Section 1.1:

1. For each of the following equations, draw the direction field along the horizontal lines $y = -1$, 0, 1, 2, and 3. In each case, describe the behaviour of the solutions as x goes to ∞.

(a) $y' = y$.

(b) $y' = y - 1$.

(c) $y' = 2 - y$.

(d) $y' = y(2 - y)$.

Section 1.2:

2. For each equation in Problem 1, solve the initial value problem $y(0) = 1$.

Section 2.1:

3. Solve $2y' + y = 3t$.

4. Solve $y' + 2y = te^{-2t}$.

5. Solve $y' + \frac{2}{t}y = \frac{1}{t^2}\cos t,\ \ y(\pi) = 0$.

6. Solve $ty' + (t + 1)y = t,\ \ y(\ln 2) = 1$.

Section 2.2:

7. Solve $y' = \frac{x^2}{y(1+x^3)}$.

8. Solve $y' = \frac{2x}{y+x^2y},\ \ y(0) = -2$.

9. Solve $y' = \frac{e^{-x}-e^x}{3+4y},\ \ y(0) = 1$.

10. Solve $y' = \frac{3x^2}{3y^2-4},\ \ y(1) = 0$. On what interval is the solution valid?

11. Solve $y' = \frac{2-e^x}{3+2y},\ \ y(0) = 0$. Where does the solution reach its maximum?

Section 2.6:

12. Decide if exact, and if exact, solve: first, $x(xy + 1)y' + y(xy + 1) = 0$; second, $xy' + y = 0$.

13. Decide if exact, and if exact, solve: $e^x \sin y + 3y = (3x - e^x \sin y)y'$.

14. Solve $9x^2 + y - 1 = (4y - x)y'$, $y(1) = 0$. Find a cubic equation in x whose roots bound the domain of the solution. Given that this cubic has only one real root (which you can quickly learn by searching on "cubic solver"), decide whether the domain extends to $+\infty$ or to $-\infty$.

15. For what b is this exact? Solve it for that b: $ye^{2xy} + x + bxe^{2xy}y' = 0$.

16. Show that this is exact after multiplication by xe^x, and solve it: $(x+2)\sin y + xy' \cos y = 0$.

Section 3.1:

17. Solve: $y'' + 4y' + 3y = 0$, $y(0) = 2$, $y'(0) = -1$.

18. Solve: $y'' + 3y' = 0$, $y(0) = -2$, $y'(0) = 3$.

19. Solve: $4y'' - y = 0$, $y(-2) = 1$, $y'(-2) = -1$.

20. Solve: $2y'' - 3y' + y = 0$, $y(0) = 2$, $y'(0) = \frac{1}{2}$. Find its maximum and its zeroes.

Section 3.2:

21. Find the Wronskian of each pair of functions: $\{\cos t, \sin t\}$, $\{x, xe^x\}$, $\{\cos^2\theta, 1+\cos 2\theta\}$.

22. Find the domain of the solutions of $ty'' + 3y = t$ defined at $t_0 = 1$.

23. What is the domain of the solutions of $y''\cos t + ty' + y\ln t = 0$ defined at $t_0 = 1$? $t_0 = 2$?

24. Consider $yy'' + (y')^2 = 0$. Check that 1 and $\sqrt{t}$ are solutions, but $1 + \sqrt{t}$ is not.

25. Suppose that $W(f, g) = 3e^{4t}$ and $f = e^{2t}$. Find all possible g.

26. Find $W(2f - g, f + 2g)$ in terms of $W(f, g)$.

Definition. The *standard* solutions at $t = t_0$ of a linear second order homogeneous differential equation are the solutions $y_1(t)$ and $y_2(t)$ with initial values

$$y_1(t_0) = 1, \quad y_1'(t_0) = 0, \qquad\qquad y_2(t_0) = 0, \quad y_2'(t_0) = 1.$$

27. Consider $y'' + 5y' + 6y = 0$. Find its standard solutions y_1 and y_2 at $t_0 = 0$.

28. Consider $y''\cos t + y'\sin t - y = 0$. What are the possible Wronskians of pairs of solutions?

29. Consider $(1 - t^2)y'' - 2ty' + 2y = 0$. What is the Wronskian $W(y_1, y_2)$ of the standard solutions at $t_0 = 0$? Of the standard solutions at $t_0 = 2$? Of the standard solutions at $t_0 = 1$?

Section 3.3:

30. Consider $y'' + 9y = 0$. Find its standard solutions y_1 and y_2 at $t_0 = \pi/6$.

31. Consider $y'' + 4y' + 8y$. Find its standard solutions at $t_0 = 0$.

32. Consider $y'' + 4y' + 8y$. Find the Wronskian of its standard solutions at $t_0 = \pi/8$.

Section 3.4:

33. Solve $y'' + 4y' + 4y = 0$, $y(-1) = 2$, $y'(-1) = 1$.

34. Consider $y'' - y' + \frac{1}{4}y = 0$, $y(0) = 2$. What value of $y'(0)$ separates solutions that go to $+\infty$ from solutions that go to $-\infty$?

