The External Ultrapower of HOD via W_1^1

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- (iii) means that $HOD|\Theta$ is a model of the form $L[\vec{E}]$ where \vec{E} is a coherent sequence of extenders.

Steel used the proof of (iii) to show

(iv) If κ is a regular cardinal in HOD such that $cof^{L(\mathbb{R})}(\kappa) > \omega$, then κ is measurable in HOD.



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- Besides the results mentioned above, little is known about the large cardinal structure of HOD. In particular- what large cardinal properties does ω_2 have in HOD?. Conjecture: ω_2 is strong to ω_{ω} in HOD and ω_{ω} is the least α s.t. α is Woodin in $L[HOD|\alpha]$.
 - More generally, how does the large cardinal structure of HOD interact with the structural theory of $L(\mathbb{R})$ under AD?
- (ii) There is a class of naturally arising embeddings of HOD called external ultrapowers. It is interesting to consider these maps from the inner model point of view- can they be viewed as iterations of HOD?

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In attempting to study the large cardinal structure of HOD, it is natural to study such an external ultrapower embedding and attempt show that the embedding witnesses some large cardinal property.

This is how Woodin originally showed that δ_1^2 is strong to Θ and that Θ is Woodin in HOD.



Recall that AD implies that ω_1 is measurable in $L(\mathbb{R})$ and let W_1^1 denote the unique normal measure.

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Definition

The external ultrapower of HOD via W_1^1 , $Ext(HOD, W_1^1)$, is the model of set theory with universe

$$\{ [f]_{W_1^1} \mid f \in L(\mathbb{R}) \land f : \omega_1 \to HOD \}$$

and the ϵ -relation defined in the usual ultrapower way.



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Note that the universe of $Ext(HOD, W_1^1)$ consists of equivalence classes of all such functions in $L(\mathbb{R})$.

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Let $j_{W_1^1}: HOD \to Ext(HOD, W_1^1)$ denote the canonical embedding (i.e. $j_{W_1^1}(\alpha) = [c_{\alpha}]_{W_1^1}$ where c_{α} is the constant α function).



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We use our analysis to answer a question of Jackson-Ketchersid about which ordinals less than ω_{ω} are coded by functions in HOD.



Let W_1^n be the n-fold product of W_1^1 .



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Notation

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Below ω2

Let $\mu = W_1^1 \cap HOD$ (μ is the unique normal measure on ω_1 in HOD).

We define the iterated ultrapower:

For $\alpha + 1$ a successor ordinal, let

$$Ult^{\alpha+1}(HOD,\mu) = Ult(Ult^{\alpha}(HOD,\mu),i_{\alpha}(\mu))$$

for α a limit ordinal let

$$Ult^{\alpha}(HOD, \mu) = DirLim_{\beta < \alpha}(Ult^{\beta}(HOD, \mu))$$
 where

 $i_{\alpha}: HOD \rightarrow Ult^{\alpha}(HOD, \mu)$ are the canonical embeddings.

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Theorem

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(Steel) HOD and $Ext(HOD, W_1^1)$ have a successful comparison.

We will also make frequent use of the following Lemma, which is a generalization of a result of Jackson-Ketchersid.

Lemma

Let M be an iterate of HOD, and let $\kappa \in M$ be s.t. there are no total extenders in M, E, with $crit(E) < \kappa$ and $Lh(E) > \kappa$. Let Γ be a proper class of ordinals. Then every $A \in (P(\kappa))^M$ is definable from parameters in $\kappa \cup \Gamma$.

Sketching the proof:

Let

$$HOD \xrightarrow{\mathcal{T}} M$$

$$Ext(HOD, W_1^1) \xrightarrow{S} N$$

be the comparison given by Steel's Theorem.

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We first show that in the first ω_2 -many steps of the comparison, $Ext(HOD, W_1^1)$ doesn't move while HOD iterates μ ω_2 -many times.

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First the minimality of HOD implies that M=N.

