Indiscernibles for $L[T_2]$

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1 Introduction and Review

Our goal is to present a theory of indiscernibles for the model $L[T_2]$ and to use that theory to give a new proof of the weak partition property on $\boldsymbol{\delta}^1_3$. This represents joint work with Steve Jackson.

Martin originally proved the strong partition property on ω_1 using the theory of indiscernibles for models of the form L[x], for x a real. This theory, in conjunction with the fact that every subset of ω_1 is in L[x] for some real x, provided a good coding for subsets of ω_1, and this coding was sufficient to prove the strong partition property. Becker and Kechris subsequently proved that every subset of $\boldsymbol{\delta}_{2n+1}$ was in L[T_{2n+1},x], for some real x, and the hope was that Martin's techniques would generalize to these models, proving the strong partition property on δ_{2n+1}. Unfortunately, a theory of indiscernibles for the models $L[T_n, x]$ was not forthcoming. Led by Kunen, other methods were developed for providing good codings of subsets of $\delta_{\mathbf{2n+1}}$. These methods, which hinge on an analysis of measures on $\delta_{\mathbf{2n+1}}$, crystallized into Jackson's theory of descriptions. This theory suffices to prove the strong partition property on all $\delta_{\mathbf{2n+1}}$, as well as for odd projective ordinals in projective-like hierarchies bounded by some wadge rank (below the first inaccesible?). It is hoped that the methods presented here might be generalized to prove the strong partition property for odd projective ordinals in scaled projective-like hierarchies below the supremum of the suslin cardinals.

The proof we will present will be in the spirit of Martin's original proof. Jackson's recent result that every weakly homogeneous tree has a stabilization (i.e. the tree T can be restricted to large sets such that if S is the tree constructed from the Martin-Solovay construction over T, then the leftmost branches of S are a scale), allows one to replace the L[T_n,x] of Becker-Kechris with $L[S_n, x]$ where S_n is a homogeneous tree. The fact that S_n is homogeneous allows one to develop a theory of indiscernibles for the models $L[S_n, x]$, which in turn provide good codings. We now proceed with the proof.

We review some basic notation. The reals $\mathbb{R}$ will be identified with the Baire space ω^ω. By a tree on a set A we mean a subset of $A^{<\omega}$ closed under initial segments. If T is a tree on $\omega \times R$, p[T]=$\{x \in \omega^\omega : \exists r \in R^\omega \forall n (x|n, r|n) \in T\}$. If $s_0, t_0, s_1, t_1 \in \omega^{<\omega}$, we write $(s_0, t_0) < (s_1, t_1)$ to mean $s_0 \subset t_0$ and $s_1 \subset t_1$. For T a tree we define the Kleene-Brouwer ordering on T $<^T_{KB}$ ($<_{KB}$ when T is clear from context) for s,t$\in T$ by

$$s <_{KB} t \leftrightarrow (t \subset s) \vee \exists n(s|n = t|n \wedge s(n) < t(n))$$

note that if T is a tree on a wellfounded set, then T is wellfounded iff $<_{KB}$ is a wellfounded relation. For T a tree on $A \times B$, we define T_a for $a \in A^{<\omega}$ to be $\{b \in B^{|a|} | (a,b) \in T\}$. Likewise, for $x \in \omega^\omega$ define $T_x = \bigcup_{n\in\omega} T_{x \restriction n}$. Finally, we fix a bijection $\pi : \omega \to \omega^{<\omega}$ such that if $s \subset t$, then $\pi^{-1}(s) < \pi^{-1}(t)$.

Definition 1.1. If T is a tree on $\omega \times \omega \times \kappa$, we say T is weakly homogeneous if there are measures $\{\mu_{s,t} | s,t \in \omega^{<\omega} \wedge |s| = |t|\}$ such that

(i) $\mu_{s,t}$ is a measure on $\kappa^{|s|}$ such that $\mu_{s,t}(T_{s,t}) = 1$.

(ii) if $s_0 \subset s_1$ and $t_0 \subset t_1$ then μ_{s_1,t_1} projects onto μ_{s_0,t_0}

(iii) if $x, y \in \omega^\omega$ are such that $\exists \alpha \in \kappa^\omega \forall n((x \restriction n, y \restriction n, \alpha \restriction n) \in T$ and $\{A_n\}_{n\in\omega}$ are such that $A_n \in \mu_{x\restriction n, y \restriction n}$, then $\exists \alpha \in \kappa^\omega \forall n(\alpha \restriction n \in A_n))$

Definition 1.2. If T is a weakly homogeneous tree on $\omega \times \omega \times \kappa$ as witnessed by $\{\mu_{s,t}\}$, then we define the Martin-Solovay construction over T to be a tree S projecting to $p[T]^c$ by

$(s, \alpha) \in S \leftrightarrow \exists f : \prod_{i \leq s}(\{\pi(i)\} \times T_{s\restriction|\pi(i)|,\pi(i)}) \to \kappa^+$ such that f is order preserving with respect to the KB ordering on $(\{\pi(i)\} \times T_{s|\pi(i)|,\pi(i)})$ (we order tuples lexicographically) and $\forall i < n(\alpha(i) = [f^{\pi(i)}]_{\mu_{s\restriction|\pi(i)|,\pi(i)}})$. Where, for $(\pi(i), \alpha) \in T$, $f^{\pi(i)}(\alpha) = f((\pi(i), \alpha))$

Definition 1.3. If T is a tree on $\omega^n \times \kappa$, then we say T is good if $\forall (s, \alpha) \in (\omega^n \times \kappa)^{<\omega}((s, \alpha) \in T \to \forall n < |\alpha|(\alpha(0) > \alpha(n)))$

For example, we can arrange for the Schoenfield tree for any $\mathbf{\Sigma^1_2}$ set to be good. Notice that if T is a good, weakly-homogeneous tree, then we can take the f as in Definition 1.2 to be into κ. Further, if κ has the strong partition property then we can define measures $\{\nu_s\}_{s\in\omega^{<\omega}}$ on the Martin-Solovay tree S as follows:

$A \in \nu_s \leftrightarrow (\exists C \subset \kappa$ club such that $\forall \alpha \in \kappa^{|s|}$(if $(s, \alpha) \in S$ and the function f that witness $(s, \alpha) \in S$ as in Definition 1.2 can be taken so that f is of the correct type and $f : T_s \to C$, then $\alpha \in A$))

It is not difficult to see that in this situation, the $\{\nu_s\}$ witness that S is homogenous.

