Math 4450/5500 **Practice problems for the Final Exam** Conley

The exam will be comprehensive: roughly a quarter of the problems will be on the material of Exam 1, a quarter on the material of Exam 2, and half on the material covered after Exam 2. It will be formatted like the midterms:

- Notes, books, and calculators will not be allowed.
- For full credit, show enough work to explain your method.
- The problems will mainly be computational.

Here is a large collection of practice problems, several of them drawn from the midterm exams. Redoing the midterm review problems might also be helpful.

1. Let $A := \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

(a) Find a factorization $A = LT$, where L is lower triangular with 1's on the diagonal, and T is upper triangular.

(b) Find a factorization $A = LDU$, where L is lower triangular, U is upper triangular, both L and U have 1's on the diagonal, and D is diagonal.

2. For each of these three matrices, do the following:

(a) Find the LPU-factorization.

(b) Find a lower triangular matrix $\tilde{L}$ giving the $P\tilde{L}U$-factorization.

$$\begin{pmatrix} 0 & 0 & 2 \\ 2 & -1 & -2 \\ -6 & 2 & 7 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 2 & -1 & -2 \\ 2 & -1 & 0 \\ -2 & 0 & 3 \end{pmatrix}.$$

3. Let $A := \begin{pmatrix} 2 & -4 & 4 & 6 \\ 3 & -6 & -1 & 2 \\ -5 & 10 & -3 & -8 \end{pmatrix}$.

(a) Find R, the reduced row-echelon form of A with zero rows deleted.

(b) Find the CR-factorization of A.

(c) Find the $CW^{-1}B$-factorization of A.

(d) Find a basis of the row space $\mathrm{Row}(A)$.

(e) Find the special solution basis of the null space $\mathrm{Null}(A)$.

(f) Find the pivot column basis of the column space $\mathrm{Col}(A)$.

(g) Find a basis of the left null space $\mathrm{Null}(A^T)$.

(h) $Ax = b$ is solvable for which of the following two vectors? For that one, give the complete solution.

$$b = \begin{pmatrix} 2 \\ 11 \\ -9 \end{pmatrix}, \qquad b = \begin{pmatrix} -2 \\ 11 \\ -9 \end{pmatrix}.$$

4. Let $A := \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}$.

(a) Find $(A^T A)^{-1}$.

(b) Find the matrix P which projects to the column space $\mathrm{Col}(A)$.

(c) Find the vector in $\mathrm{Col}(A)$ closest to the standard basis vector e_3.

(d) Find the matrix $\hat{P}$ which projects to the left null space $\mathrm{Null}(A^T)$.

5. Consider $v_1 = \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$, $v_3 = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$.

(a) Find orthogonal vectors p_1, p_2, p_3 according to the Gram-Schmidt process:

$$p_1 = v_1, \qquad p_2 = v_2 - P_{p_1} v_2, \qquad p_3 = v_3 - P_{p_1} v_3 - P_{p_2} v_3.$$

(b) Set $A := \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}$ and $P := \begin{pmatrix} | & | & | \\ p_1 & p_2 & p_3 \\ | & | & | \end{pmatrix}$. Find an upper triangular matrix U with 1's on the diagonal such that $A = PU$.

(c) Find Q orthogonal and R upper triangular such that $A = QR$.

(d) Find Q^{-1}.

6. Suppose that A is a 2×2 matrix such that

- $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of eigenvalue 2,
- $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is an eigenvector of eigenvalue 1.

Find X and Λ such that $A = X \Lambda X^{-1}$, then find A, and then find an explicit formula for A^k.

7. Suppose that B is 3×3, $\mathrm{trace}(B) = 6$, $\det(B) = 6$, and $\mathrm{Col}_1(B) = e_1$.

(a) Find one eigenvector and all eigenvalues of B.

(b) Give an example of such a matrix that is not diagonal.

(c) Are *all* such matrices diagonalizable? Explain.

8. Let $B := \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}$.

(a) Find a diagonalization $B = X \Lambda X^{-1}$.

(b) Find bases of B's column space, null space, and eigenspaces.

9. Consider $\begin{pmatrix} 4 & -8 \\ -8 & 25 \end{pmatrix}$ and $\begin{pmatrix} 4 & 12 \\ 12 & 25 \end{pmatrix}$.

(a) Is either of these matrices positive definite symmetric? Explain.

(b) Factor both matrices into products of the form LDL^T, where L is lower triangular with 1's on the diagonal and D is diagonal.

(c) If the matrix is PDS, factor it as $C^T C$, where C is upper triangular.

10. Find the Cholesky decomposition $C^T C$ of $\begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 6 \end{pmatrix}$.

11. Let $S := \begin{pmatrix} 3 & 1 & -2 \\ 1 & 3 & 2 \\ -2 & 2 & 0 \end{pmatrix}$.

(a) Find orthonormal bases of S's eigenspaces.