This implies that, if b is the branch through \mathcal{T} leading to M and c is the branch through \mathcal{S} leading to N, then neither b nor c drop.



$$HOD \xrightarrow{\mathcal{T}} M$$
$$Ext(HOD, W_1^1) \xrightarrow{\mathcal{S}} N$$

The fact that c does not drop implies that, if $Ext(HOD, W_1^1)$ moves at all in the comparison, then the first extender applied to $Ext(HOD, W_1^1)$ has $length > \omega_2$

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Similarly, the fact that b does not drop implies that when HOD movies in the comparison, then the first extender applied to HOD has $length > \omega_1$.

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Similarly, the fact that b does not drop implies that when HOD movies in the comparison, then the first extender applied to HOD has $length > \omega_1$.

This is because ω_1 is the least measurable cardinal of HOD. We exploit these facts repeatedly to show that, in the first $\omega_2 - many$ steps of the comparison, $Ext(HOD, W_1^1)$ doesn't move, while HOD iterates μ .

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We conclude the proof by showing $\mu_{\alpha}=i_{\alpha}(\mu)$ and

$$\xi_{\alpha}+1=i_{\alpha+1}(\omega_1).$$



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We conclude the proof by showing $\mu_{\alpha} = i_{\alpha}(\mu)$ and

 $\xi_{\alpha} + 1 = i_{\alpha+1}(\omega_1)$.

As a demonstration of our methods, we will show the proof of $i_{\alpha}(\mu) = \mu_{\alpha}$.

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$$Ult^{\alpha}(HOD, \mu) \xrightarrow{\mathcal{T}} M$$

$$\downarrow_{j_{\alpha}} \downarrow$$

$$Ext(HOD, W_1^1) \xrightarrow{\mathcal{S}} N$$

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This means that $i_{\alpha}(\omega_1) \in i_{\mathcal{T}}(A)$ but $i_{\alpha}(\omega_1) \notin j_{\alpha}(A)$.

Let Γ be a proper class of ordinals fixed by all the above embeddings and let $\bar{\beta} \in (\Gamma \cup i_{\alpha}(\omega_{1}))^{<\omega}$ be s.t. A is definable in $Ult(HOD, E \upharpoonright \xi_{\alpha})$ from $\bar{\beta}$.



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Say $A = \tau^{Ult(HOD,E \upharpoonright \xi_{\alpha})}(\bar{\beta})$.



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but $i_{\mathcal{T}}(A) = \tau^{M}(\bar{\beta}) = \tau^{N}(\bar{\beta}) = i_{\mathcal{S}}(j_{\alpha}(A))$, a contradiction!

$$\mu_{\alpha} = i_{\alpha}(\mu)$$

$$Ult(HOD, E \upharpoonright \xi_{\alpha}) \xrightarrow{\mathcal{T}} M$$

$$j_{\alpha} \bigvee_{j_{\alpha}} Ext(HOD, W_{1}^{1}) \xrightarrow{\mathcal{S}} N$$

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This shows that $\mu_{\alpha} = i_{\alpha}(\mu)$.



Before moving on to a discussion of $Ext(HOD, W_1^1)$ above ω_2 , we use our analysis thus far to answer a question of Jackson-Ketchersid.

Ultrapowers via the W_1^n provide a way of coding ordinals less than ω_ω via functions $f:\omega_1^n\to\omega_1$.

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i.e., given such a function, view f as coding the ordinal $[f]_{W_1^n}$. Jackson-Ketchersid asked "what ordinals are coded by functions in $f \in HOD$?"

Definition

Let $f: \omega_1^n \to \omega_1$, $f \in HOD$. We say "there is a gap at f" if

$$Sup([g]_{W_1^n} \mid g \in HOD \ \land \ [g] < [f]) < [f]_{W_1^n}$$

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Jackson-Ketchersid showed that for all $f: \omega_1 \to \omega_1$, $f \in HOD$, there is not a gap at f.

Let $f: \omega_1^n \to \omega_1$, $f \in HOD$. We say "there is a gap at f' if

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i.e. the ordinals less than ω_2 coded by functions in HOD constitute an initial segment of ω_2 .

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i.e. the ordinals less than ω_2 coded by functions in HOD constitute an initial segment of ω_2 .

We complete the analysis by showing the following:

Theorem

Let $f \in HOD$, $f : \omega_1^n \to \omega_1$. Then f begins a gap iff $cof^{Ult(HOD,\mu^n)}([f]_{\mu^n}) \in \{i_1(\omega_1),...,i_{n-1}(\omega_1)\}$



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We will sketch the proof of the representative case n=2.



