The Sound of a Guitar

Purpose: The purpose of this project is to demonstrate an application of integrals in physics and show you how Mathematica can help you solve problems involving calculus.
Directions: Mick and Tina are musicians. One day while discussing the theory of the guitar Mick said that he understood that when a string vibrates it has certain normal modes which give the guitar its harmonics.

Tina agreed and explained to Mick how to understand the normal modes. She said, "All the normal modes correspond to sin curves." She then drew the diagram labeled Figure 1 and went on to explain "These pictures give the shape of the string when the string is vibrating in its first few normal modes. In the first picture, which shows the first normal mode, the string vibrates back and forth so that when it is up as far as it will go it has the indicated shape. The next picture gives the second normal mode, and so on. These normal modes give the harmonics of the guitar."

Mick asks, "Those pictures are nice but how do you say more precisely what the shapes are?"

"Oh, that is an interesting question," said Tina. "In fact, if we pick one end of the string as the origin and choose our units so the string has length π then the equations for the curves are simply $\sin(x)$, $\sin(2x)$, $\sin(3x)$, and so on which give the first, second, third, and so on harmonics."

"But when I pluck the guitar don't I just pull the string out so it looks like a V and then release it?" asked Mick as he pointed to Figure 2. "It really doesn't look like any of the pictures you drew, does it? Which of the harmonics do I hear then?" asked Mick.

Tina thought for a moment and then replied "Well, I am not exactly sure, but I think you get a combination of all of them, although I really do not know how to compute the relative magnitudes of the harmonics you would hear. It has been so long since I studied calculus!"

"Yeah, I never realized how important calculus would be to my career as a musician when I studied mathematics in college," said Mick.

Tina then remembered that you are currently taking calculus. She therefore turns to you for a solution. Do the following steps to answer the question. You will probably wish to use Mathematica starting on Step 5. Do not use Mathematica on Steps 1-4.

1. Let $f(x)$ be a continuous function on the interval $[-a, a]$. Show that if f is an odd function then $\int_{-a}^{a} f(x)dx = 0$. (Hint: Look at a picture and use the property of integrals which says $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$.)
2. Suppose that n and m are non-negative integers. Show that $\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0$.
3. Suppose that n and m are non-negative integers. Compute $\int_{-\pi}^{\pi} \sin nx \sin mx dx$ and $\int_{-\pi}^{\pi} \cos nx \cos mx dx$. (Hint : Start with identities like $\cos(nx+mx) = \cdots$ and $\cos(nx-mx) = \cdots$ and do some algebra to rewrite the functions you are to integrate. If you are not sure how to do it, you may try looking at your old precalculus book. It probably gives the formula you are looking for.)

4. Suppose that $f(x) = a_1 \sin(x) + a_2 \sin(2x) + \cdots + a_n \sin(nx) + b_0 + b_1 \cos(x) + b_2 \cos(2x) + \cdots + b_n \cos(nx)$. A function of this form is called a **trigonometric polynomial**. Compute $\int_{-\pi}^{\pi} f(x) \sin(mx) dx$ and $\int_{-\pi}^{\pi} f(x) \cos(mx) dx$ for every nonnegative integer m.
5. Consider the function $f(x) = \sin^3(x)$. By using trigonometric identities it is possible to rewrite $f(x)$ as a trigonometric polynomial. Use what you learned above (and possibly Mathematica) to decide what the coefficients a_i and b_i are. Then use trigonometric identities to verify that the answer you obtained is correct. Do the same with $\sin^5(x) + 2\cos^2(x)$. (In Mathematica the command to integrate a function is Integrate[f[x],{x,xmin,xmax}] in order to compute $\int_{\text{xmin}}^{\text{xmax}} f(x) dx$. You may wish to try a few integrals you know before doing this part just to be sure you are using the command properly.)
6. Consider the function $f(x) = x^3 - \pi^2 x$. Is it possible to approximate $f(x)$ with a trigonometric polynomial? In particular, you are to find a trigonometric polynomial which approximates $f(x)$ to within 0.01 for every value of x between $-\pi$ and π. After figuring out trigonometric polynomials to approximate $f(x) = x^3 - \pi^2 x$ plot the difference between your function and $y = x^3 - \pi^2 x$ to see how close they are together. Next try to approximate $g(x) = x$ just like you did for $f(x)$. Can you get a good approximation this way?
7. Now you are ready to answer Tina and Mick's question. First think of the harmonics that Tina drew to be only half the picture. So think of the string representing only the part of the curve from 0 to π but the whole curve goes from $-\pi$ to π as in Figure 3. Then extend the shape of the string when it is plucked as shown in Figure 4. Use what you learned above to determine the relative sizes of the coefficients of the $\sin nx$ terms in the approximating trigonometric polynomials. You should make reasonable assumptions about the equation of the curve in Figure 4. Tina probably would like to know for each n the square of the ratio of the coefficient corresponding to the n^{th} harmonic over the coefficient corresponding to the first harmonic since the energy is proportional to the square of the amplitude.
8. **Extra credit!** Determine the limit given below. You need not prove your answer, but say why you believe your answer is what you claim. Use what you learned in some of the parts above.

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)}$$

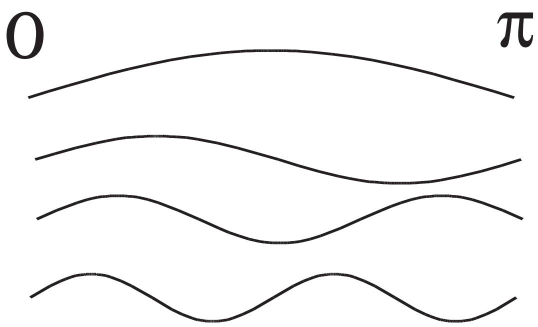

Figure 1.

Figure 2.

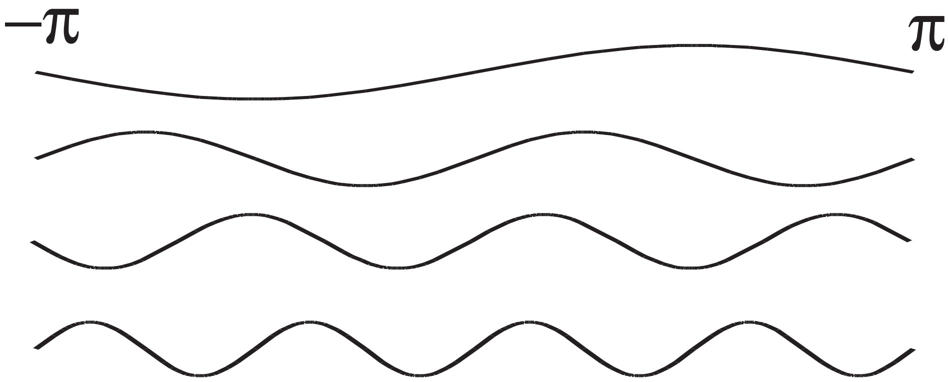

Figure 3.

Figure 4.