During the exam you will not have access to notes, books, or calculators. For full credit you will need to show enough work so that I know what you did. The problems will be computational, with the focus being on matrix factorizations.

Here is a list of topics, with the relevant sections and practice problems, some drawn from past homework sets. There are many problems, but they are only for practice and will not be collected. Just do the ones you have time for.

Inverses and the LU-factorization

Given an invertible square matrix A:

- Be able to compute its inverse by row operations.
- Be able to compute its LU-factorization (and whether or not it has one).
- *Problems:* Section 2.2: 3, 35; Section 2.3: 3, 4, 6.

The LPU- and PLU-factorizations

Given an invertible square matrix A:

- Be able to compute its unique LPU-factorization according to the procedure given in class: work column by column from left to right, and from top to bottom in each column, using the highest non-zero entry, ignoring pivot rows from previous columns.
- Be able to find the corresponding $P\tilde{L}U$-factorization: P and U are the same, and $\tilde{L} = P^{-1}LP$. If the procedure has been carried out correctly, $\tilde{L}$ will be lower triangular.
- *Problems:* Section 2.4: 13, 19: find LPU and $P\tilde{L}U$ for all three matrices.

R_0, the CR- and $CW^{-1}B$-factorizations, and special solutions

Given an arbitrary $m \times n$ matrix A:

- Be able to compute its rank r and its reduced row echelon form R_0.
- Be able to compute the $r \times n$ matrix R (remove the zero rows from R_0).
- Be able to compute the special solution basis of $\mathrm{Null}(A)$.
- Be able to compute the CR-factorization: the columns of C are the pivot columns of A.
- Be able to compute the $CW^{-1}B$-factorization from page 99. In class we described this as the $A^J(A^J_I)^{-1}A_I$-factorization, where
  - $J = (J_1, \ldots, J_r)$ is the set of indices of any collection of columns of A that form a basis of its column space,
  - A^J is the $m \times r$ matrix given by the J-columns of A,
  - $I = (I_1, \ldots, I_r)$ is the set of indices of any collection of rows of A that form a basis of its row space,
  - A_I is the $r \times n$ matrix given by the I-rows of A,
  - A^J_I is the $r \times r$ matrix given by the intersection of the J-columns with the I-rows.
- *Problems:* Section 1.4: 7, 12; Section 3.2: 4, 33, 35, 36, 45: for the matrices in all these problems, find the CR-factorization, the special solutions, and at least one $A^J(A^J_I)^{-1}A_I$-factorization (you do not need to invert A^J_I).
- *Additional Problems:* Section 3.2: 2, 3, 23, 24.

The solution of $Ax = b$ and the four subspaces of A

Given an arbitrary $m \times n$ matrix A:

- For any $b \in \mathbb{R}^m$, be able to find the complete solution of $Ax = b$.
- Be able to find bases of the subspaces $\mathrm{Col}(A)$, $\mathrm{Row}(A)$, $\mathrm{Null}(A)$, $\mathrm{Null}(A^T)$.
- *Problems:* Section 3.3: 27, 29;
  - Section 3.5: for the A in 19, find bases of the four subspaces.

Orthogonal subspaces and projections

Let V be any subspace of $\mathbb{R}^m$:

- Given a basis of V, be able to find a basis of $V^\perp$.
- Be able to find the matrix projecting to V: it is $P_A = A(A^TA)^{-1}A^T$, where A is any matrix whose columns are a basis of V.
- *Problems:* Section 4.1: 21; Section 4.2: for the matrices in 12, 33, find P_A.

