

PSEUDODIFFERENTIAL JACOBI FORMS AND GEOMETRIC RANKIN-COHEN BRACKETS

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ABSTRACT. Cohen, Manin, and Zagier recovered the Rankin-Cohen bracket for modular forms from an action of the modular group on pseudodifferential operators whose coefficients are holomorphic functions on the Poincaré upper half plane. We investigate pseudodifferential operators on the Jacobi upper half space with respect to the elliptic variable instead of the modular variable typically considered. We introduce a family of actions of the Jacobi group and show that a space of invariant pseudodifferential operators is isomorphic to the space of Jacobi forms by producing an equivariant map. Our construction arises from the explicit action of a Casimir operator for the complexified Lie algebra of the real Jacobi Lie group. As an application, we identify new families of Rankin-Cohen brackets with geometric origin indexed by a complex parameter. In particular, we isolate a subvariety of lines of Rankin-Cohen brackets in each degree of expected dimension 1 reflecting the geometry of the Jacobi upper half space.

CONTENTS

1. Introduction	1
2. Jacobi plane and groups	5
3. Pseudodifferential operators in the elliptic variable	6
4. Slash action on pseudodifferential operators	7
5. An equivariant splitting of filtered modules	9
6. Raising and lowering operators and a Casimir operator	11
7. Equivariant splitting for pseudodifferential operators	14
8. Applications to Jacobi forms	17
9. Rankin-Cohen brackets for Jacobi forms	20
Appendix A. Jacobi Lie algebra and finding the Casimir operator	23
Appendix B. Noncommutative multiplication of pseudodifferential operators	26
References	27

1. INTRODUCTION

In 1997, Cohen, Manin, and Zagier [16] connected Rankin-Cohen brackets to the noncommutative multiplication of certain pseudodifferential operators. Various authors have considered pseudodifferential operators in special cases with the intent of connecting elliptic modular forms to the deformation quantization of Poisson manifolds advertised by Kontsevich [30]. Consider Bieliavsky, Tang, and Yao [4], Dumas and Royer [23], and Ovsienko [33], for example. One asks if this approach may yield canonical families of Rankin-Cohen brackets reflecting inherent geometry for other modular and automorphic forms.

To address this question, we turn to an approach based on equivariant operators on graded noncommutative algebras advertised by Beilinson to Cohen, Manin, and Zagier. Specifically, Beilinson suggested using a Casimir operator to link pseudodifferential operators to modular forms, see [16, p. 23]. This idea

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previously has not been pursued with rigor in the literature. In particular, we identify a non-vanishing condition which reflects the reducibility of certain Harish-Chandra modules. This condition is required to obtain an equivariant direct sum decomposition of the space of pseudodifferential operators. In the case of elliptic modular forms, this may be derived via formal combinatorial arguments, see [16]. In cases where this vanishing condition fails, indecomposable constituents akin to the quasi-modular form E_2 can appear, explaining in part the focus of Dumas and Royer [23] on quasi-modular forms as opposed to modular forms. The resulting decomposition result applies to other settings beyond elliptic modular forms in which formal combinatorial proofs seem prohibitively intricate. We examine in detail the case of Jacobi forms for its historical prominence, importance in combinatorial number theory, and a subtle non-uniqueness in its theory of Rankin-Cohen brackets. While we focus on Jacobi forms, our theory is conducive to studying pseudodifferential operators or Taylor expansions (see Choie and Lee [13, 14]) in a general setting. This approach should allow for an extension of results here to weakly holomorphic forms, quasi-modular forms, and forms with singularities at torsion points, for example.

A natural question arises for Jacobi forms: What notion of Rankin-Cohen bracket reflects key properties? For elliptic modular forms, the only ambiguity in defining Rankin-Cohen brackets arises from rescaling. In contrast, for Jacobi forms, Böcherer [6] showed that the space of (all possible) ν -th Rankin-Cohen brackets grows in dimension as ν grows (see Subsection 9.1). Choie [9] described a 1-dimensional subspace for each ν given by polynomials in the heat operator (see [6]). Choie and Eholzer [12] gave a basis for the space of all Rankin-Cohen brackets by further leveraging the heat operator, which via the theta decomposition reflects the action on elliptic modular forms. Nevertheless, it is not immediately clear how to define privileged families of brackets reflecting intrinsic traits of Jacobi forms.