35. Consider $t^2y'' - t(t+2)y' + (t+2)y = 0$. Check that t is a solution. Find all solutions.

36. Consider $ty'' - y' + 4t^3y = 0$. Given: $\sin t^2$ is a solution. Find all solutions.

37. Consider $(t-1)y'' - ty' + y = 0$. Check that e^t is a solution. Find all solutions.

Section 3.5:

38. Find the general solution: $y'' - 2y' - 3y = 3e^{2t}$.

39. Find the general solution: $y'' - 2y' - 3y = -3e^{-t}$.

40. Find the general solution: $y'' + 9y = t^2e^{3t} + 6$.

41. Find the general solution: $2y'' + 3y' + y = t^2 + 3\sin t$.

42. Find the general solution: $y'' + y = \cos t$.

Section 3.6:

43. Find the general solution: $y'' + y = \tan t$.

44. Find the general solution: $y'' + 9y = 9\sec^2 3t$.

45. Find the general solution: $y'' + 4y' + 4y = t^{-2}e^{-2t}$.

46. Find the general solution: $y'' + 4y = 3\csc 2t$.

47. Find the general solution: $y'' - 2y' + y = e^t/(1+t^2)$.

Section 5.1:

48. Find the ROC: $\sum^\infty \frac{n}{2^n}x^n$.

49. Find the ROC: $\sum^\infty \frac{1}{n^2}(2x)^n$.

50. Find the ROC: $\sum^\infty n(2x^2)^n$.

51. Find the ROC: $\sum^\infty \frac{(2n)!}{(n!)^2}x^n$.

52. Find the Airy ROC: $1 + \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 + \frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}x^9 + \cdots$.

53. Find the Taylor series of e^{x^2}. Show that its derivative is equal to its product with $2x$.

54. Find the Taylor series of the antiderivative $\int e^{x^2}\,dx$. What is its ROC?

55. Find the Taylor series of $\cos\sqrt{x}$.

56. Find the Taylor series of $\frac{1}{1+x^3}$. What is its ROC?

57. Find the Taylor series of $\frac{3x^2}{(1+x^3)^2}$. What is its ROC?

58. Find the Taylor series of $(1-x)^{-1/2}$. What is its ROC?

Section 5.2: In these problems, find the recursion relation and the first four non-zero terms of the series of the two standard solutions. When a_n depends only on a_{n-2}, find its closed formula.

59. $y'' - xy' - y = 0$

60. $y'' + xy' + 2y = 0$

61. $(1+x^2)y'' - 4xy' + 6y = 0$

62. $(1-x)y'' + y = 0$

63. $(1-x)y'' + xy' - y = 0$

64. $y'' + x^2 y = 0$

Section 5.3:

65. For each equation, find the minimum guaranteed ROC of its Taylor series solutions:

(a) $(x^2+2x-2)y'' + xy' - y = 0$

(b) $(x^2-2x-2)y'' - xy' - y = 0$

(c) $(x^2+2x+2)y'' + xy' + y = 0$

(d) $y''\cos x + y\cos 2x = 0$

In these problems, find the series of the two standard solutions out to the x^4 term:

66. $y'' + y'\sin x + y\cos x = 0$

67. $y''e^x + xy = 0$

Section 5.4: In these problems, find the general solution:

68. $x^2y'' + 3xy' + 5y = 0$

69. $x^2y'' + 4xy' + 2y = 0$

70. $x^2y'' + 7xy' + 9y = 0$

71. $x^2y'' - 4xy' + 4y = 0$

Section 5.5: In these problems, find the recursion relation, the indicial equation and its roots, and the first four non-zero terms of the series of the solution corresponding to the larger root. If the roots do not differ by an integer, repeat for the smaller root.

Hint: In treating equations whose second derivative term is a multiple of xy'', it may be easiest to begin by multiplying the whole equation by x.

72. $2xy'' + y' + xy = 0$

73. $xy'' + y = 0$

74. $3xy'' + 2y' + xy = 0$

75. $x^2y'' + xy' + (x^2 - \frac{1}{9})y = 0$

76. $xy'' + y' - y = 0$

77. $2x^2y'' + 3xy' + (2x^2 - 1)y = 0$

Section 6.1:

78. Find the Laplace transform of each function:

(a) t^7

(b) $e^{-7t} \sin 4t$

(c) $2te^{-3t}$

(d) $-t \sin 2t$

(e) $t^4 e^{-4t}$

Sections 6.2-6.3:

79. For $y'' - y' - 6y = 0$, $y_0 = 1$, and $y_0' = -1$, find $Y(s)$ and $\mathcal{L}^{-1}[Y]$.

80. For $y'' - 4y' + 4y = 0$, $y_0 = 1$, and $y_0' = 1$, find $Y(s)$ and $\mathcal{L}^{-1}[Y]$.

81. For $y'' + 2y' + y = e^{-2t}$, $y_0 = 0$, and $y_0' = 0$, find $Y(s)$ and $\mathcal{L}^{-1}[Y]$.

82. For $y'' + 2y' + 5y = 0$, $y_0 = 2$, and $y_0' = -1$, find $Y(s)$ and $\mathcal{L}^{-1}[Y]$.

83. For $y'' - 2y' + 2y = \cos t$, $y_0 = 1$, and $y_0' = 0$, find $Y(s)$ and $\mathcal{L}^{-1}[Y]$.

84. For $y'' - 2y' + 2y = e^{-t}$, $y_0 = 0$, and $y_0' = 1$, find $Y(s)$ and $\mathcal{L}^{-1}[Y]$.

85. For $y'' + 4y = u_0(t) - u_\pi(t)$, $y_0 = 1$, and $y_0' = 0$, find $Y(s)$.

86. Graph $tu_0 - (t-1)u_1$ and find its Laplace transform.

87. Graph $u_0 - u_1 + u_2 - u_3 + u_4 - u_5 + \cdots$ and find its Laplace transform.

88. Find $\mathcal{L}^{-1}[(s-2)^{-4}]$.

89. Find $\mathcal{L}^{-1}[e^{-2s}/(s^2 + s - 2)]$.

90. Find and graph $\mathcal{L}^{-1}[(e^{-s} + e^{-2s} - e^{-3s} - e^{-4s})/s]$.

91. Express $\mathcal{L}[t^{-1} \sin t]$ as a power series in s^{-1}, and sum the series.

92. Express $\mathcal{L}[\int_0^t f(x)dx]$ in terms of $\mathcal{L}[f]$.

93. Express $\mathcal{L}[f(cx)]$ in terms of $\mathcal{L}[f]$.

Section 7.1:

94. Rewrite $u'' + 3u' + 2u = 0$ as $x' = Ax$ via $x_1 = u$, $x_2 = u'$. Find A.

95. Rewrite $u'' + 3u' + 2u = e^{-3t}$, $u(0) = 1$, $u'(0) = 2$ as a system $x' = Ax + G$, $x(0) = c$.

Section 7.2:

96. Set $A = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}$ and $B = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix}$. Compute $A - 2B$, AB, and BA.

97. Set $A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix}$. Which is singular? Invert the other one.

98. Set $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Find $f(t)$ so that $x = vf(t)$ solves $x' = Ax$.

Section 7.3:

99. Compute the eigenvalues and eigenvectors:

$$\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

100. Compute the eigenvalues and eigenvectors:

$$\begin{pmatrix} -3 & \frac{3}{4} \\ -5 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

Section 7.5:

101. Consider $A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$.

(a) Find the general solution of $x' = Ax$.

(b) Find the solution with $x(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(c) Draw a direction field, on graph paper. Make it LARGE: a half a page. Indicate the eigenlines, labelled with their directions and eigenvalues. Draw three arrows on each coordinate ray, as well as on the four rays along the lines through $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

(d) In another color, add to your direction field two trajectories in each of the four regions bounded by the eigenlines, and indicate their directions.

102. Consider $B = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$.

(a) Find the general solution of $x' = Bx$.

(b) Find the solution with $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(c) Draw a direction field. Again, use graph paper and make it large: a half a page. Indicate the eigenlines, labelled with their directions and eigenvalues. Draw three arrows on each coordinate ray, as well as on the four rays along the lines through $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(d) In another color, add to your direction field two trajectories in each of the four regions bounded by the eigenlines, and indicate their directions.

103. Consider $C = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$, $E = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$.

(a) Find the general solutions of $x' = Cx$, $x' = Dx$, and $x' = Ex$.

(b) Draw a direction field, large, on graph paper. Indicate the eigenlines, labelled with their directions and eigenvalues. Draw two arrows on each coordinate ray, as well as on the four rays along at least two lines of your choice, but separating the eigenlines.

(c) In another color, add to your direction field two trajectories in each of the four regions bounded by the eigenlines, and indicate their directions.

Section 7.6:

104. Consider $F = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $G = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $H = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$.

(a) Find the general solutions of $x' = Fx$, $x' = Gx$, and $x' = Hx$.

(b) Draw a direction field: as always, large and on graph paper. Draw two arrows on each coordinate ray, as well as on the four rays along the lines through $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(c) In another color, add to your direction field two trajectories.

105. Consider $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$, $D = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}$.

(a) Find the general solutions of $x' = Ax$, $x' = Bx$, $x' = Cx$, and $x' = Dx$.

(b) Draw a direction field, with arrows on the axes and the lines through $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(c) Add to your direction field at least three trajectories.

Section 7.8:

106. Consider $E = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix}$, $F = \begin{pmatrix} 6 & -4 \\ 1 & 2 \end{pmatrix}$, $G = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}$, $H = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}$.

(a) Find the general solutions of $x' = Ex$, $x' = Fx$, $x' = Gx$, and $x' = Hx$.

(b) Draw a direction field. Include the eigenline, with its direction. Include arrows on the axes, on the line where they are horizontal, and on the line where they are vertical.

(c) Add to your direction field four trajectories, two on each side of the eigenline.

(d) For G, solve the IVP $x(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. For H, solve the IVP $x(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.