We recall some definitions from [?]. W^1_1 is the club measure on ω_1 and W^m_1 is its m-fold product. The following definitions come from Definition 4.25 and the paragraphs following it in [?]. Using the notation above, these definitions describe how the measures $\{\nu_s\}$ on S come from the measures $\{\mu_{s,t}\}$ on T. Recall that in this context, we are only interested in permutations f of $\{0, 1, ..., n-1\}$ such that $f(0) = n-1$.

Definition 1.4. A type-1 tree of uniform cofinalities (of depth n) is a function $\mathcal{R}$ satisfying the following:

(i) $(p_1, i_1) \in dom(\mathcal{R})$ for $0 \le i_1 \le a$ for some integer a, and $p_1 =$ the unique permutation of length 1, namely $p_1 = (1)$. for $i_1 = 0$, $\mathcal{R}((p_1, i_1)) = (s)$, and for $i_1 > 0, \mathcal{R}((p_1, i_1))$ is either (ω), or a permutation p_2 of length 2 (hence $p_2 = (2,1)$). Also, (p_1, i_1) is maximal in dom($\mathcal{R}$) iff $\mathcal{R}((p_1, i_1)) =$ (ω) or (s).

(ii) In general, dom($\mathcal{R}$) consists of tuples $(p_1, i_1, ..., i_{m-1}, p_m, i_m), m \leq n$, and such a tuple is maximal in dom($\mathcal{R}$) iff $\mathcal{R}((p_1, i_1, ..., p_m, i_m)) = (\omega)$ or (s) (there are the only values permitted therefore if m=n). $\mathcal{R}((p_1, i_1, ..., p_m, i_m)) =$ (s) iff $i_m = 0$. if $\mathcal{R}((p_1, i_1, ..., p_m, i_m)) \neq (\omega)$ or (s), then $\mathcal{R}((p_1, i_1, ..., p_m, i_m))$ is a permutation p_{m+1} immediately extending p_m. In this case we have $(p_1, i_1, ..., p_m, i_m, p_{m+1}, i_{m+1}) \in dom(\mathcal{R})$ for some integers $0 \leq i_{m+1} \leq a$ (a≥ 0 and depends on $(p_1, i_1, ..., p_m, i_m, p_{m+1})$).

Definition 1.5. For $\mathcal{R}$ a type-1 tree of uniform cofinalities, we define $<^{\mathcal{R}}$ to be the lexicographic ordering on sequences $(\alpha_1, i_1, ..., i_{m-1}, \alpha_m, i_m)$

(i) $\alpha_1, ..., \alpha_m < \omega_1$

(ii) $(\alpha_1, ..., \alpha_m)$ is of type p_m where $(p_1, ..., p_m)$ is the unique sequence such that $(p_1, i_1, ..., p_m, i_m) \in dom(\mathcal{R})$ (Here we say $(\alpha_1, i_1, ..., \alpha_m, i_m)$ is of type $(p_1, i_1, ...p_m, i_m)$).

Definition 1.6. Say a function $f : dom(<^{\mathcal{R}}) \to \omega_1$ is of type $\mathcal{R}$ if it is order-preserving and

(i) f($(\alpha_1, i_1, ..., \alpha_m, i_m)$) has uniform cofinality ω if either $(\alpha_1, i_1, ..., \alpha_m, i_m)$ has successor rank in $<^{\mathcal{R}}$ or if $\mathcal{R}((p_1, i_1, ..., p_m, i_m)) = (\omega)$ (for $(\alpha_1, i_1, ..., \alpha_m, i_m)$ of type $(p_1, i_1, ..., p_m, i_m)$).

(ii) Otherwise, f($(\alpha_1, i_1, ..., \alpha_m, i_m)$) $= sup\{f(\overline{s}) : \overline{s} <^{\mathcal{R}} (\alpha_1, i_1, ..., \alpha_m, i_m)$

Definition 1.7. To each tree of uniform cofinalities $\mathcal{R}$ we associate a measure $M^{\mathcal{R}}$ by

$A \in M^{\mathcal{R}} \leftrightarrow$ There is a club $C \subset \omega_1$ such that $\forall f : dom(<^{\mathcal{R}}) \to C$ of type R $[f]_{\mathcal{R}} \in A$ (where $[f]_{\mathcal{R}} = (..., [f^{(p_1,i_1,...,p_m,i_m)}]_{W_1^m}, ...)$)

where $f^{(p_1,i_1,...,p_m,i_m)}(\alpha_1, ..., \alpha_m) = f((\alpha_1, i_1, ..., \alpha_m, i_m))$ (here $(\alpha_1, ..., \alpha_m)$ appear in the correct order).