(b) Find an orthogonal diagonalization $Q\Lambda Q^T$ of S.

(c) Find the matrices projecting orthogonally to S's eigenspaces.

12. Consider $E := \frac{1}{4}\begin{pmatrix} 7 & 3\sqrt{3} \\ 3\sqrt{3} & 13 \end{pmatrix}$.

(a) Find the PDS square root $E^{1/2}$.

(b) Graph $x^T E x = 4$ as precisely as possible.

(c) Find an SVD $U\Sigma V^T$ of E. How are U and V related? Explain why.

(d) Find *all* orthogonal M such that $M\Sigma M^T = E$. How do they act on $\mathbb{R}^2$?

13. Find the PDS square root of $\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix}$.

14. Let $S := \begin{pmatrix} 7 & -\sqrt{3} \\ -\sqrt{3} & 5 \end{pmatrix}$.

(a) Find an orthogonal diagonalization $S = Q\Lambda Q^T$.

(b) Find orthonormal bases of S's eigenspaces.

(c) Find the matrices projecting orthogonally to S's eigenspaces.

15. Let $F := \begin{pmatrix} 2 & 0 \\ 1 & -5 \end{pmatrix}$.

(a) Find a factorization $F = PU$, where P has orthogonal columns and U is upper triangular with 1's on the diagonal.

(b) Find a factorization $F = QDU$, where Q is orthogonal, D is diagonal with positive entries, and U is upper triangular with 1's on the diagonal.

16. Let $C := \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}$.

(a) Find a factorization $C = LDL^T$, where L is lower triangular with 1's on the diagonal, and D is diagonal with positive entries.

(b) Find a factorization $C = TT^T$, where T is lower triangular.

17. Let $G := \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix}$.

(a) Find an orthogonal diagonalization of G^TG.

(b) Find the modulus $|G|$, which is by definition $(G^TG)^{1/2}$.

(c) Find the SVD $G = U\Sigma V^T$ (U, V orthogonal, Σ positive diagonal).

(d) Find the polar decomposition $G = Y|G|$, where Y is orthogonal.

18. Consider the matrix B from Problem 8 above.

(a) Write B as xy^T for vectors x and y.

(b) Find B's singular value.

(c) Find B's Schmidt decomposition.

(d) Find B's pseudo-inverse, B^+.

(e) Compute B^+B and BB^+, and also give them in terms of x and y.

19. Let $u_1 := \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $u_2 := \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $w_1 := \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $w_2 := \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, and

$$H := w_1u_1^T + w_2u_2^T.$$

(a) Find the singular values and the Schmidt decomposition of H.

(b) Find the reduced SVD $H = U_{\mathrm{r}}\Sigma_{\mathrm{r}}V_{\mathrm{r}}^T$.

(c) Compute H and its pseudo-inverse H^+ explicitly.

20. Find block diagonalizations $P\Gamma P^{-1}$ of the following matrices. Include bases of the eigenspaces and generalized eigenspaces. Explain all steps.

$$\begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ & & 0 & 1 \\ & & & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & 0 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 0 \end{pmatrix}.$$

The next problem is at the level of an extra-credit problem:

21. Let Ξ be an invertible anti-upper triangular matrix:

$$\Xi = \begin{pmatrix} & & & & \xi_1 \\ & * & & \xi_2 & \\ & & \iddots & & \\ & \xi_{n-1} & & & \\ \xi_n & & & & \end{pmatrix}. \qquad \text{Describe } \Xi^{-1}.$$

The last three problems are on 3×3 orthogonal matrices. We did not have time for that topic this year because of the snow, but I left the problems on the review sheet in case anyone is interested in seeing them:

22. One of these matrices is a rotation and one is not:

$$K := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L := \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Which one is a rotation?

(b) Give the axis and angle of the rotation.

(c) Find the real eigenvector v of the non-rotation.

(d) Describe the action of the non-rotation on v, and also on the plane $v^{\perp}$.

23. Define $\Theta_z := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & \\ 1 & 1 & \\ & & \sqrt{2} \end{pmatrix}$, $\Theta_x := \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & & \\ & 1 & -1 \\ & 1 & 1 \end{pmatrix}$.

(a) Give the angles and axes of rotation of Θ_z and Θ_x.

(b) Give the angle (as an arc cosine) and the axis of rotation of $\Theta_z\Theta_x$.

(c) Give the angle and the axis of rotation of $\Theta_z\Theta_x\Theta_z^T$.

24. Let Φ be a 2×2 reflection matrix. Define $\hat{\Phi} := \begin{pmatrix} \Phi & \\ & -1 \end{pmatrix}$, a 3×3 matrix. Describe the action of $\hat{\Phi}$ on $\mathbb{R}^3$ in terms of features of Φ.