Choie, Dumas, Martin, and Royer [10, 11] found families of Rankin-Cohen brackets for *weak* Jacobi forms using the theory of Connes and Moscovici [19], which leverages Hopf symmetries. Weak Jacobi forms lend themselves to an extension of methods used for elliptic modular forms via their Taylor expansions. Note that Dumas and Martin [22] considered pseudodifferential operators on $\mathbb{H}$ for examining Rankin-Cohen brackets of modular forms using Jacobi-like forms, which exhibit invariance only under $SL_2(\mathbb{Z})$ as opposed to the full Jacobi group. Also see Choie and Lee [13] and [14].

We define here families of Rankin-Cohen brackets for Jacobi forms respecting the geometry of the Jacobi upper half space $\mathbb{H}_J$. We leverage the noncommutative multiplication of pseudodifferential operators to identify 1-complex-parameter's worth of families of Rankin-Cohen brackets of geometric origin. This gives in each degree ν a subvariety of lines of expected dimension 1 in the space of all possible Rankin-Cohen brackets as described by Böcherer [6]. Our brackets also capture the inherent growth condition of Jacobi forms. The difference between weak and non-weak Jacobi forms is reflected in the support of the Fourier expansion. Methods from elliptic modular forms prove fruitful in the weak case, but for some applications in combinatorial number theory, this growth condition is crucial. Our brackets may serve as a useful tool in such settings.

1.1. Cohen-Manin-Zagier. Cohen, Manin, and Zagier [16] considered the natural noncommutative multiplication on a space of pseudodifferential operators induced from the Leibniz rule, see also Zagier [41]. Specifically, they considered the space

$$\Psi DO(\partial_\tau, C^\infty(\mathbb{H})) \quad \text{of pseudodifferential operators in } \partial_\tau := \partial/\partial\tau$$

whose coefficients are smooth functions $\mathbb{H} \rightarrow \mathbb{C}$ on the Poincaré upper half plane $\mathbb{H}$ in the variable τ . Their work hinged on an *equivariant mapping*

$$(1.2) \quad C^\infty(\mathbb{H}) \xleftarrow{SL_2(\mathbb{R})\text{-equivariant}} \Psi DO(\partial_\tau, C^\infty(\mathbb{H})).$$

They carried the noncommutative multiplication of pseudodifferential operators over to $C^\infty(\mathbb{H})$ to define two parameters' worth of (associative) noncommutative multiplications on $C^\infty(\mathbb{H})$ and thus on modular forms. They then showed how to recover the Rankin-Cohen bracket for modular forms from such multiplications and gave combinatorial conversion formulas of independent interest. Their work describes the Rankin-Cohen bracket as a natural bracket arising ultimately from the Leibniz rule.

1.3. Equivariant algebra. For other automorphic forms, analogs of the equivariant map of (1.2) prove challenging to identify. We establish an elementary algebraic result to find such equivariant mappings in terms of the graded module $\text{gr } M$ associated to any filtered module M . We use a subscript i to indicate the i -th filtered or graded component and work over a field $\mathbb{F}$.

Proposition 5.6 *Suppose M is a $\mathbb{Z}$ -filtered $\mathbb{F}[G][x]$ -module for a group G with filtration complete over $\mathbb{F}[x]$. Say x acts by a scalar on each graded component of $\text{gr}(M)$ with mutually distinct eigenvalues. For each i in $\mathbb{Z}$, the short exact sequence of $\mathbb{F}[G][x]$ -modules splits via a unique G -equivariant map Υ :*

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \xrightarrow{\Upsilon} (\text{gr } M)_i \longrightarrow 0.$$