Now, assume we are applying the Martin Solovay construction (Definition 1.2) to a weakly homogeneous tree T such that p[T] is a $\mathbf{\Sigma^1_2}$ complete set. Then the $\prod_{i \leq s}(\{\pi(i)\} \times T_{s\restriction|\pi(i)|,\pi(i)})$ of Definition 1.2 is precisely dom($<^{\mathcal{R}}$) for some tree of uniform cofinalities $\mathcal{R}$. Thus the measures $\{\nu_s\}$ on the resultant tree S are all measures of the form $M^{\mathcal{R}}$.

Finally, we will need one result from the descriptions analysis of [??]. Let $\mathcal{R}$ be a tree of uniform cofinalities and let $f : dom(<_{\mathcal{R}}) \to \omega_1$ be of type $\mathcal{R}$. Let $(\alpha_1, ..., \alpha_n) \in \omega_1^n$, we define f$[(\alpha_1, ..., \alpha_n)]$ to be the set of all possible ways to apply f to a subset of $(\alpha_1, ..., \alpha_n)$. Formally, let $f[(\alpha_1, ..., \alpha_n)]$ be the set

$\{\beta \ : \ \exists m \leq n \ \exists i_1, j_1, ..., i_m, j_m \leq n \ (\alpha_{j_1}, i_1, ..., \alpha_{j_m}, i_m) \in dom(<_{\mathcal{R}}) \ \wedge f((\alpha_{j_1}, i_1, ..., \alpha_{j_m}, i_m)) = \beta\}$

Theorem 1.8. *Let $\mathcal{R}$ be a tree of uniform cofinalities and let $F : dom(M^{\mathcal{R}}) \to \omega_n$. Then there is a $g{:}\omega_1^m \to \omega_1$ (m is the number of descriptions defined over $\mathcal{R}$ relative to W_1^{n-1}) s.t.*

$$\forall^*_{M^{\mathcal{R}}}[f] \ F([f]) = [\bar{\alpha} \mapsto g(f[\bar{\alpha}])]_{W_1^{n-1}}$$

2 Indiscernibles for L[S_2,x]

For T a weakly homogeneous tree, we say T is stable iff if S is the tree coming from the Martin Solovay construction over T, then the leftmost branches of S are a scale. I.e. for $x \in p[S]$, if $\bar{\alpha}$ is the leftmost branch of S_x, then the functions $\phi_n(x) = \bar{\alpha}(n)$ are a scale on p[S]. By [?], for every weakly homogeneous tree T on $\omega \times \omega \times \kappa$, there is a weakly homogeneous T' on $\omega \times \omega \times \kappa$ such that p[T]=p[T'] and T' is stable.

Fix a stable tree T on $\omega \times \omega \times \omega_1$ such that $p[T]$ is a $\mathbf{\Sigma^1_2}$ complete set. Let T be weakly homogeneous as witnessed by the measures $\{\mu_{s,t}\}_{(s,t)\in(\omega\times\omega)^{<\omega}}$ (recall each $\nu_{s,t}$ is of the form W_1^m for some m). Let S_2 come from the Martin Solovay construction over T and be homogeneous as witnessed by the measures $\{\mu_s\}_{s\in\omega^{<\omega}}$ (recall each μ_s is of the form $M^{\mathcal{R}}$ for some tree of uniform cofinalities $\mathcal{R}$). Fix a real $x \in \omega^\omega$, for the rest of this section we will develop a theory of indiscernibles for the model L$[S_2, x]$.

Definition 2.1. By a class of indiscernibles for $L[S_2, x]$ we will mean a proper class of ordinals Γ such that $min(\Gamma) > \omega_\omega$ and $\forall \gamma_1, ..., \gamma_n, \gamma\prime_1, ..., \gamma\prime_n \in \Gamma$, if $\gamma_1 < ... < \gamma_n$ and $\gamma\prime_1 < ... < \gamma\prime_n$ and $\alpha_1, ..., \alpha_m < min(\gamma_1, \gamma\prime_1)$ then for all $(m+n)-ary$ formulas ϕ, $L[S_2, x] \models \phi(\alpha_1, ..., \alpha_m, \gamma_1, ..., \gamma_n) \leftrightarrow L[S_2, x] \models \phi(\alpha_1, ..., \alpha_m, \gamma\prime_1, ..., \gamma\prime_n)$.

It is straightforward to show that this definition is equivalent the standard definition of indiscernibles in terms of embeddings.

Definition 2.2. Given Γ a class of indiscernibles, a homogeneous set for Γ (in the language $\mathcal{L}$) is a club $C \subseteq \omega_1$ such that for all trees of uniform cofinalities $\mathcal{R}$, for all $f, g : dom(<_{\mathcal{R}}) \to C$ of type $\mathcal{R}$, for all $\gamma_1 < ... < \gamma_n \in \Gamma$, and for all formulas ψ (in the language $\mathcal{L}$) we have

$$L[S_2, x] \models \psi([f]_{\mathcal{R}}, \gamma_1, ..., \gamma_n) \leftrightarrow L[S_2, x] \models \psi([g]_{\mathcal{R}}, \gamma_1, ..., \gamma_n)$$

For $A \subset \omega_1$ we will write $A^{\uparrow}$, the "lift-up of A"', to denote the set

$$\{\alpha : \exists f : \omega_1^m \to A \ \wedge \ [f]_{W_1^m} = \alpha\}$$

For Γ a class of indiscernibles for $L[S_2, x]$ and C a homogeneous set for Γ, we will call (Γ, C) an indiscernible pair. In general, for (Γ, C) an indiscernible pair, we will be interested in $\mathcal{H} = hull^{L[S_2,x]}(\Gamma \cup C^{\uparrow})$. The transitive collapse of $\mathcal{H}$ will be some model $L[S, x]$ and the natural question is whether $S = S_2$. This motivates the following definition:

