These notes prove some basic strucure theorems about extenders. Given an extender E that witnesses the strength of some embedding, we use our theorems to show that the generators of E have a property we call "full". Though we have no proof, it seems likely that the converse is true as well, i.e. if E is an extender with full generators, then E comes from a strong embedding.

Let E be an extender over HOD with $crit(E) = \kappa$ and let $\{\xi_\alpha\}_{\alpha<\gamma}$ enumerate the generators of E in increasing order. For all $\alpha < \gamma$, let μ_α be the measure on ξ_α derived from the embedding $i_\alpha : Ult(HOD, E \restriction \xi_\alpha) \to Ult(HOD, E \restriction (\xi_\alpha + 1))$. Note that, in general, $\mu_\alpha \notin Ult(HOD, E \restriction \xi_\alpha)$. In fact, if E is to witness any large cardinal properties, there must be many α for which this is the case.

We demonstrate the connection between μ_α and $E \restriction (\xi_\alpha + 1)$. Let $A \subset \xi_\alpha$, A$\in$ $Ult(HOD, E \restriction \xi_\alpha)$. Say $A = [a, f]_{E\restriction\xi_\alpha}$, with $a \in [\xi_\alpha]^{<\omega}$ and $f : \kappa^{|a|} \to 2^\kappa$. Define $A^* \subset \kappa^{|a|+1}$ to be the set $\{(\overline{\alpha}, \beta) : \beta \in f(\overline{\alpha})\}$. Then $A \in \mu_\alpha \leftrightarrow A^* \in E_{a^\frown\xi_\alpha}$. To see this, note that (using notation from the diagram below)

$$\begin{aligned} A \in \mu_\alpha &\leftrightarrow \xi_\alpha \in \tilde{i}_\alpha(A) \\ &\leftrightarrow \xi_\alpha \in i_\alpha(A) \\ &\leftrightarrow \xi_\alpha \in [a, f]_{E_{a^\frown\xi_\alpha}} \\ &\leftrightarrow \forall^*_{E_{a^\frown\xi_\alpha}}(\overline{\alpha}, \beta)\ \beta \in f(\overline{\alpha}) \\ &\leftrightarrow A^* \in E_{a^\frown\xi_\alpha} \end{aligned}$$

Where the second "$\leftrightarrow$" holds because $crit(k) > \xi_\alpha$.

$$\begin{tikzcd}
Ult(HOD, E \restriction \xi_\alpha) \arrow[r, "i_\alpha"] \arrow[dr, "\tilde{i}_\alpha"] & Ult(HOD, E \restriction (\xi_\alpha + 1)) \\
& Ult(Ult(HOD, E \restriction \xi_\alpha), \mu_\alpha) \arrow[u, "k"]
\end{tikzcd}$$

Next, we show how E can be decomposed via $\{\mu_\alpha\}_{\alpha<\gamma}$. For $\alpha \leq \gamma$, we define by induction the "decomposition of E up to ξ_α on HOD", $Dec_\alpha(HOD)$, by

(i): $Dec_0(HOD) = HOD$

(ii): $Dec_{\alpha+1}(HOD) = Ult(Dec_\alpha(HOD), \mu_\alpha)$

(iii): For α a limit ordinal $Dec_\alpha(M) = DirLim_{\beta<\alpha}(HOD)$.

Note that while μ_α is ostensibly a measure on $Ult(HOD, E \restriction \xi_\alpha)$, it is always the case that $Dec_\alpha(HOD) \subset Ult(HOD, E \restriction \xi_\alpha)$, so that (ii) makes sense.

Let $\tilde{i}_{\alpha,\alpha+1} : Dec_\alpha(HOD) \to Dec_{\alpha+1}(HOD)$ be the ultrapower embedding.

Claim 0.1. *Let $\tilde{E}$ be the $(\kappa, lh(E))$-extender derived from the embedding $\tilde{i} : HOD \to Dec_\gamma(HOD)$, then $\tilde{E} = E$.*

Proof. It is clear that E and $\tilde{E}$ have the same generators. We show by induction on $\alpha < \gamma$ that

$$\tilde{E} \restriction (\xi_\alpha + 1) = E \restriction (\xi_\alpha + 1) \tag{1}$$

For $\alpha = 0$, (1) is immediate from the fact that $\mu_0 = E_\kappa$.

Now, assume we have shown (1) for $\beta < \alpha$. Let $a \in [\xi_\alpha]^{<\omega}$ and let $A^* \in \kappa^{|a|+1}$. Define $f : \kappa^{|a|} \to 2^\kappa$ by $f(\overline{\alpha}) = \{\beta : (\overline{\alpha}, \beta) \in A^*\}$ and let $A = [a, f]_{E\restriction\xi_\alpha}$.

