Tippe Top Design

Purpose. Tippe tops are very interesting tops. When given a good spin, they flip over and spin up-side-down. In order for them to work properly, the center of mass must be low in the top. In this project you will determine how deep to drill a hole in a tippe top in order to make the center of mass as low as possible.

The story. John makes tippe tops, but he wishes to improve his design. He thinks that to make a better tippe top, he needs to make the top spherical and drill out a hole to a depth that makes the center of mass as low as possible. He reasons that if he doesn't drill a hole at all, by symmetry the center of mass would be at the center of the sphere. As he drills the hole deeper, he is removing mass above the center of mass and therefore lowering the center of mass. So it seems that the deeper he drills the lower the center of mass. On the other hand, if he drills the hole all the way through the sphere, then symmetry would say that the center of mass is back at the center of the sphere. This seems like a contradiction since as he drills down, he would remove material from above the center of mass and therefore lower the center of mass.

John knows you are taking calculus and turns to you to help him work out a few computations. Assume the radius of the tippe top is 1 unit. (John can scale your results for whatever radius he makes.) He uses different size drill bits and he wants to know how deep to drill for any given size. The top is a sphere which can be modeled as the graph of $x^2 + y^2 = 1$ rotated about the $y-$axis and the drilled hole can be thought of as being drilled from the top of the sphere down. The question is, what $y-$value should one drill to in order to make the center of mass as low as possible?

Procedure. Your first goal is to make a chart with three columns. The first column should be the radius r of the drilled hole, the second should be the $y-$position h where the optimally drilled hole stops, and the third column should be the $y-$coordinate $\bar{y}$ of the center of mass. Let the values in the r column start at 0.05 and increase by 0.05 up to a radius of 0.95. You can use a spreadsheet to make your chart or you can use other software such as Mathematica or Maple. There is a fair amount of computation and if you wish to make some intermediate columns to facilitate the computations, that is fine. Use calculus to determine the formulas for h and the center of mass and let a software package compute the values for you.

Look at the chart and see if there is a radius where the chart would indicate that the optimal depth hole would break through the bottom of the sphere and therefore not work! If there is, use the chart to estimate the maximum radius where the optimal depth hole does not break through the bottom.

Finally, look at the chart and see if there is a simple relation between the center of mass and the optimal depth of the hole (at least for the hole radii and optimal depths that work). Prove the relation you found and say why it makes sense intuitively.