The Exponential Function

The purpose of this project is to define the number e and to develop a definition for e^r. We all know that $e^3 = eee$, $e^{-2} = \frac{1}{ee}$, and $e^{3/4} = \sqrt[4]{e^3}$. But what is $e^{\sqrt{2}}$? We will use power series to make a reasonable definition of e^r for any real number r.

We begin by defining the function $E(x)$ using the series

$$E(x) = \Sigma_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Follow the outline below to derive properties of the function $E(x)$ and to to eventually define e^r for any number r. Keep in mind that the steps are given in a particular order for a reason. In other words, if you want a hint on a particular part, look at the parts that came before.

1. Find the interval of convergence for $E(x)$ to establish its domain.

2. Compute $E(0)$. Define $e = E(1)$ and use this definition to show $2 < e < 3$.

3. Compute $E'(x)$. Your final answer should not be a power series.

4. Show that for any real numbers a and x, $E(a+x) = E(a)E(x)$. (Hint: Do the Taylor expansion for $f(x) = E(a+x)$.)

5. Show that $E(-x) = \frac{1}{E(x)}$ for any real number x.

6. Show that the range of $E(x)$ is all positive real numbers. (Note, you need to show that $E(x)$ is positive and that for any positive number, say y, there is an x so that $E(x) = y$. It may be helpful to use the Intermediate value Theorem to do this.

7. Use induction to show that $e^n = E(n)$ for all integers $n \geq 0$.

8. Show that $e^{-n} = E(-n)$ for all positive integers n.

9. Show that $e^{\frac{1}{n}} = E\left(\frac{1}{n}\right)$ for all integers $n \neq 0$.

10. Show that $e^{\frac{p}{q}} = E\left(\frac{p}{q}\right)$ for all positive integers p and q.

11. Based on the last four parts, give a reasonable definition of e^r any real number r.

Extra credit (5 points): Prove that e is irrational. That is, you are to prove that the number e cannot be written in the form $\frac{a}{b}$ where both a and b are integers.