Definition 2.3. For (Γ, C) an indiscernible pair, we will say (Γ, C) is a full pair if there is a countable language $\mathcal{L}$ (containing $\{\underline{\epsilon}, \underline{S_2}\}$ and with an interpretation over $L[S_2, x]$) and a set $A \subset \omega_1$ such that $hull_{\mathcal{L}}^{L[S_2,x]}(\Gamma \cup C^{\uparrow}) \cap \omega_\omega = \ A^{\uparrow}$

Theorem 2.4. *Let (Γ, C) be a full pair, and let $\mathcal{M} = L[S, x]$ be the transitive collapse of $\mathcal{H} = hull^{L[S_2,x]}(C^{\uparrow})$. Then $S = S_2$. In particular $M = L[S_2, x]$.*

Proof. S is the collapse of S_2 with the second coordinates restricted to $A^{\uparrow}$. I.e. S is the collapse of the set

$$\{(s, \alpha) : \ (s, \alpha) \in S_2 \ \wedge \ \alpha \in (A^{\uparrow})^{<\omega}\}$$

This means that, if S_2 comes from the Martin Solovay construction applied to T, then S is the transitive collapse of the tree S' where S' comes from the Martin Solovay construction applied to T except that the functions f as in Definition 1.2 are required to be into A. Note that the transitive collapse of A is ω_1.

Let $\rho : A \to \omega_1$ and $\theta : \mathcal{H} \to L[S, x]$ be the collapse maps. For $f : \omega_1^n \to A$ let $(\rho(f))(\alpha) = \rho(f(\alpha))$. Note that

$$\forall f : \omega_1^n \to A,\ [\rho(f)]_{W_1^n} = \theta([f]_{W_1^n}) \tag{$\star$}$$

Above, we observed that S is the transitive collapse of the tree S'. By $(\star)$, the transitive collapse of S' is the same as the Martin Solovay construction applied to T with the functions f as in Definition 1.2 required to be into the transitive collapse of A, which is the full Martin Solovay construction. Thus $S = S_2$.

$\square$

Theorem 2.5. *Let Γ be a class of indiscernibles for $L[S_2, x]$ with $min(\Gamma) > \omega_{\omega+1}$. Then there is a $C \subset \omega_1$, a homogeneous set for $L[S_2, x]$, such that (Γ, C) is a full pair.*

Proof. Fix Γ a class of indiscernibles with $min(\Gamma) > \omega_{\omega+1}$. The construction of C will be an ω length induction. At each stage n we will have a countable language $\mathcal{L}_n$ containing $\{\underline{\epsilon}, \underline{S_2}\}$, a club $C_n \subset \omega_1$, and a countable set of functions $\mathcal{F}_n \subset {}^{\omega_1}\omega_1$ such that the following properties hold:

(i) C_n is a homogeneous set for Γ in the language $\mathcal{L}_n$

(ii) $\mathcal{L}_{n+1}$ is $\mathcal{L}_n$ with countable many function symbols $\{\underline{g}_m\}_{m\in\omega}$ added for each function $g \in \mathcal{F}_n$.

(iii) For every term τ in the language $\mathcal{L}_n$, for all $\gamma_1 < ... < \gamma_n \in \Gamma$, and for all trees of uniform cofinalities $\mathcal{R}$ such that $\forall^*_{M^{\mathcal{R}}}[f]\ \tau([f], \gamma_1, ..., \gamma_n) < \omega_{m+1}$ there exists a function $g \in \mathcal{F}_n$ such that for all $f : dom(<_{\mathcal{R}}) \to C_n$

$$\tau([f], \gamma_1, ..., \gamma_n) = [\bar{\alpha} \to g(f[\bar{\alpha}])]_{W_1^m}$$

(iv) $\mathcal{F}_n$ is closed under composition

Although (iv) is not necessary, it will make aspects of our proof notationally easier. Our construction will also guarantee that $C_{n+1} \subset C_n$.

We expand upon (ii): for g$\in \mathcal{F}_n$, $\ g : \omega_1^m \to \omega_1$, for each $l \in \omega$ we add a function symbol $\underline{g_l}$ to $\mathcal{L}_{n+1}$ to be interpreted as a function $\underline{g_l}^{L[S_2,x]} : \omega_{l+1}^m \to \omega_{m+1}$ by

$$\underline{g_l}^{L[S_2,x]}([f_1], ...[f_m]) = [\bar{\alpha} \mapsto g(f_1(\bar{\alpha}), ..., f_n(\bar{\alpha}))]_{W_1^l}$$

This is independent of the choices of representatives $f_1, ..., f_m$. Note also that each $\underline{g_l}^{L[S_2,x]}$ is essentially a subset of ω_ω. Thus, due to the fact that $min(\Gamma) > \omega_{\omega+1}$, Γ is still a class of indiscernibles with respect to the language $\mathcal{L}_{n+1}$.

Assume we have such a construction and set $C = \bigcap_{n\in\omega} C_n, \mathcal{L} = \bigcup_{n\in\omega} \mathcal{L}_n, \mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$. We show C is a full set of indiscernibles. Specifically, let A be the closure of C under the functions in $\mathcal{F}$, we will show that $Hull^{L[S_2,x]}_{\mathcal{L}}(\Gamma \cup C^{\uparrow}) \cap \omega_\omega = A^{\uparrow}$.

Let $\mathcal{H} = Hull_{\mathcal{L}}^{L[S_2,x]}(\Gamma \cup C^{\uparrow})$.

