Here are 11 practice problems and 1 extra-credit style problem. For full credit, you will need to simplify all answers. The following formulas will be given.

$$y = e^{-\int p} \int g e^{\int p}, \qquad W = e^{-\int p}, \qquad y_2 = y_1 \int \frac{W}{y_1^2}, \qquad Y = y_2 \int \frac{g y_1}{W} - y_1 \int \frac{g y_2}{W}$$

$$\mathcal{L}\big[f\big](s) = \int_0^\infty e^{-st} f(t)dt, \qquad \mathcal{L}\big[e^{bt} f\big] = \mathcal{L}\big[f\big](s-b)$$

$$\mathcal{L}\big[tf\big] = -\tfrac{d}{ds}\mathcal{L}\big[f\big], \qquad \mathcal{L}\big[u_c(t) f(t-c)\big](s) = e^{-sc}\mathcal{L}\big[f\big](s)$$

$$\mathcal{L}\big[y'' + by' + cy\big] = (s^2 + bs + c)Y - (s+b)y(0) - y'(0)$$

$$v_\lambda e^{\lambda t}, \qquad \big(w + (Aw - \lambda w)t\big)e^{\lambda t}$$

In Problems 1-3 you are given a matrix A. In each of them, find the following:

(a) The general solution of the system $x'(t) = Ax(t)$.

(b) A picture of the solution trajectories. Include any eigenlines (with their directions), one solution trajectory (with direction) in each of the regions into which the eigenlines divide the plane, and the axis directions.

(c) In Problem 1 only, the solution $x(t)$ with $x(0) = \binom{1}{0}$.

1. $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ **2.** $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ **3.** $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

4. Consider $y'' - y' - 2y = g(t)$, with ICs $y(0) = -1$, $y'(0) = 1$.

(a) Let $g = 6e^t$. Find $Y(s)$, apply partial fractions, and then find $y(t)$.

(b) Let $g = 9e^{2t}$. Find $Y(s)$, apply partial fractions, and then find $y(t)$.

5. Compute $\mathcal{L}[te^t \sin 3t]$.

6. Consider the equation $(x+1)y'' + xy' + y = 0$.

(a) Find the recursion relation for the power series at $x = 0$ of its solutions.

(b) Find the first three non-zero terms of each of its fundamental solutions.

7. Suppose that $2x^2y'' + 3xy' + (x-6)y = 0$ has solution

$$y_r = x^r + a_{r+1}x^{r+1} + a_{r+2}x^{r+2} + \cdots .$$

(a) What are the possible values of r?

(b) What is the recursion relation for the a_μ? (It does not depend on r.)

(c) For each possible value of r, compute a_{r+1} and a_{r+2}.

8. e^{2x} is a solution of $xy'' - y' + (2-4x)y = 0$. Find the general solution.

9. Find the general solution of $y'' + 4y = g$ for:

(a) $g = t^2 + e^{2t}$, (b) $g = t^2 + \sin 2t$, (c) $g = \sec 2t$.

10. For what value of a is the following equation exact? Solve it for that value.

$$\Big(x + (xy+a)e^{xy}\Big)\,dx + \Big(x^2e^{xy} - y\Big)\,dy = 0, \quad y(2) = 0.$$

11. Solve $y' - y\tan x = \sec x$, $y(0) = -1$. Hint: if you have forgotten $\int \tan x$, do it by udu substitution with $u = \cos x$.

Extra Credit. Suppose that y_1 is a solution of $y'' + py' + qy = 0$. Prove that

$$y_2 = y_1 \int \frac{e^{-\int p}}{y_1^2} \quad \text{is another solution.}$$