Fibonacci Numbers.

You may have heard about the Fibonacci sequence of numbers. It starts with the numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$$

Even if you haven't seen them before you may recognize the pattern. To get each of the numbers after the second one you simply add the two previous numbers. This gives an easy way to write out the Fibonacci sequence as far as you wish. Suppose that you only wanted to know the 100^{th} term of the sequence and you did not necessarily want to know all the numbers that came before it. Is there a formula which would give you this number directly? In this project you will discover the formula. Follow the steps below:

a. Let $a_1, a_2, a_3, \cdots$ represent the Fibonacci sequence. Consider the power series $f(x) = a_1 x + a_2 x^2 + a_3 x^3 + \cdots$. Show that the radius of convergence of this power series is at least $\frac{1}{2}$.
b. Find a polynomial $p(x)$ such that $p(x)f(x) = x$.
c. Find the power series expansion for $\frac{1}{ax+b}$ and determine its radius of convergence.
d. Find the partial fraction decomposition for $\frac{x}{p(x)}$ and use it to determine the coefficients for $f(x)$.
e. What is the radius of convergence for $f(x)$?
f. Use your result from d) to determine a formula for a_k for each value of k. Use a calculator or computer to verify that the formula you derived is correct for small values of k. Then determine the value of a_{100} and a few other values of a_k for k large.

You will be expected to justify every statement you make. For example, it is not good enough to just find a polynomial in part b). You are required to find the polynomial and then convince the skeptical reader that $p(x)f(x) = x$. As always, if you get stuck on a point then see your instructor for a hint.