The Gram-Schmidt process and the QR decomposition

- Given any r independent vectors $v_1, \ldots, v_r$ in $\mathbb{R}^m$, be able to use the Gram-Schmidt process to find orthonormal vectors $q_1, \ldots, q_r$ with the same span.
- Given any $m \times r$ matrix A of rank r, be able to factor it as QR, where
  - Q is an $m \times r$ matrix with orthonormal columns,
  - R is an invertible upper triangular $r \times r$ matrix.
- *Problems:* Section 4.4:
  - In 17, factor the 2×3 matrix with columns a, b into QR;
  - In 18, factor the 4×4 matrix with columns a, b, c, d into QR;
  - In 21, factor the 4×3 matrix $(A|b)$ into QR;
  - In 22, factor the 3×3 matrix with columns a, b, c into QR;
  - Do 24, find the projections P_S to S and $P_{S^\perp}$ to $S^\perp$, and find $P_S + P_{S^\perp}$.

Here is a copy of a previous exam which you may use for practice. This year's exam will cover the same material in the same format. You will be allowed to do the problems in any order. For full credit, you will have to show all work and explain all answers.

Problems 1-3 are LPU-factorization problems. In each of them, do the following:

(a) Find the LPU-factorization of A.

(b) Find a lower triangular matrix $\tilde{L}$ such that $A = P\tilde{L}U$.

Hint: in exactly one of these three problems, P is the identity. Therefore, in that problem the LPU-factorization is actually just an LU-factorization.

1. $A := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$.

2. $A := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix}$.

3. $A := \begin{pmatrix} 1 & -1 & 2 & -1 \\ -1 & 1 & -3 & 2 \\ 2 & -2 & 6 & -2 \\ -3 & 1 & -5 & 6 \end{pmatrix}$.

4. Let $A := \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 3 & 1 \\ -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{pmatrix}$.

(a) Find R, the reduced row-echelon form of A with zero rows deleted.

(b) Find the CR-factorization of A.

(c) Find the $CW^{-1}B$-factorization of A.

(d) Find a basis of the row space $\text{Row}(A)$.

(e) Find the special solution basis of the null space $\text{Null}(A)$.

(f) Find the pivot column basis of the column space $\text{Col}(A)$.

(g) Find a basis of the left null space $\text{Null}(A^T)$.

5. Let $A := \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(a) Find $(A^T A)^{-1}$.

(b) Find the matrix P which projects to the column space $\mathrm{Col}(A)$.

(c) Find the vector in $\mathrm{Col}(A)$ closest to the vector $e_3 + e_4$.

(d) Find the matrix $\hat{P}$ which projects to the left null space $\mathrm{Null}(A^T)$.

6. For any vector u, let P_u denote projection to the line through u. Consider

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix}.$$

(a) Find orthogonal vectors p_1, p_2, p_3 according to the Gram-Schmidt process:

$$p_1 = v_1, \qquad p_2 = v_2 - P_{p_1} v_2, \qquad p_3 = v_3 - P_{p_1} v_3 - P_{p_2} v_3.$$

(b) Define q_j to be the vector $p_j/|p_j|$ for $j = 1, 2, 3$. Write out the matrices

$$A = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}, \qquad P := \begin{pmatrix} | & | & | \\ p_1 & p_2 & p_3 \\ | & | & | \end{pmatrix}, \qquad Q := \begin{pmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{pmatrix}.$$

(c) Find Q^{-1}.

(d) Find an upper triangular matrix R such that $A = QR$.

(e) Find a diagonal matrix D such that $P = QD$.

(f) Find an upper triangular matrix U such that $A = PU$.

7. For A as in Problem 4, the equation $Ax = b$ is solvable for which of the following two vectors? For that one, give the complete solution.

$$b = \begin{pmatrix} 2 \\ 6 \\ 1 \\ 3 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 7 \\ 1 \\ 3 \end{pmatrix}.$$

8. **(Extra credit)** Find the $CW^{-1}B$ decomposition of $A := \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 2 \end{pmatrix}$.

9. **(Extra credit)** Let A be an $m \times n$ matrix of rank r. Let P and $\hat{P}$ be the matrices projecting to $\mathrm{Row}(A)$ and $\mathrm{Null}(A)$, respectively. Find their sizes, their ranks, their sum $P + \hat{P}$, and their products $P\hat{P}$ and $\hat{P}P$. Explain.