Trigonometry without Geometry

or

Non–Circular Trigonometry

While reviewing trigonometry Ted decides that the definitions for trigonometric functions are circular. It bothers Ted that the definitions are in terms of the arc length of a circle. He prefers definitions of functions which do not depend on geometry. After all, when you define $f(x) = x^2$ you do not rely on geometry, you only multiply x by itself. Ted approaches you for assistance in developing definitions for trigonometric functions which do not depend on the circle. You point out that there are several ways to define the trigonometric functions and suggest the following procedure.

You are to complete the steps below. When you write your report, explain carefully what each of the functions you define should be in terms of the trigonometric functions. You should also point out that formulas you derive are consistent with the usual trigonometric identities. Keep in the spirit that this is an alternate approach to defining the trigonometric functions. You are not to use identities unless you prove them first. Be sure to show your calculations.

1. Define $A(x) = \int_0^x \frac{1}{1+t^2}dt$ for all x. Note that we know (or will know) from class that $A(x) = \arctan(x)$. You are not to use this fact since you are trying to give alternate definitions for the trigonometric functions. Using this definition compute $A(0)$ and $A'(x)$ for every x. What is $A'(0)$?
2. You may assume that $p = \lim_{x\to\infty} A(x)$ exists. With this assumption compute $\lim_{x\to-\infty} A(x)$ in terms of p. Then show that p is between 1 and 2. State the exact value of p without proof.
3. Show that $A(x)$ has an inverse function. Let $T(x)$ be the inverse function for $A(x)$. State what the domain and range are for both $A(x)$ and $T(x)$.
4. Compute the derivative $T'(x)$. (Your answer should be in terms of $T(x)$.) What is $T'(0)$?
5. Now, define $C(x) = \frac{1}{\sqrt{T'(x)}}$. Compute $C(0)$, $C'(x)$ for every x (in terms of $T(x)$), and $C'(0)$.
6. Next define $S(x) = -C'(x)$. Compute $S(0)$, $S'(x)$ in terms of $T(x)$, and $S'(0)$.
7. Find a relation between $S^2(x)$ and $C^2(x)$ and prove it.
8. Find $S'(x)$ in terms of $C(x)$ and prove your answer.
9. How would you extend the domain of the functions $S(x)$ and $C(x)$ to be consistent with the trigonometric functions?
10. For extra credit you may estimate π numerically by approximating $\lim_{x\to\infty} A(x)$. You may either use a computer or calculator to do the estimate. You should also give some indication of how close your estimate is to the correct value and how you know it is as close as you say it is.