s-m-n theorem and AD

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We first recall the statements of the *smn* theorem and of the recursion theorem:

Theorem 0.1 (smn-theorem, recursion theorem, Kleene). *Let $\underset{\sim}{\Gamma}$ be a pointclass with a universal set. Then there are universal sets $U_{\mathcal{X}} \subseteq \mathbb{R} \times \mathcal{X}$, for all perfect product spaces $\mathcal{X}$ with the following properties:*

1. *(smn-theorem)*

   For every $\mathcal{X} = X_1 \times ... \times X_n$ and $\mathcal{Y} = X_1 \times ... \times X_n \times ... \times X_m$, where $m > n$, there is a continuous function $s_{\mathcal{Y},\mathcal{X}} : \mathbb{R} \times \mathcal{X} \to \mathbb{R}$ such that

   $$U_{\mathcal{Y}}(y, x_1, ..., x_n, .., x_m) \longleftrightarrow U_{\mathcal{X}'}(s_{\mathcal{Y},\mathcal{X}}(y, x_1, .., x_n), x_{n+1}, .., x_m),$$

   where $\mathcal{X}' = X_{n+1} \times ... \times X_m$

2. *(Recursion theorem) For every perfect product space $\mathcal{X} = X_1 \times ... \times X_n$ and $\underset{\sim}{\Gamma}$ set $A \subseteq \mathbb{R} \times \mathcal{X}$, there is a $y^* \in \mathbb{R}$ such that for all $\vec{x} \in \mathcal{X}$,*

   $$U_{\mathcal{X}}(y^*, \vec{x}) \longleftrightarrow A(y^*, \vec{x})$$

We thank Jackson for suggesting to prove the following.

Theorem 0.2 (uniform smn-theorem). *Assume AD. Let $\underset{\sim}{\Gamma}$ be a nonselfdual pointclass. Then there is a continuous smn-function s such that $s : \mathbb{R} \times \mathcal{X} \to \mathbb{R}$ such that for any universal sets $U_{\mathcal{X}} \subseteq \mathbb{R} \times \mathcal{X}$ we have :*

$$U_{\mathcal{Y}}(y, x_1, ..., x_m) \leftrightarrow U_{\mathcal{X}'}(s(y, x_1, ..., x_n), x_{n+1}, ..., x_m)$$

where $\mathcal{Y} = \mathcal{X} \times \mathcal{X}'$ and $\mathcal{X}' = X_{n+1} \times ... \times X_m$

Proof. The proof will use Wadge's Lemma. For any perfect product space $\mathcal{X}$, let $U_{\mathcal{X}} \subseteq \mathbb{R} \times \mathcal{X}$ be univesal sets. We need to define the *smn* function between $U_{\mathcal{Y}}$ and $U_{\mathcal{X}'}$. Since $U_{\mathcal{Y}} \subseteq \mathbb{R} \times \mathcal{Y}$ and $U_{\mathcal{X}'} \subseteq \mathbb{R} \times \mathcal{X}'$ are universal sets they will give $\underset{\sim}{\Gamma}$ subsets of $\mathcal{Y}$ and $\mathcal{X}'$ respectively. So let

$$U_{\mathcal{Y}}(y, x_1, ..., x_m) \leftrightarrow A(x_1, ..., x_m)$$

and

$$U_{\mathcal{X}'}(s, x_{n+1}, ..., x_m) \leftrightarrow B(x_{n+1}, ..., x_m).$$

Using recursive substitution we have

$$U_{\mathcal{Y}}(y, x_1, ..., x_m) \leftrightarrow A'(y_0, \langle y_1, x_1, ..., x_m \rangle)$$

and

$$U_{\mathcal{X}'}(s, x_{n+1}, ..., x_m) \leftrightarrow B'(s_0, \langle s_1, x_{n+1}, ..., x_m\rangle).$$

Suppose B' is $\underset{\sim}{\Gamma}$-complete. We now define some sets A'' and B''. B'' will be Γ-complete. Let $\tau_x : \mathbb{R} \to \mathbb{R}$ be a lipschitz continuous function such that $\tau_x(s) = x(\langle s\rangle)$. Let

$$A'(y_0, \langle y_1, x_1, ..., x_m\rangle) \leftrightarrow \tau_{(y)_0}(\langle y_1, x_1, ..., x_m\rangle) \in A''$$

and

$$B'(s_0, \langle s_1, x_{n+1}, ..., x_m\rangle) \leftrightarrow \tau_{(s)_0}(\langle s_1, x_{n+1}, ..., x_m\rangle) \in B''.$$

Notice that A'' and B'' are sets of reals and they are in $\underset{\sim}{\Gamma}$. By Wadge's Lemma we then can transform A'' into B'' so that B'' involves $y_0, y_1, x_1, ..., x_m$. Recall that

$$A \leq_W B \leftrightarrow (A(x) \leftrightarrow B(\rho(x)))$$

for some ρ which can be taken to be a winning strategy for II in the Wadge game, so consider ρ to be Lipschitz continuous. By Wadge's Lemma, we have that

$$A''(\tau_{(y)_0}(\langle y_1, x_1, ..., x_m\rangle)) \leftrightarrow B''(\rho(\tau_{(y)_0}(\langle y_1, x_1, ..., x_m\rangle)$$

This hold because B'' is $\underset{\sim}{\Gamma}$ complete so we can't have B'' Wadge reducible to A^c, this would allow computing B'' in $\check{\underset{\sim}{\Gamma}}$). This means we need to solve the following equation:

$$B''(\rho(\tau_{(y)_0}(\langle y_1, x_1, ..., x_m\rangle))) \leftrightarrow B''(\tau_{(s)_0}(\langle s_1, x_{n+1}, ..., x_m\rangle)).$$

So we just let as usual $s_{\mathcal{Y},\mathcal{X}}(y_1, x_1, ..., x_n) = \langle \epsilon, \langle y, x_1, ..., x_n\rangle\rangle$, where ϵ is such that

$$B''(\tau_\epsilon(\langle\langle y, x_1, ..., x_n\rangle, x_{n+1}, ..., x_m\rangle)) \leftrightarrow B''(\rho(\tau_{(y)_0}(\langle y_1, x_1, ..., x_m\rangle))).$$

That is we choose ϵ such that

$$B''(\tau_\epsilon(z) \leftrightarrow B''(\rho(\tau_{z_{0,0,0}}(\langle z_{0,0,1}, z_{0,1}, ..., z_{0,n}, z_1, ..., z_{m-n}\rangle))).$$

This can be done by recursive substitution. So we have defined $s_{\mathcal{Y},\mathcal{X}}(y_1, x_1, ..., x_n)$ for all Lipschitz continuous functions τ and ρ given by Wadge's Lemma and for all $U_{\mathcal{X}} \subseteq \mathbb{R} \times \mathcal{X}$ good universal Γ sets for Γ subsets of $\mathcal{X}$.

□