1.4. Jacobi group and pseudodifferential operators. We apply this proposition to the case when G is the Jacobi group, x is a Casimir operator, and M is a space of pseudodifferential operators on the Jacobi upper half plane $\mathbb{H}_J = \{(\tau, z) : \tau \in \mathbb{H}, z \in \mathbb{C}\}$. In contrast to previous work, we take operators in ∂_z instead of ∂_τ to directly incorporate the geometry of $\mathbb{H}_J$ instead of relying on properties of modular forms; see [Subsection 2.2](#), [Subsection 3.1](#), and [Definition 6.7](#). Here, M_i is the space of operators with highest power i on ∂_z . These carry slash actions that are not merely parametrized by a weight k and a Jacobi index m , but by a weight k and a decomposition of m as $m_l + m_r$ for left and right Jacobi indices m_l and m_r in $\mathbb{C}$, see [Subsection 4.1](#). We obtain a G -equivariant map (see [Theorem 7.8](#))

$$(1.5) \quad \Upsilon : C^\infty(\mathbb{H}_J) \xrightarrow{G\text{-equivariant}} \Psi\text{DO}(\partial_z, C^\infty(\mathbb{H}_J))$$

that plays the role of the map in (1.2) in the theory of Cohen-Manin-Zagier using [Proposition 5.6](#) and a Casimir operator (see [Subsection 1.7](#) below and [Corollary 6.12](#)) with distinct eigenvalues on the filtered components M_i for weights $k \geq 2$. Here, Υ splits the projection map $\Psi\text{DO}(\partial_z, C^\infty(\mathbb{H}_J)) \rightarrow C^\infty(\mathbb{H}_J)$ onto the top coefficient.

1.6. Jacobi forms. We define the space $\text{J}\Psi_{(m_l, m_r)}$ of *Jacobi pseudodifferential operators* by taking the pseudodifferential operators invariant under the slash action of the Jacobi group Γ with a natural growth condition reflecting that for the space of Jacobi forms $J_{k, m}$, see [Subsection 8.3](#). We take invariants on both sides of (1.5) and impose growth conditions to obtain a splitting map for Jacobi forms, see [Corollary 8.9](#):

$$\Upsilon : J_{k, m} \longrightarrow \Psi\text{J}_{(m_l, m_r)}.$$

We combine these splitting maps for $k \geq 2$ to decompose the space of pseudodifferential Jacobi forms:

Corollary 8.14 *There is a $\mathbb{C}$ -linear isomorphism from the formal direct product over k of all Jacobi forms $J_{k, m}$ of weight $k \geq 2$ and index $m > 0$ to the space of Jacobi pseudodifferential operators $\text{J}\Psi_{(m_l, m_r)}$ with $m = m_l + m_r$ with highest power -2 on $\partial/\partial z$:*

$$\prod_{k \geq 2} J_{k, m} \cong \left(\Psi\text{J}_{(m_l, m_r)} \right)_{-2}.$$

This corollary then gives a 1-complex-parameter family of Rankin-Cohen brackets for Jacobi forms in the sense of Böcherer [6] as the pull-back of the noncommutative multiplication of pseudodifferential operators under the map Υ (see [Subsection 9.1](#) and [Theorem 9.2](#)):

Corollary. *For any parameter $\mathbf{t} \in \mathbb{C}$ and $v \geq 0$, there is a $\mathbb{C}$ -linear Rankin-Cohen bracket*

$$[\ , \]_{(v)}^{\mathbf{t}} : J_{k, m} \otimes J_{k', m'} \longrightarrow J_{k+k'+v, m+m'}$$

for any weights $k, k' \geq 2$ and indices $m, m' > 0$ leveraging the noncommutative multiplication of pseudodifferential operators.

See [Subsection 9.8](#) for the first few brackets given explicitly in terms of partials $\partial/\partial z$ and $\partial/\partial \tau$.

1.7. Casimir operators. A key step in Cohen, Manin, and Zagier [16] uses direct calculation to construct the equivariant map (1.2) explicitly. The analogous calculation appears more complicated for other automorphic forms. Cohen, Manin, and Zagier explain an alternate approach (see [16, Section 2]) suggested by Beilinson using a *Casimir element* and the fact that the relevant Lie group is *reductive*. This idea trades direct computation of the equivariant map (1.2) for the potentially easier computation of a Casimir operator. Two problems arise in trying to deploy this idea for other automorphic forms:

1. The relevant Lie group may be *non-reductive*. Existence of an analogous equivariant map must follow from other facts.
2. We may ultimately trade one calculation for another just as complicated: explicit determination of the action of the Casimir operator. (In [16], the Casimir operator had a simple formula.)