First we show that $\mathcal{H} \cap \omega_\omega \subseteq A^{\uparrow}$. To see this, let $\beta \in \mathcal{H} \cap \omega_\omega$. Then $\beta = \tau([f], \gamma_1, ..., \gamma_n)$ for some tree of uniform cofinalities $\mathcal{R}$ and $f : dom(<_{\mathcal{R}}) \to C$, $\gamma_1 < .... < \gamma_n \in \Gamma$ and τ a term in $\mathcal{L}_m$. By (iii), there is a $g \in \mathcal{F}_m$ such that $\beta = [\bar{\alpha} \to g(f[\bar{\alpha}])]_{W_1^m}$. Thus $\beta \in g[C]^{\uparrow}$ and clearly $g[C] \subset A$.

Next, we show $\mathcal{H} \cap \omega_\omega \supseteq A^{\uparrow}$. To this end, let $f : \omega_1^n \to A$ be such that [f] is not representable by a $f' : \omega_1^n \to C$. A simple partition argument shows that, WLOG, there is a single $g \in \mathcal{F}$ such that $f : \omega_1^n \to g[C]$. Say dom(g)=ω_1^m. For $i < m$, define $f_i : \omega_1^n \to C$ by $f_i(\bar{\alpha})$ is the ith coordinate of $g^{-1}(f(\bar{\alpha}))$. Then

$$[f] = [\bar{\alpha} \mapsto g(f_1(\bar{\alpha}), ..., f_m(\bar{\alpha}))]_{W_1^n} = \underline{g_n}^{L[S_2,x]}([f_1], ..., [f_m])$$

$\underline{g_n}$ is a term in $\mathcal{L}_{n+1}$ and $[f_1], ..., [f_m] \in C^{\uparrow}$, so [f]$\in Hull_{\mathcal{L}}^{L[S_2,x]}(C^{\uparrow})$ as required.

Finally, we describe how to construct $\{\mathcal{L}_n\}, \{C_n\}, \{\mathcal{F}_n\}$ so that (i)-(iv) hold. Set $\mathcal{L}_1 = \{\underline{\epsilon}, \underline{S_2}\}$ $\mathcal{F}_0 = \emptyset$. By the strong partition property on ω_1, for each formula ψ in $\mathcal{L}_n$, for each tree of uniform cofinalities $\mathcal{R}$, and for each $m \in \omega$, let $C_{\psi,\mathcal{R},m} \subset \omega_1$ be club such that for all $f_1, f_2 : dom(<_{\mathcal{R}}) \to C_{\psi,\mathcal{R},m}$ of type $\mathcal{R}$ and for all $\gamma_1 < ... < \gamma_m \in \Gamma$

$$L[S_2, x] \models \psi([f_1], \gamma_1, ..., \gamma_m) \leftrightarrow L[S_2, x] \models \psi([f_2], \gamma_1, ..., \gamma_m)$$

Further, by theorem 1.8 ,for each tree of uniform cofinalities $\mathcal{R}$, and $\gamma_1 < ... < \gamma_m \in \Gamma$, and each term τ in $\mathcal{L}_n$ such that $\forall^*_{\mathcal{R}}\bar{\alpha}\ \tau(\bar{\alpha}, \gamma_1, ..., \gamma_m) < \omega_\omega$, let $g^{\tau,\mathcal{R},m} : \omega_1^r \to \omega_1$ and $C_{\tau,\mathcal{R},m} \subset \omega_1$ club be so that for all $f : dom(<_{\mathcal{R}}) \to C_{\tau,\mathcal{R},m}$ of type $\mathcal{R}$

$$\tau([f], \gamma_1, ..., \gamma_m) = [\bar{\alpha} \mapsto g^{\tau,\mathcal{R},m}(f[\alpha])]_{W_1^l}$$

Then set $C_n = \bigcap_{\psi,\mathcal{R},m} C_{\psi,\mathcal{R},m} \cap \bigcap_{\tau,\mathcal{R},m} C_{\tau,\mathcal{R},m}$ and let $\mathcal{F}_n$ the be the closure under compositions of $\mathcal{F}_{n-1} \cup \bigcup_{\tau,\mathcal{R}} \{g^{\tau,\mathcal{R},m}\}$. To complete the induction, set $\mathcal{L}_{n+1} = \mathcal{L}_n \cup \bigcup_{\tau,\mathcal{R},m} \{\underline{g_l^{\tau,\mathcal{R}m}}\}_{l \in \omega}$. It is clear that this construction has the desired properties. $\square$

3 A good coding of subsets of ω_ω

Our goal in this section is to show that the weak partition property holds at δ^1_3. We first show how to adapt the main theorem of [??] to our current situation.

Let $\{\phi_n\}$ be the scale on p$[S_2]$ coming from the leftmost branch, we now compute the complexity of the ϕ_n.

Theorem 3.1. *$\{\phi_n\}$ is a $\mathbf{\Sigma^1_3}$ scale on $p[S_2]$.*

Proof. We must show that the relations $<^*_n$ and $\leq^*_n$ are $\mathbf{\Sigma^1_3}$ where

$$x <^*_n y \leftrightarrow (x \in p[S_2] \wedge y \notin p[S_2]) \vee (\phi_n(x) < \phi_n(y))$$

$$x \leq^*_n y \leftrightarrow (x \in p[S_2] \wedge y \notin p[S_2]) \vee (\phi_n(x) \leq \phi_n(y))$$

We will compute the relation $\leq^*_n$, the proof for $<^*_n$ is nearly identical.

