The External Ultrapower of HOD via W_1^1

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(iii) means that $HOD|\Theta$ is a model of the form $L[\vec{E}]$ where $\vec{E}$ is a coherent sequence of extenders.
Steel used the proof of (iii) to show

(iv) If κ is a regular cardinal in HOD such that $cof^{L(\mathbb{R})}(\kappa) > \omega$, then κ is measurable in HOD.

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More generally, how does the large cardinal structure of HOD interact with the structural theory of $L(\mathbb{R})$ under AD?

(ii) There is a class of naturally arising embeddings of HOD called external ultrapowers. It is interesting to consider these maps from the inner model point of view- can they be viewed as iterations of HOD?

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This is how Woodin originally showed that δ^2_1 is strong to Θ and that Θ is Woodin in HOD.

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Definition

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$$\{ [f]_{W_1^1} \mid f \in L(\mathbb{R}) \ \wedge \ f : \omega_1 \to HOD \}$$

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Note that the universe of $Ext(HOD, W_1^1)$ consists of equivalence classes of all such functions in $L(\mathbb{R})$.
Let $j_{W_1^1} : HOD \to Ext(HOD, W_1^1)$ denote the canonical embedding (i.e. $j_{W_1^1}(\alpha) = [c_\alpha]_{W_1^1}$ where c_α is the constant α function).

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We show that, for a large initial segment of the embedding, the answer is "yes".
We use our analysis to answer a question of Jackson-Ketchersid about which ordinals less than ω_ω are coded by functions in HOD.

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We define the iterated ultrapower:
For $\alpha + 1$ a successor ordinal, let
$Ult^{\alpha+1}(HOD, \mu) = Ult(Ult^{\alpha}(HOD, \mu), i_\alpha(\mu))$
for α a limit ordinal let
$Ult^{\alpha}(HOD, \mu) = DirLim_{\beta<\alpha}(Ult^{\beta}(HOD, \mu))$ where
$i_\alpha : HOD \to Ult^{\alpha}(HOD, \mu)$ are the canonical embeddings.

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Theorem

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We will also make frequent use of the following Lemma, which is a generalization of a result of Jackson-Ketchersid.

Lemma

Let M be an iterate of HOD, and let $\kappa \in M$ be s.t. there are no total extenders in M, E, with $crit(E) < \kappa$ and $Lh(E) > \kappa$. Let Γ be a proper class of ordinals. Then every $A \in (P(\kappa))^M$ is definable from parameters in $\kappa \cup \Gamma$.

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First the minimality of HOD implies that M=N.
This implies that, if b is the branch through $\mathcal{T}$ leading to M and c is the branch through $\mathcal{S}$ leading to N, then neither b nor c drop.

$$HOD \xrightarrow{\quad\mathcal{T}\quad} M$$

$$Ext(HOD, W_1^1) \xrightarrow{\mathcal{S}} N$$

The fact that c does not drop implies that, if $Ext(HOD, W_1^1)$ moves at all in the comparison, then the first extender applied to $Ext(HOD, W_1^1)$ has *length* $> \omega_2$

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We exploit these facts repeatedly to show that, in the first $\omega_2 - many$ steps of the comparison, $Ext(HOD, W_1^1)$ doesn't move, while HOD iterates μ.

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As a demonstration of our methods, we will show the proof of $i_\alpha(\mu) = \mu_\alpha$.

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Let Γ be a proper class of ordinals fixed by all the above embeddings and let $\bar{\beta} \in (\Gamma \cup i_\alpha(\omega_1))^{<\omega}$ be s.t. A is definable in $Ult(HOD, E \restriction \xi_\alpha)$ from $\bar{\beta}$.

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Say $A = \tau^{Ult(HOD, E\restriction\xi_\alpha)}(\bar{\beta})$.

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but $i_\mathcal{T}(A) = \tau^M(\bar{\beta}) = \tau^N(\bar{\beta}) = i_\mathcal{S}(j_\alpha(A))$, a contradiction!

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This shows that $\mu_\alpha = i_\alpha(\mu)$.

Before moving on to a discussion of $Ext(HOD, W_1^1)$ above ω_2, we use our analysis thus far to answer a question of Jackson-Ketchersid.

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i.e., given such a function, view f as coding the ordinal $[f]_{W_1^n}$.
Jackson-Ketchersid asked "what ordinals are coded by functions in $f \in HOD$?"

Definition

Let $f : \omega_1^n \to \omega_1$, $f \in HOD$. We say "there is a gap at f" if

$$Sup([g]_{W_1^n} \mid g \in HOD \ \wedge \ [g] < [f]) < [f]_{W_1^n}$$

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Theorem

Let $f \in HOD$, $f : \omega_1^n \to \omega_1$. Then f begins a gap iff $cof^{Ult(HOD,\mu^n)}([f]_{\mu^n}) \in \{i_1(\omega_1), ..., i_{n-1}(\omega_1)\}$

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We will sketch the proof of the representative case n=2.

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F is the extender that comes from iterating $i_1(\mu)$ $\omega_2 - many$ times and there is a gap at f iff k is discontinuous at $[f]_{\mu\times\mu}$

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So f begins a gap iff i_F is discontinuous at $[f]_{\mu\times\mu}$.

It is straightforward to check that i_F is discontinuous at $[f]_{\mu\times\mu}$ iff $cof^{Ult(HOD,\mu\times\mu)}([f]_{\mu\times\mu}) = i_1(\omega_1)$.

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Thus, the next thing that happens in the comparison is that $Ult^{\omega_2}(HOD, \mu)$ iterate μ_2 until the image of κ_2 "lines up" with the second measurable cardinal of $Ext(HOD, W^1_1)$ (an open question is how long this iteration lasts).

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The limiting factor is our inability to prove the following Lemma for such κ

Lemma

Let M be an iterate of HOD, and let $\kappa \in M$ be s.t. there are no total extenders in M, E, with $crit(E) < \kappa$ and $Lh(E) > \kappa$. Let Γ be a proper class of ordinals. Then every $A \in (P(\kappa))^M$ is definable from parameters in $\kappa \cup \Gamma$.

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(iii) What ordinals less than ω_3 are coded by $f : \omega_2 \to \omega_2$, $f \in HOD$?

Thank you!