We show how to exploit Beilinson's idea even when the relevant Lie group is not reductive and/or direct computation of the Casimir operator bears little enlightenment. We construct a unique equivariant map like in (1.2) for functions on the Jacobi upper half space $\mathbb{H}_J$ by constructing a Casimir operator directly as a product of *raising and lowering operators*. Conley and Raum [18] and also Bringmann, Conley, and Richter [7] use Helgason's theories in differential geometry (see [26]) to analyze operators covariant under the action of a nonreductive Lie algebra. We adapt this approach to the setting of infinite-dimensional vector bundles. Note that we construct the Casimir operator as composition of operators of degree 1 instead of using Helgason's theory [26, page 267, Theorem 10] for universal enveloping algebras.

1.8. Why Jacobi forms? We concentrate on the setting of Jacobi forms and functions $\mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$ of a modular variable τ in $\mathbb{H}$ and elliptic variable z in $\mathbb{C}$ with focus on z for several reasons.

First, geometric constructions suggest considering pseudodifferential operators in ∂_z instead of ∂_τ . A decomposition of the universal enveloping algebra for the corresponding Jacobi Lie group (see [18]) gives

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}[Z]} \mathbb{C}(Z) \supset \mathcal{U}(\mathfrak{sl}_2(\mathbb{R})) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{h}_2),$$

where $\mathfrak{h}_2$ is the complexified Heisenberg Lie algebra with center generated by Z . We view this as giving two different kinds of pseudodifferential operators: the first in ∂_τ arising from $\mathcal{U}(\mathfrak{sl}_2(\mathbb{R}))$ and the second in ∂_z from $\mathcal{U}(\mathfrak{h}_2)$. The first kind in ∂_τ may be related to modular forms via the heat operator, see [9, 12]. Historically, phenomena attached to $\mathcal{U}(\mathfrak{h}_2)$ suggest that the second kind in ∂_z yields interesting applications. Indeed, modular completions of indefinite theta functions may be viewed through the lens of representations of $\mathcal{U}(\mathfrak{h}_2)$ (see [8, 37]). Indefinite theta functions and their restrictions to modular forms capture combinatorial data in their coefficients; for example, they are used to determine combinatorial congruences, asymptotics, and identities, see Ono [32]. They also lie at the foundation of modern understandings of mock theta series, see [36, 43].

Second, our focus on the copy of $\mathcal{U}(\mathfrak{h}_2)$ requires a new action on pseudodifferential operators (see [Corollary 4.8](#)), in contrast to a mere extension of the action in Cohen, Manin, and Zagier [16] (see Lee [31]).

Third, Jacobi forms often help establish new arguments for other automorphic forms. For example, see [5], [15], [20], [25], [28], and [42].

Thus by exploring pseudodifferential operators in the elliptic variable ∂_z instead of the modular variable ∂_τ , we aim to establish new techniques that go beyond extending tools for elliptic modular forms. We hope this may open doors to new avenues of research.

1.9. Outline. In [Section 2](#), we recall the action of the Jacobi group Γ and (extended) real Jacobi Lie group G on the Jacobi upper half plane $\mathbb{H}_J := \mathbb{H} \times \mathbb{C}$. We introduce pseudodifferential operators $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ in the elliptic variable z in $\mathbb{C}$ in [Section 3](#) and define an action of the Jacobi group in [Section 4](#) preserving the noncommutative multiplication of pseudodifferential operators, see [Proposition 4.4](#). A closed form for this action appears in [Corollary 4.8](#).

We turn to filtered modules in [Section 5](#) and give a general algebraic condition for G -equivariant splittings in terms of distinct eigenvalues, see [Proposition 5.6](#). In [Section 6](#), we construct a Casimir operator on

the space of pseudodifferential operators as a product of covariant raising and lowering operators (see [Definition 6.3](#)) with closed form given in [Proposition 6.10](#) and [Corollary 6.12](#). We show that this Casimir operator is equivariant (see [Theorem 6.8](#)) with respect to the slash action. Note that in Appendix A, we explain how these new raising, lowering, and Casimir operators were found through an analysis of operators covariant under the action of the complex Lie algebra $\mathfrak{g}$ attached to the Jacobi group G .

