Exam 2 will be formatted like Exam 1:

- Notes, books, and calculators will not be allowed.
- For full credit, show enough work to explain your method.

Here is a list of topics, with the relevant sections. For preparation, redo some of the problems from Quizzes 6, 7, 8, and 9, and if you have time, do some of the additional practice problems given here.

Pseudoinverses

These will not be tested on this exam, as we will cover them more completely when we study the singular value decomposition.

Determinants

- Know the cofactor formula for the determinant.
- Know the behaviour of the determinant under row and column operations.
- The determinant distributes over matrix multiplication.
- The determinant of a triangular matrix is the product of the pivots.
- *Problems:* Section 5.1: 2, 3, 5, 8; Section 5.2: 2, 4, 10, 12, 13.

Eigenvectors and eigenvalues

- Be able to compute the characteristic polynomial and the eigenvalues.
- The trace and determinant are the sum and product of the eigenvalues.
- Be able to compute eigenvectors and bases of eigenspaces.
- *Problems:* Section 6.1: 9, 11, 18, 21, 23, 25, 27, 29, 30, 31.

Diagonalization

- A is diagonalizable if and only if for every eigenvalue λ, the dimension of $\mathrm{Null}(A - \lambda)$ is the multiplicity of λ as a root of $\mathrm{char}_A(t)$.
- For A diagonalizable, be able to find X and Λ such that $A = X\Lambda X^{-1}$.
- Be able to use $A = X\Lambda X^{-1}$ to compute A^k.
- If $A = YBY^{-1}$, then A and B have the same eigenvalues.
- *Problems:* Section 6.2: 4, 5, 6, 11, 12, 16, 17, 25, 28.

Symmetric matrices

Given a symmetric matrix S:

- Be able to diagonalize it by an orthogonal matrix: $S = Q\Lambda Q^T$.
- Know how to compute the projection form of the spectral theorem:
  $$S = \lambda_1 P_{V_1} + \lambda_2 P_{V_2} + \cdots + \lambda_r P_{V_r},$$
  where $\lambda_1, \ldots, \lambda_r$ are the *distinct* eigenvalues, and V_s is the λ_s-eigenspace.
- *Problems:* Section 6.3: 14, 15, 16, 17, 18.

Positive definite and positive semidefinite matrices

Let S be a symmetric matrix.

- Definition: S is PDS (PSDS) if all eigenvalues are positive (nonnegative).
- Test: S is PDS (PSDS) iff $x^T Sx$ is positive (nonnegative) for all $x \neq 0$.
- Test: S is PDS iff all upper left subdeterminants are positive.
- If S is PDS, then all diagonal subdeterminants are positive.
- Test: S is PDS iff it has an LU-decomposition with positive pivots.
- $A^T A$ is always PSDS, and it is PDS iff A's columns are independent.
- For S PDS, be able to compute its Cholesky decomposition $C^T C$.
- For S PSDS, be able to compute its PSDS square root.
- For S 2×2 PDS, be able to graph $x^T Sx = \gamma$ for any $\gamma > 0$.
- *Problems:* Section 6.3: 4, 24, 26, 29, 30, 34, 35, 37, 40, 41, 42.

Orthogonal motions of $\mathbb{R}^2$

- Know Proposition 6.3 in the supplemental notes.
- For S 2×2 symmetric, there are in general eight orthogonal matrices diagonalizing it. Be able to find them and to describe geometrically the way they act on $\mathbb{R}^2$.

Here is a copy of a previous exam which you may use for practice. This year's exam will cover the same material. You will be allowed to do the problems in any order. For full credit, you will have to show all work and explain all answers.

1. Suppose that A is a 3×3 matrix, with eigenvectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ (eigenvalue 2)}; \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ (eigenvalue 4)}; \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ (eigenvalue 3)}.$$

(a) Find X and Λ such that $A = X\Lambda X^{-1}$.

(b) Find A.

(c) Find a basis of $\mathbb{R}^3$ of eigenvectors of A^k, and give their eigenvalues.

(d) Is A invertible? Is it symmetric? Is it positive definite?

2. Consider $\begin{pmatrix} 2 & 4 & -2 \\ 4 & 12 & 8 \\ -2 & 8 & 37 \end{pmatrix}$ and $\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -1 \\ -2 & -1 & 15 \end{pmatrix}$.

(a) Factor both matrices into products of the form LDL^T, where L is lower triangular with 1's on the diagonal and D is diagonal.

(b) Which matrix is PDS? Factor it as C^TC, where C is upper triangular.

3. Consider $E := \begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}$.

(a) Find the characteristic polynomial $\text{char}_E(t)$.

(b) Find the eigenvalues of E.

(c) Find an orthonormal basis of $\mathbb{R}^2$ of eigenvectors of E.

(d) Find an orthogonal diagonalization $Q\Lambda Q^T$ of E.

(e) Find the matrices projecting orthogonally to E's eigenspaces.

(f) Write E as a linear combination of the projection matrices from (e).

4. Find $\sqrt{E}$, the PDS square root of the matrix E in Problem 3.

5. Graph $x^TEx = 36$, where E is as in Problem 3. Draw a large, precise graph.

6. Let E be as in Problem 3.

(a) How many orthogonal matrices Q are there such that $Q^T E Q$ is diagonal?

(b) Find one of determinant 1 and one of determinant -1.

(e) Describe the way the two matrices you found act on $\mathbb{R}^2$.

7. Suppose that S is a real symmetric matrix with characteristic polynomial

$$\mathrm{char}_S(t) = (t-1)(t-2)^2(t-3).$$

For each eigenvalue λ of S, let P_λ be projection to the λ-eigenspace of S.

(a) What is the size of S? Is it invertible? Is it diagonalizable?

(b) For each eigenvalue λ, give the rank of P_λ.

(c) Write S as a linear combination of the P_λ's.

(d) Write I as a linear combination of the P_λ's.

(e) Is S PDS? If so, give $\sqrt{S}$ as a linear combination of the P_λ's.

(f) What is P_λ^2? For $\lambda \neq \lambda'$, what is $P_\lambda P_{\lambda'}$?

8. Let $T := \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Repeat Problem 3 for T:

(a) Find the characteristic polynomial $\mathrm{char}_T(t)$.

(b) Find the eigenvalues of T. (Hint: $\mathrm{char}_T(t)$ factors.)

(c) Find orthonormal bases of T's eigenspaces.

(d) Find an orthogonal diagonalization $Q\Lambda Q^T$ of T.

(e) Find the matrices projecting orthogonally to T's eigenspaces.

(f) Write T as a linear combination of the projection matrices from (e).

9. (Extra credit) We had to skip the material of this problem this year, so it will not be on the exam even as an extra-credit problem. If it interests you, see Proposition 6.4 of the supplementary notes.

$$\text{Set } X = \tfrac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & & \\ & 1 & -1 \\ & 1 & 1 \end{pmatrix} \quad \text{and} \quad Z := \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & \\ 1 & 1 & \\ & & \sqrt{2} \end{pmatrix}$$

Given: X and Z are rotation by $\frac{\pi}{4}$ around the x-axis and z-axis, respectively.

(a) Explain why XZ is orthogonal and rotates $\mathbb{R}^3$ around some axis.

(b) Consider the angle of rotation of XZ. Place it between two of these values:

$$0 < \tfrac{\pi}{6} < \tfrac{\pi}{4} < \tfrac{\pi}{3} < \tfrac{\pi}{2} < \tfrac{2\pi}{3} < \tfrac{3\pi}{4} < \tfrac{5\pi}{6} < \pi.$$

(c) Find the axis of rotation of XZ.