For $x \in \omega^\omega$ and $(s, \alpha) \in (\omega \times \kappa)^{<\omega}$, we will say "the tree T_x below (s, α)" to refer to the set $\{(s', \alpha') : (s \cup s', \alpha \cup \alpha') \in T_x\}$. Examining Definition

1.2, it is clear that

$$x \leq_n^* y \leftrightarrow (x \in p[S_2] \wedge y \notin p[S_2]) \vee \forall^*_{W_1^{|\pi(n)|}} \bar{\alpha}(\text{ The tree } T_x \text{ below}$$

$(\pi(n), \sigma_1(\bar{\alpha}))$ embeds into the tree T_y below $(\pi(n), \sigma_2(\bar{\alpha})))$

where σ_1, σ_2 are the unique permutations of length $|\pi(n)|$ such that $(\pi(n), \sigma_1(\bar{\alpha})) \in T_x$ and $(\pi(n), \sigma_2(\bar{\alpha})) \in T_y$.

We will first show that "The tree T_x below $(\pi(n), \sigma_1(\bar{\alpha}))$ embeds into the tree T_y below $(\pi(n), \sigma_2(\bar{\alpha}))$" is $\mathbf{\Delta^1_3}$, and then show that $\mathbf{\Delta^1_3}$ is closed under the quantifier $\forall^*_{W_1^m}$

To this end, first notice that the coding lemma gives us a $\mathbf{\Sigma^1_2}$ coding of subsets of ω_1(better codings exist, but we will not need them here). That is, there is a map $\phi : \omega^\omega \to P(\omega_1)$ such that

(i) $\forall A \subset \omega_1$ there is a real z s.t. $\phi(z) = A$.

(ii) $\forall \alpha \{z : \alpha \in \phi(z)\} \in \mathbf{\Sigma^1_2}$

We can view such a coding as coding subsets of $(\omega \times \omega_1)^{<\omega}$.

To say "z codes an embedding from the tree T_x into the tree T_y is to say

$$\forall (s, \alpha) \in (\omega \times \omega_1)^{<\omega} \exists ! (t, \beta) \in (\omega \times \omega_1)^{<\omega} (\phi(z)((s, \alpha), (t, \beta)))$$

$$\wedge \forall (s_0, \alpha_0), (s_1, \alpha_1), (t_0, \beta_0), (t_1, \beta_1) \in (\omega \times \omega_1)^{<\omega}$$

$$((\phi((s_0,\alpha_0),(t_0,\beta_0)) \wedge \phi((s_1,\alpha_1),(t_1,\beta_1)) \wedge ((s_0,\alpha_0) <_{KB}^{T_x} (s_1,\alpha_1))) \rightarrow$$

$$((t_0,\beta_0) <_{KB}^{T_y} (t_1,\beta_1)))$$

hence is $\mathbf{\Delta^1_3}$ ($\mathbf{\Delta^1_3}$ is closed under unions and intersections of length ω_1). The above easily generalizes to show that "z codes an embedding from the tree T_x below $(\pi(n),\sigma_1(\bar{\alpha}))$ into the tree T_y below $(\pi(n),\sigma_2(\bar{\alpha}))$" is also $\mathbf{\Delta^1_3}$.

"The tree T_x below $(\pi(n),\sigma_1(\bar{\alpha}))$ embeds into the tree T_y below $(\pi(n),\sigma_2(\bar{\alpha}))$" can be computed as $\mathbf{\Sigma^1_3}$ by $\exists z$(z codes an embedding from the tree T_x below $(\pi(n),\sigma_1(\bar{\alpha}))$ into the tree T_y below $(\pi(n),\sigma_2(\bar{\alpha}))$). It can also be computed as $\mathbf{\Pi^1_3}$ by $\forall z$ (z does not compute an embedding from the tree T_y below $(\pi(n),\sigma_2(\bar{\alpha}))$ into a proper initial segment of the tree T_x below $(\pi(n),\sigma_1(\bar{\alpha}))$).

Thus, it remains to be shown that $\mathbf{\Delta^1_3}$ is closed under the quantifier $\forall^*_{W^1_m}$. We will show it is closed under $\forall^*_{W^1_1}$, the general case is only a complication of notation. To this end, let $\{A_\alpha : \alpha < \omega_1\}$ be a sequence of $\mathbf{\Delta^1_3}$ sets. It suffices to show that $\forall^*_{W^1_1}\alpha\ A_\alpha$ is $\mathbf{\Sigma^1_3}$, as an identical computation would show that $\neg(\forall^*_{W^1_1}\alpha\ A_\alpha)(= \forall^*_{W^1_1}\alpha\ (A_\alpha)^c)$ is $\mathbf{\Sigma^1_3}$. Let T' denote the Kunen tree. Then

$\forall^*_{W^1_1}\alpha(x \in A_\alpha) \leftrightarrow \exists z((T'_z$ is wellfounded) $\wedge \forall\alpha < \omega_1((\alpha$ is closed under $T'_z) \rightarrow x \in A_\alpha))$

hence is $\exists^{\mathbb{R}}(\mathbf{\Pi^1_2} \wedge \mathbf{\Delta^1_3})$ which is $\mathbf{\Sigma^1_3}$. $\square$

A trivial modification to S_2 ensures that, for

$$((a_0,...,a_n),(\xi_0,...,\xi_n)),((a_0,...,a_n),(\xi'_0,...\xi'_n)) \in S_2((\xi_n < \xi'_n) \rightarrow \forall i < n(\xi_i \leq \xi'_i))$$

For x,y$\in p[S_2]$ and n,m$\in \omega$, define $(n,x) <_{S_2} (m,y)$ iff $(n = m \wedge \phi_n(x) < \phi_n(y))$. Then $<_{S_2} \in \mathbf{\Sigma^1_3}$ and is well-founded of rank ω_ω. The following is a generalization of the main result of [??].