In [Section 7](#), we use the Casimir operator to show existence of an equivariant mapping as in (1.5) (see [Theorem 7.8](#) and [Corollary 8.8](#)) and give an explicit closed form. We give applications in [Section 8](#) to Jacobi forms. We define Jacobi pseudodifferential operators and give an isomorphism to the product space of Jacobi forms in [Corollary 8.13](#). This gives rise to new families of Rankin-Cohen brackets for Jacobi forms in [Section 9](#) using the pull-back of the noncommutative multiplication on pseudodifferential operators. We also include an iterative approach for computing Rankin-Cohen brackets.

1.10. Notation. For a group G and field $\mathbb{F}$, we say an $\mathbb{F}$ -vector space M is a G -module when M is an $\mathbb{F}[G]$ -module, and we say any function $f : M \rightarrow M'$ of G -modules is G -equivariant when f is an $\mathbb{F}[G]$ -module homomorphism. All slash actions are right group actions. We assume all rings are unital, all algebras are associative $\mathbb{C}$ -algebras, and tensor products are taken over $\mathbb{C}$ unless otherwise indicated: $\otimes = \otimes_{\mathbb{C}}$.

2. JACOBI PLANE AND GROUPS

In this section, we recall the setting of Jacobi forms; see the reference texts by Eichler and Zagier [\[24\]](#) and Berndt and Schmidt [\[3\]](#).

2.1. The Jacobi upper half plane. The *Jacobi upper half plane* (of cogenus 1) is the space $\mathbb{H}_J := \mathbb{H} \times \mathbb{C}$, for $\mathbb{H} := \{\tau \in \mathbb{C} : \text{Im}(\tau) > 0\}$ the usual Poincaré upper half plane. For (τ, z) in $\mathbb{H}_J$, we call

$$\begin{aligned} \tau &\in \mathbb{H} \text{ the } \textit{modular variable}, \text{ and} \\ z &\in \mathbb{C} \text{ the } \textit{elliptic variable}. \end{aligned}$$

We write $C^\infty(\mathbb{H}_J)$ for the smooth functions, i.e., $\phi : \mathbb{H}_J \rightarrow \mathbb{C}$ which are smooth as functions of four real variables.

2.2. The Jacobi groups Γ and G . Recall the associated Jacobi groups:

$$\begin{aligned} \text{the (full) Jacobi group is} \quad \Gamma &:= \text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2; \\ \text{the (centrally extended) real Jacobi group is} \quad G &:= (\text{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2) \times \mathbb{R}. \end{aligned}$$

Here, the semidirect product is defined via the right action of SL_2 on $\mathbb{Z}^2$ or $\mathbb{R}^2$ (written as row vectors),

$$(M, X)(M', X') = (MM', XM' + X') \quad \text{for } M, M' \in \text{SL}_2 \quad \text{and } X, X' \text{ in } \mathbb{Z}^2 \text{ or } \mathbb{R}^2,$$

and G arises from a nontrivial central extension by the additive group $\mathbb{R} \cong \{(I, 0, \kappa) : \kappa \in \mathbb{R}\}$,

$$0 \longrightarrow \mathbb{R} \hookrightarrow G \longrightarrow \text{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2 \longrightarrow 0,$$

giving the multiplication

$$(M, X, \kappa)(M', X', \kappa') := (MM', XM' + X', \kappa + \kappa' + \det \begin{pmatrix} X & X' \\ X' & X \end{pmatrix}).$$

In fact, $G \cong \text{SL}_2(\mathbb{R}) \ltimes H_3(\mathbb{R})$, where $H_3(\mathbb{R})$ is the real Heisenberg Lie group of upper triangular 3×3 matrices. We identify Γ with the quotient group of $(\text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2) \times \mathbb{Z} \subset G$ by its central subgroup $\mathbb{Z} \cong \{(I, 0, \kappa) : \kappa \in \mathbb{Z}\}$ throughout. This implies the group actions of G defined below (see [Subsection 2.5](#) and [Definition 4.2](#)) factor through Γ if they are trivial on $\mathbb{Z} \subset G$.