By our work above,

$$A^* \in \tilde{E}_{a^\frown \xi_\alpha} \leftrightarrow \ \xi_\alpha \in \tilde{i}_{\alpha,\alpha+1}(A),$$

and by the definition if $\tilde{i}_{\alpha,\alpha+1}$,

$$\xi_\alpha \in \tilde{i}_{\alpha,\alpha+1}(A) \leftrightarrow A \in \mu_\alpha$$

again, by our work above

$$A \in \mu_\alpha \leftrightarrow A^* \in E_{a^\frown \xi_\alpha}$$

so $\tilde{E} \restriction (\xi_\alpha + 1) = E \restriction (\xi_\alpha + 1)$ as required.

$\square$

Corollary 0.2. $Dec_\alpha(HOD) = Ult(HOD, E \restriction \xi_\alpha)$

Let E be an extender on HOD that witnesses that κ is $(\kappa+2)$-strong - what are the generators of E? As before, let $\{\xi_\alpha\}_{\alpha<\gamma}$ enumerate the generators of E. The strength of E implies that $(\kappa^{++})^{ULT(HOD,E)} = (\kappa^{++})^{HOD}$. Further, the embedding $Ult(HOD, E \restriction \xi_\alpha) \to Ult(HOD, E)$ is elementary with critical point ξ_α. Thus, if $(\kappa^{++})^{Ult(HOD,E\restriction\xi_\alpha)} < (\kappa^{++})^{Ult(HOD,E)}$, then $\xi_\alpha = (\kappa^{++})^{Ult(HOD,E\restriction\xi_\alpha)}$. This reasoning gives rise to the following lemma:

Lemma 0.3. *Let E be an extender on HOD with critical point κ. Let γ be a HOD-cardinal less than the least inaccessible above κ such that $ULT(HOD, E) \restriction \gamma = HOD \restriction \gamma$. Then $\xi_1 = (\kappa^{++})^{Ult(HOD,E\restriction\xi_1)}$, and in general, for $\alpha < \gamma$, $\xi_\alpha = (|sup_{\beta<\alpha}(\xi_\beta)|^+)^{Ult(HOD,E\restriction\xi_\alpha)}$.*

The proof of the Lemma is repeated application of the following two facts:

(i): If ξ_α is a successor cardinal in $Ult(HOD, E \upharpoonright \xi_\alpha)$, then ξ_α is not a cardinal in $Ult(HOD, E \upharpoonright (\xi_\alpha + 1))$.

(ii): If we know $\xi_\alpha \geq \lambda$ and λ is a cardinal in $Ult(HOD, E \upharpoonright (\xi_\alpha+1))$ such that $(\lambda^+)^{Ult(HOD,E\upharpoonright\xi_\alpha)} < (\lambda^+)^{Ult(HOD,E\upharpoonright(\xi_\alpha+1))}$, then $\xi_\alpha = (\lambda^+)^{Ult(HOD,E\upharpoonright\xi_\alpha)}$.

The lemma leaves open the following question;

Question 0.4. Let E be an extender on HOD with critical point κ. Assume $Ult(HOD, E) \upharpoonright \gamma = HOD \upharpoonright \gamma$ where $\gamma > \eta$ and η is the ηth inaccessible in HOD. Is it possible that η is a generator of E? Is it possible that η is not a generator of E?

Let E be an extender on HOD with critical point κ. , Let γ be a HOD-cardinal greater than κ. Let $(\xi_\alpha)_{\alpha<\gamma}$ enumerate (in increasing order) E's generators. If $\xi_1 = (\kappa^{++})^{Ult(HOD,E\upharpoonright\xi_1)}$ and for $\alpha > 1$ $\xi_\alpha = (|sup_{\beta<\alpha}(\xi_\beta)|^+)^{Ult(HOD,E\upharpoonright\xi_\alpha)}$, then we say that E is γ-saturated.

Fact (i) above implies that if E is γ-saturated, then for all HOD-cardinals $\rho < \gamma$, $(\rho^+)^{Ult(HOD,E)} = (\rho^+)^{HOD}$. This motivates the following question:

Question 0.5. Let E be an extender that is γ-saturated, then is it the case that $Ult(HOD, E)|\gamma = HOD|\gamma$?

If the answer is "yes" then this could be viewed as a generalization of Farmer Schlutzenberg's theorem that all measure in HOD are on the sequence.

In view of Lemma 0.3, the weakest possible embedding hypotheses (after measurability) that a cardinal κ can possess is that there is an extender E on κ such that E has two generators and the second generator of E, say ξ, is $(\kappa^{++})^{Ult(V,E\restriction\xi)}$. We will call such a κ a 2-generator cardinal.

Question 0.6. Let κ be a 2-generator cardinal. Does κ have Mitchell order κ^{++}?