Limits of Polynomials

Tom is taking Calculus 1710. Tom's sister, Mary, took calculus a few years ago and did quite well. Unfortunately for Tom she has forgotten some of the material on limits. Tom claims that if you want to compute the limit of a function then all you have to do is plug in the value for x into the function and you can compute the limit. Mary argues that that does not work for all functions. For example, $\lim_{x\to 0} \frac{x^2+x}{x} = 1$, but you can not plug in $x = 0$ to compute the limit.

Tom then says, "Well, yes. But, after you factor and cancel, then you can just plug in $x = 0$. I'll bet that for any function that you can plug in $x = a$, the limit $\lim_{x\to a} f(x)$ is the same as $f(a)$."

"What about the greatest integer function?" asks Mary. "$\lim_{x\to 0} \lfloor x \rfloor$ does not exist, but $\lfloor 0 \rfloor = 0$."

"You keep bringing in strange functions! I am talking about regular functions that you see every day. I still think that for any 'reasonable' function $\lim_{x\to a} f(x) = f(a)$."

"Oh, do you mean polynomial functions are continuous?" asks Mary.

"Yeah! Aren't they?"

"I really don't remember. But I do remember being surprised to hear my calculus teacher say that most functions are not continuous. So I think that some polynomial functions may not be continuous," Mary replied.

Tom and Mary go to you for help in determining the answer to this question. They realize that to show polynomials are continuous they only need to show that if $f(x)$ is a polynomial then $\lim_{x\to a} f(x) = f(a)$ for every real number a. Below is an outline of how you may prove that polynomials are continuous. You are not to use the properties of limits discussed in class such as $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$, unless you prove them.

There is a fact involving inequalities which is quite useful in proving limits. First prove the following fact:

Fact: If a and b are real numbers then $|a + b| \leq |a| + |b|$. To prove this it may be useful to look at different cases, depending on whether a or b is positive or negative.

The goal of this project is to show that if f is a polynomial then $\lim_{x\to a} f(x) = f(a)$ for every real number a. (That is, f is continuous.) Follow the outline below and you should be successful.

(The numbers a and c are constants, while f and g are functions.)

1. Prove: $\lim_{x\to a} x = a$.
2. Prove: $\lim_{x\to a} c = c$ where c is a constant.
3. Prove: If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} cf(x) = cL$ where c is a constant. (You may wish to do a special case where $c = 0$ and then do it for $c \neq 0$.)
4. Prove: If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = K$, then $\lim_{x\to a}(f(x) + g(x)) = L + K$. (Hint: Given an ϵ find a δ so that when x is within δ of a, $f(x)$ is within $\frac{\epsilon}{2}$ of L and $g(x)$ is within $\frac{\epsilon}{2}$ of K.)
5. Prove: If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} xf(x) = aL$.
6. Now use the above facts to prove by induction on the degree of the polynomial that if $f(x)$ is a polynomial, then $\lim_{x\to a} f(x) = f(a)$.