2.3. Jacobi cocycles. Fix a *Jacobi index* $m \in \mathbb{Z}$. Let β be the usual scalar cocycle with respect to the left action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ and let α_m be the usual *Jacobi cocycle*:

$$\begin{aligned} \beta : G \times \mathbb{H}_J &\rightarrow \mathbb{C}, & \beta(g, (\tau, z)) &= (c\tau + d)^{-1}, \quad \text{and} \\ \alpha_m : G \times \mathbb{H}_J &\rightarrow \mathbb{C}, & \alpha_m(g, (\tau, z)) &= \exp\left(m 2\pi i \left(\frac{-c(z + \lambda\tau + \mu)^2}{(c\tau + d)} + \lambda^2\tau + 2\lambda z + \lambda\mu + \kappa \right)\right) \end{aligned}$$

for $g = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, (\lambda, \mu), \kappa \right)$ in G . For the function defined by holding a group element g fixed, we simply write

$$\beta(g) : \mathbb{H}_J \rightarrow \mathbb{C} \quad \text{and} \quad \alpha_m(g) : \mathbb{H}_J \rightarrow \mathbb{C}.$$

More generally, for k in $\mathbb{Z}$, we define the *weight k , index m cocycle*

$$(2.4) \quad \alpha_{k,m} : G \times \mathbb{H}_J \rightarrow \mathbb{C} \quad \text{by} \quad \alpha_{k,m} := \beta^k \cdot \alpha_m.$$

Then $\alpha_{k,m} : G \rightarrow C^\infty(\mathbb{H}_J)$ gives a *factor of automorphy*.

2.5. Slash group actions. The real Jacobi group G acts on $\mathbb{H}_J$ by

$$g(\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right) \in \mathbb{H}_J$$

for any $g = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, (\lambda, \mu), \kappa \right) \in G$. (Note that κ plays no role in this action.) The action of G on $\mathbb{H}_J$ extends to a family of actions on complex-valued functions parameterized by both a *weight* $k \in \mathbb{Z}$ and a (*Jacobi*) *index* $m \in \mathbb{C}$: The *slash operator* $|_{k,m}$ of weight k and index m maps a function $\phi : \mathbb{H}_J \rightarrow \mathbb{C}$ and element g in G to the function $(\phi|_{k,m}g) : \mathbb{H}_J \rightarrow \mathbb{C}$ defined by

$$\phi|_{k,m}g := \alpha_{k,m}(g) (\phi \circ g),$$

i.e.,

$$(\phi|_{k,m}g)(\tau, z) := \beta^k(g) \alpha_m(g, (\tau, z)) \phi(g(\tau, z)) \quad \text{for all } (\tau, z) \in \mathbb{H}_J, \quad g \in G.$$

3. PSEUDODIFFERENTIAL OPERATORS IN THE ELLIPTIC VARIABLE

We consider pseudodifferential operators on the Jacobi upper half space $\mathbb{H}_J$. Note that Cohen, Manin, and Zagier [16] consider pseudodifferential operators in the modular variable ∂_τ , while we consider pseudodifferential operators in the elliptic variable ∂_z . Results on modular forms previously have been adapted to larger sets of automorphic forms by focusing on the modular variable τ , but we turn attention to the elliptic variable z instead to capture structures intrinsic to Jacobi forms versus elliptic modular forms. Coefficients for our pseudodifferential operators are either smooth or holomorphic, i.e., taken from either

$$C^\infty(\mathbb{H}_J) = \{\phi : \mathbb{H}_J \rightarrow \mathbb{C} \text{ smooth as a function of 4 real variables}\} \quad \text{or} \quad \mathcal{O}(\mathbb{H}_J) := \{\phi : \mathbb{H}_J \rightarrow \mathbb{C} \text{ holomorphic}\}.$$