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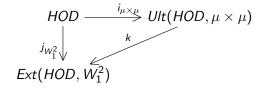
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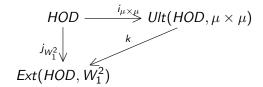


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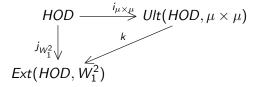
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One can show that F is the $(i_1(\omega_1), \omega_2)$ extender derived from k. Further, $Ult(Ult(HOD, \mu \times \mu), F) = Ult^{\omega_2+1}(HOD, \mu)$



Key Idea

This yields the following picture:

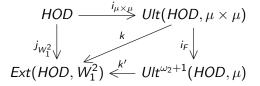
$$HOD \xrightarrow{I_{\mu \times \mu}} Ult(HOD, \mu \times \mu)$$

$$\downarrow_{J_{W_1^2}} \downarrow \qquad \qquad \downarrow_{I_F} \downarrow$$

$$Ext(HOD, W_1^2) \xleftarrow{k'} Ult^{\omega_2 + 1}(HOD, \mu)$$

F is the extender that comes from iterating $i_1(\mu)$ ω_2-many times and there is a gap at f iff k is discontinuous at $[f]_{\mu\times\mu}$

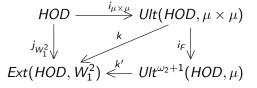
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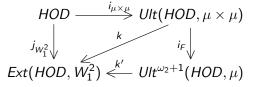
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F is the extender that comes from iterating $i_1(\mu)$ $\omega_2 - many$ times and there is a gap at f iff k is discontinuous at $[f]_{\mu \times \mu}$ One can show that $crit(k') = i_F(i_2(\omega_1))$ So f begins a gap iff i_F is discontinuous at $[f]_{\mu \times \mu}$.

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Below ω2



F is the extender that comes from iterating $i_1(\mu)$ ω_2-many times and there is a gap at f iff k is discontinuous at $[f]_{\mu\times\mu}$ One can show that $crit(k')=i_F(i_2(\omega_1))$

So f begins a gap iff i_F is discontinuous at $[f]_{\mu \times \mu}$. It is straightforward to check that i_F is discontinuous at $[f]_{\mu \times \mu}$ iff $cof^{Ult(HOD,\mu \times \mu)}([f]_{\mu \times \mu}) = i_1(\omega_1)$.



$$HOD \xrightarrow{\mathcal{T}} M$$
$$Ext(HOD, W_1^1) \xrightarrow{\mathcal{S}} N$$

$Ext(HOD, W_1^1)$ above ω_2

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Let κ_2 denote the second measurable cardinal of $Ult^{\omega_2}(HOD, \mu)$ and let μ_2 denote it's normal measure.

ickground Below

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Thus, the next thing that happens in the comparison is that $Ult^{\omega_2}(HOD, \mu)$ iterate μ_2 until the image of κ_2 "lines up" with the second measurable cardinal of $Ext(HOD, W_1^1)$ (an open question is how long this iteration lasts).

Indeed, using various methods, one can show that this process continues for quite a ways.

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The limiting factor is our inability to prove the following Lemma for such κ

Lemma

Let M be an iterate of HOD, and let $\kappa \in M$ be s.t. there are no total extenders in M, E, with $crit(E) < \kappa$ and $Lh(E) > \kappa$. Let Γ be a proper class of ordinals. Then every $A \in (P(\kappa))^M$ is definable from parameters in $\kappa \cup \Gamma$.

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Some open questions:

- (i) What is a full analysis of $j_{W_1^1}$?
- (ii) Let S_1^1 be the measure on ω_2 coming from W_1^1 and the strong partition property on ω_1 . Do HOD and $Ext(HOD, S_1^1)$ have a successful comparison? what is $j_{S_1^1}: HOD \to Ext(HOD, S_1^1)$?

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- (iii) What ordinals less than ω_3 are coded by $f:\omega_2\to\omega_2$, $f\in HOD$?

Thank you!