Theorem 3.2. *For every $A \subset \omega_\omega$ there is a real x such that $A \in L[S_2, x]$.*

Proof. The argument is identical to that of [??], so we will only show how to modify the definitions to fit our situation. The only differences are that (i) we will use $<_{S_2}$ to code ordinals instead of ϕ_0 (ii) S_2 is not the tree of a scale. We replace the following definitions in [??]:

$$T^\eta = \{(m, (a_0, ..., a_n), (\xi_0, ..., \xi_n)) : ((a_0, ..., a_n), (\xi_0, ..., \xi_n)) \in S_2 \wedge (n < m \vee \xi_m \leq \eta(m))\}$$

where $\eta(m)$ is such that if $|(m,x)|_{<_{S_2}} = \eta$ then x(m)=$\eta(m)$. If there is no such x then $\eta(m) = \omega_1$. note that, as in [??], $(m,y) \in p[T^\eta] \to (|(m,y)|_{<_{S_2}} \leq \eta)$

For all $n \in \omega$ let $Q_n \subseteq \omega \times \mathbb{R} \times (\omega \times \omega^{n+1} \times \omega_\omega^{n+1})$ be the following set:

$$\{(m_0, x, m_1, (a_0, ..., a_n), (\xi_0, ..., \xi_n)) : x \in p[S_2] \wedge (m_1, (a_0, ..., a_n), (\xi_0, ..., \xi_n)) \in T^{|(m_0,x)|_{<_{S_2}}}\}$$

As in [??], let Q_n^* be the code set of Q_n where the ith coordinate is encoded using ϕ_i. Formally, $Q_n^* \subseteq \omega \times \mathbb{R} \times (\omega \times \omega^{n+1} \times \mathbb{R}^{n+1})$ is the set of tuples

$$\{(m_0, x, m_1, (a_0, ..., a_n), (z_0, ..., z_n)) : \forall i \leq n(z_i \in p[S_2]) \wedge$$

$$(m_0, x, m_1, (a_0, ..., a_n), (\phi_0(z_0), ..., \phi_n(z_n))) \in Q_n\}$$

The main complication in our generalization is that in [??] their T (our S_2) is the tree of a scale. This allows them to compute Q_n^* as $\mathbf{\Sigma^1_3}$ in an easy way.

However, we are able to use the homogeneity of S_2 to give a similar, albeit more complicated, computation. Namely,

$$(m_0, x, m_1, (a_0, ..., a_n), (z_0, ..., z_n)) \in Q_n^* \leftrightarrow x \in p[S_2] \wedge \forall i \leq n(z_i \in p[S_2]) \wedge$$

$\exists y[y \in p[S_2] \wedge \forall i \leq n(y(i) = a_i) \wedge (\phi_0(y), ..., \phi_n(y))$ has the same R-type as $(\phi_0(z_0), ..., \phi_n(z_n)) \wedge (m_1, y) <_{S_2} (m_0, x)]$

Where $(\alpha_0, ..., \alpha_n)$ has the same R-type as $(\beta_0, ..., \beta_n)$ iff there is a tree of uniform cofinalities $\mathcal{R}$ and functions $f_1, f_2 : dom(<_{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$ such that $[f_1]_{\mathcal{R}} = (\alpha_0, ...\alpha_n)$ and $[f_2]_{\mathcal{R}} = (\beta_0, ..., \beta_n)$. A straightforward computation shows that this is $\mathbf{\Sigma^1_3}$, thus Q_n^* is $\mathbf{\Sigma^1_3}$. Finally, given $A \subset \omega_\omega$, let A^* be the codeset for A using $<_{S_2}$ and let x be such that $A^*, <_{S_2}, Q_n^* \in \Sigma^1_3(x)$. The argument of [??] shows that A$\in L[S_2, x]$.

$\square$

We now prove the weak partition property holds at δ^1_3.

Fix a countable language $\mathcal{L} \supset \{\underline{\epsilon}, \underline{S_2}\}$ such that, for each n, $\mathcal{L}$ has countably many n-ary function symbols. Note that for any full set of indiscernibles C as constructed above, we can assume that C is a full set of indiscernibles as witnessed by $\mathcal{L}$ under some appropriate interpretation of $\mathcal{L}$.

Note that for any sequence of ordinals $\bar{\alpha} \in \omega_\omega^{<\omega}$ there is a smallest tree of uniform cofinalities $\mathcal{R}_{\bar{\alpha}}$ such that there is a function f: $dom(<_{\mathcal{R}_{\bar{\alpha}}}) \to \omega_1$ of type $\mathcal{R}_{\bar{\alpha}}$ and $\bar{\alpha}$ is a subset of [f].

For C a full set of indiscernibles for $L[S_2, x]$, we can code the $\mathcal{L}$-theory of $Hull_{\mathcal{L}}^{L[S_2,x]}(C^{\uparrow})$ by a real y. Specifically (viewing y as a subset of ω), for a tree of uniform cofinalities $\mathcal{R}$ and for a formula ψ in the language $\mathcal{L}$, we put $< \mathcal{R}, \psi > \in y$ iff ψ is defined on $dom(M^{\mathcal{R}})$ and for $f : dom(<_{\mathcal{R}}) \to C, L[S_2, x] \models \psi([f])$. In this situation, we say y "is a sharp for $L[S_2, x]$".

For y a real, we say "y looks like a sharp" iff y codes a consistent theory via the coding above and y$\models$ ZFC + V=L[S]. Note that to say "y looks like a sharp" is $\mathbf{\Delta^1_1}$.