3.1. Pseudodifferential operators. Let $R = C^\infty(\mathbb{H}_J)$ or $\mathcal{O}(\mathbb{H}_J)$. We write ∂ for the formal differential operator with respect to the elliptic variable z of $\mathbb{H}_J$. A *pseudodifferential operator* with respect to $\partial := \partial_z$ with coefficients in R is a formal series

$$\sum_{-\infty < n < \infty} \phi_n \partial^n \quad \text{with each } \phi_n \in R.$$

Here, $n < \infty$ indicates that only finitely many terms with positive n are nonzero. The space of all pseudodifferential operators is then

$$\Psi\mathrm{DO}(R, \partial) := \left\{ \sum_{-\infty < n < \infty} \phi_n \partial^n : \phi_n \in R \right\}.$$

3.2. Noncommutative multiplication. The algebra of (ordinary) differential operators (in the elliptic variable z),

$$\text{DO}(R, \partial) := \left\{ \sum_{0 \leq n < \infty} \phi_n \partial^n : \phi_n \in R \right\},$$

is a subspace of $\Psi\text{DO}(R, \partial)$ with multiplication defined by letting $\partial = \partial_z$ act as the partial derivative $\partial/\partial z$,

$$\partial \phi = \phi \partial + \frac{\partial}{\partial z}(\phi) \quad \text{for all } \phi \in R,$$

so that

$$\left(\sum_{0 \leq n < \infty} \phi_n \partial^n \right) \left(\sum_{0 \leq m < \infty} \phi'_m \partial^m \right) = \sum_{n,m} \sum_{0 \leq r \leq n} \binom{n}{r} \phi_n \frac{\partial^r}{\partial z^r}(\phi'_m) \partial^{n+m-r}.$$

For fixed r , the binomial coefficient $\binom{n}{r}$ defines a polynomial in $n \geq 0$ that can be extended to all $n \in \mathbb{Z}$. Hence we may define a multiplication on $\Psi\text{DO}(R, \partial)$ by extending to negative powers on ∂ ,

$$(3.3) \quad \left(\sum_{-\infty < n < \infty} \phi_n \partial^n \right) \left(\sum_{-\infty < m < \infty} \phi'_m \partial^m \right) = \sum_{n,m} \sum_{0 \leq r} \binom{n}{r} \phi_n \frac{\partial^r}{\partial z^r}(\phi'_m) \partial^{n+m-r},$$

which equips this space with a non-commutative (associative) algebra structure (see [16]) with the space of ordinary differential operators as a subalgebra.

3.4. Filtration. We define the *order* (filtered degree) of a nonzero pseudodifferential operator to be the largest n for which the coefficient of ∂^n is nonzero, and we call that nonzero coefficient the *symbol* or *leading coefficient* of the operator. Order then defines a natural ascending $\mathbb{Z}$ -filtration,

$$\Psi\text{DO}(R, \partial) = \bigcup_k \Psi\text{DO}(R, \partial)_k \quad \text{with } \Psi\text{DO}(R, \partial)_k \subset \Psi\text{DO}(R, \partial)_{k+1} \quad \text{for all } k \text{ in } \mathbb{Z}$$

for $\Psi\text{DO}(R, \partial)_k$ the set of pseudodifferential operators of order k or less,

$$\Psi\text{DO}(R, \partial)_k = \left\{ \sum_{n \leq k} \phi_n \partial^n : \phi_n \in R \right\},$$

since $\Psi\text{DO}(R, \partial)_k \cdot \Psi\text{DO}(R, \partial)_{k'} \subset \Psi\text{DO}(R, \partial)_{k+k'}$ for any integers k, k' . When restricting to elements of $\Psi\text{DO}(R, \partial)_k$, we call the coefficient of ∂_z^k the *top coefficient* (which may be zero) to distinguish from the leading coefficient (which is always nonzero).

4. SLASH ACTION ON PSEUDODIFFERENTIAL OPERATORS

We introduce a right action of the extended real Jacobi group G on the space of pseudodifferential operators depending on a pair of (left, right) parameters m_l, m_r in $\mathbb{C}$ which decompose an index m as $m = m_l + m_r$ and a weight k in $\mathbb{Z}$. We define this action by separating a cocycle tracking the action into two parts, one acting on the left and one acting on the right. We use the parameter k corresponding to weight in order to later consider operators which change the weight.

