Which one is a rotation?

(b) Give the axis and angle of the rotation.

(c) Find the real eigenvector v of the non-rotation.

(d) Describe the action of the non-rotation on v, and also on the plane $v^{\perp}$.