For y a real that looks like a sharp, we build the model $\mathcal{M}_y$ in the following way: the universe of $\mathcal{M}_y$, M_y, is the set of all pairs ¡$\tau, (x_1, ...x_n)$¿ where τ is an nary $\mathcal{L}$-formula and $(x_1, ..., x_n)$ is a sequence of reals coding a function $f_{x_i} : \omega_1^{n_i} \to \omega_1$ (here we use our $\mathbf{\Sigma^1_2}$ coding of subsets of ω_1). Given $< \tau_1, (x^1_1, ..., x^1_{m_1}) >, ..., < \tau_n, (x^n_1 ..., x^n_{m_n}) >$ in the universe of $\mathcal{M}_y$ and ψ an n-ary $\mathcal{L}$-formula

$$\mathcal{M}_y \models \psi(< \tau_1, (x^1_1, ..., x^1_{m_1} >, ..., < \tau_n, (x^n_1 ..., x^n_{m_n}) >)) \leftrightarrow < \mathcal{R}_{([f_{x^1_1}], ..., [f_{x^1_{m_1}}], ..., [f_{x^n_1}], ..., [f_{x^n_{m_n}}])}, \psi' > \in y$$

Where, if f:dom($<_{\mathcal{R}_{([f_{x^1_1}], ..., [f_{x^1_{m_1}}], ..., [f_{x^n_1}], ..., [f_{x^n_{m_n}}])}}$) $\to \omega_1$ is such that $([f_{x^1_1}], ..., [f_{x^1_{m_1}}], ..., [f_{x^n_1}], ..., [f_{x^n_{m_n}}]) \subset [f]$, then $\psi'([f]) \leftrightarrow \psi(\tau_1([f_{x^1_1}], ..., [f_{x^1_{m_1}}]), ..., \tau_n([f_{x^n_1}] ..., [f_{x^n_{m_n}}]))$.

We make some observations about $\mathcal{M}_y$. M_y is $\mathbf{\Delta^1_3}$ as it can be defined using ordinal quantifiers over ω_1 and our $\mathbf{\Sigma^1_2}$ coding of subsets of ω_1. Also, for $< \tau_1, (x^1_1, ..., x^1_{m_1}) >, ..., < \tau_n, (x^n_1, ..., x^n_{m_n}) > \in M_y$ and ψ an $\mathcal{L}$-formula, to say $\mathcal{M}_y \models \psi(< \tau_1, (x^1_1, ..., x^1_{m_1}) >, ..., < \tau_n, (x^n_1 ..., x^n_{m_n}) >)$ is $\mathbf{\Delta^1_3}$ by a similar computation (here we again use that $\mathbf{\Delta^1_3}$ is closed under $\forall^*_{W_1^m}$). Lastly we note that, while there may be ill founded $\mathcal{M}_y$, every $\mathcal{M}_y$ contains an initial segment of the ordinals, which we denote by $WFP(\mathcal{M}_y)$. Again, standard computations show that for $\alpha \in \omega_\omega$, ¡$\tau, (x_1, ...x_n) >\in \mathcal{M}_y$, to say "$< \tau, (x_1, ...x_n) >\in WFP(\mathcal{M}_y)$ and has rank α" is $\mathbf{\Delta^1_3}$. We now provide our coding:

Theorem 3.3. *There is a coding $\Phi : \omega^\omega \to P(\omega_\omega)$ such that*

(i) $\forall\ A \subset \omega_\omega$ *there is a real x such that $\Phi(x) = A$.*

(ii) $\forall\ \alpha < \omega_\omega\ \{x : \alpha \in \Phi(x)\} \in \mathbf{\Delta^1_3}$.

Proof. For $x \in \omega_\omega$ define $\Phi(x)$ by

$\alpha \in \Phi(x) \leftrightarrow$ ($x =< \psi, y >$ for ψ a $\mathcal{L}$-formula and y a real that looks like a sharp) and ($\alpha \in WFP(\mathcal{M}_y)$ and $\mathcal{M}_y \models \psi(\alpha)$)

Of course, by "$\mathcal{M}_y \models \psi(\alpha)$" we mean "$\exists < \tau, (z_0, ..., z_n) > ((| < \tau, (z_0, ..., z_n) > |_{ON^{\mathcal{M}_y}} = \alpha) \wedge \mathcal{M}_y \models \psi(< \tau, (z_0, ..., z_n) >))$" or, equivalenty, "$\forall < \tau, (z_0, ..., z_n) > ((| < \tau, (z_0, ..., z_n) > |_{ON^{\mathcal{M}_y}} = \alpha) \to \mathcal{M}_y \models \psi(< \tau, (z_0, ..., z_n) >))$" By the remarks in the paragraph preceding the theorem and by Theorem 3.2, Φ has the desired properties.

$\square$

By [??] Theorem 3.3 implies that the weak partition property holds at δ^1_3.

Recall from the theory of indiscernibles for $L[x]$ that the set of reals $\{y : \exists x\ (y = x^\sharp)\}$ is $\mathbf{\Pi^1_2}$. We conclude by generalizing this result to sharps for $L[S_2, x]$.

Theorem 3.4. *The set of reals $A = \{y : \exists x\ (y \text{ is a sharp for } L[S_2, x])\}$ is $\mathbf{\Pi^1_3}$.*

Proof. $y \in A$ iff y looks like a sharp, $\mathcal{M}_y$ is well-founded, and $\underline{S_2}^{\mathcal{M}_y} = S_2$. Using our coding from Theorem 3.3, to say "$\underline{S_2}^{\mathcal{M}_y} = S_2$" is $\mathbf{\Delta^1_3}$. Further $\mathcal{M}_y$ is well founded iff $\forall z($ z does not code an ω-sequence of elements of $\mathcal{M}_y$ that is $\mathcal{M}_y$-decreasing), hence is $\mathbf{\Pi^1_3}$ as desired. $\square$

References