Trigonometry using Differential Equations

There are many ways that the sine and cosine functions can be defined. In high school you learned two geometric methods, one using the circle and one involving right triangles. Here you are to start with different definitions of the trigonometric functions and then use calculus to derive some of their basic properties.

In class we defined the function ln as the solution to the initial value problem $y' = \frac{1}{x}$ with $y(1) = 0$. In other words, $\ln(x) = \int_1^x \frac{1}{x} dx$. We used this definition to derive basic properties of the function $\ln(x)$. You are to use similar ideas to derive some basic properties of the sine and cosine functions when they are defined as solutions of the differential equation $y'' + y = 0$ with appropriate initial conditions. Keep in mind that for this project you are not to use the usual properties of the sine and cosine functions unless you have already derived them. In order to keep a distinction between the usual definitions of sine and cosine, we will let $s(x)$ and $c(x)$ be solutions to the differential equation $y'' + y = 0$ with initial conditions $s(0) = 0$, $s'(0) = 1$, $c(0) = 1$, and $c'(0) = 0$.

You are to assume that for any numbers a and b, there is exactly one solution to the initial value problem $y'' + y = 0$, $y(0) = a$, and $y'(0) = b$. This is an existence and uniqueness result that you will learn if you take a course in differential equations. Please keep in mind that some initial value problems have more than one solution, while others have no solutions. (An example is the differential equation $x^3 y' = y$. With the initial condition $y(0) = 1$ there is no solution, but with the initial condition y(0)=0 there are infinitely many solutions. In fact, even the initial conditions that $y^{(n)}(0) = 0$ for every integer $n \geq 0$ yields infinitely many solutions!)

Here is what you are to do:

1. Show that $s'(x) = c(x)$ and $c'(x) = -s(x)$ for any x.
2. Show that $c(x)$ is an even function. Then show that the derivative of an even function is an odd function to conclude that $s(x)$ is an odd function.
3. Show that $s^2(x) + c^2(x) = 1$ for any x. (Hint: First show $s^2(x) + c^2(x)$ is constant, then figure out what the constant value is.)
4. Show that $s(x + a) = s(x)c(a) + c(x)s(a)$ and $c(x + a) = c(x)c(a) - s(x)s(a)$ for any values of x and a. (Hint: Think of a as a fixed number and x as a variable.)
5. Draw a picture and talk about the concavity of $s(x)$ and $c(x)$ to indicate why there is some value of $x > 0$ where $s(x) = 0$. Let p be the smallest positive value of x where $s(x) = 0$. Show $c(p) = -1$.
6. Use what you derived in 3) and 4) to show that $s(p/2) = 1$.
7. Show that $s(p/2 - x) = c(x)$.
8. Show that $s(2p + x) = s(x)$ and $c(2p + x) = c(x)$ for every x.