4.1. Slash Action. Every pair of Jacobi cocycles $\alpha, \alpha' : G \rightarrow C^\infty(\mathbb{H}_J)$ as in [Subsection 2.3](#) defines a right action of G on the pseudodifferential operators $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ defined by

$$\phi \partial^n \big|_{(\alpha, \alpha')} g = \alpha(g) (\phi \circ g) \partial_{z \circ g}^n \alpha'(g) \quad \text{for } \phi \in C^\infty(\mathbb{H}_J), g \in G,$$

where $z \circ g$ is shorthand notation for the z -coordinate of $g(z, \tau)$:

$$\partial_{z \circ g} = \beta^{-1}(g) \partial = (c\tau + d) \partial \quad \text{for } g = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu), \kappa \right) \in G, (\tau, z) \in \mathbb{H}_J.$$

We next write $|_{k, (m_l, m_r)}$ for the (right) slash action of G arising from the pair of cocycles $\alpha_{k, m_l} \alpha_{m_r}$.

Definition 4.2. For each $k \in \mathbb{Z}$ and (left-right pair) $m_l, m_r \in \mathbb{C}$, define a slash action of G on $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$,

$$|_{k, (m_l, m_r)} : \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)) \times G \longrightarrow \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)),$$

by

$$\begin{aligned} \phi \partial^n \big|_{k, (m_l, m_r)} g &:= \alpha_{k-n, m_l}(g) (\phi \circ g) \partial^n \alpha_{m_r}(g) \\ &= \beta^{k-n}(g) \alpha_{m_l}(g) (\phi \circ g) \partial^n \alpha_{m_r}(g) \quad \text{for } \phi \in C^\infty(\mathbb{H}_J), g \in G. \end{aligned}$$

Note that we could instead incorporate the additional factor $\beta^{k-n}(g)$ into the second cocycle without changing the action, as $c\tau + d$ and ∂ commute. By restricting $C^\infty(\mathbb{H}_J)$ to $\mathcal{O}(\mathbb{H}_J)$, we obtain a slash action of the real Jacobi group G on $\Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))$: For each $k \in \mathbb{Z}$ and $m_l, m_r \in \mathbb{C}$,

$$|_{k, (m_l, m_r)} : \Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J)) \times G \longrightarrow \Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J)).$$

4.3. Group action on products. The group G does not act as algebra automorphisms on $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ via this slash operator. But if we consider the family of actions over all $m_l, m_r \in \mathbb{C}$, we uncover a useful transformation law which may be checked directly:

Proposition 4.4. *The family of slash actions of G preserves multiplication of pseudodifferential operators: For any ψ, ψ' in $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ and $g \in G$:*

$$(\psi |_{k, (m_1, m_2)} g) \cdot (\psi' |_{k, (-m_2, m_3)} g) = (\psi \cdot \psi') |_{k, (m_1, m_3)} g \quad \text{for all } k \in \mathbb{Z} \text{ and } m_1, m_2, m_3 \in \mathbb{C}.$$

4.5. Slash action in canonical form. In order to express the slash action on pseudodifferential operators explicitly, we require a little lemma using the multinomial coefficient

$$\binom{n}{m_1, \dots, m_\ell} := \frac{\Gamma(n+1)}{\prod_i \Gamma(m_i+1)}.$$

Note that this agrees with the combinatorial definition $\frac{n!}{\prod_i m_i!}$ for $n, m_i > 0$. For $m_1, \dots, m_{\ell-1}, p \geq 0$ and $n \in \mathbb{Z}$,

$$(4.6) \quad \binom{n}{m_1, \dots, m_{\ell-1}, n-p} = \frac{n(n-1) \cdots (n-p+1)}{\prod_i m_i!}.$$

We relegate the proof of the next lemma to [Appendix B](#).

Lemma 4.7. *In the noncommutative ring $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$, for all $c_1, c_2 \in \mathbb{C}$ and $n \in \mathbb{Z}$,*

$$(\partial + c_1 z + c_2)^n = \sum_{0 \leq p} \sum_{\substack{0 \leq t, r, s \\ t+s+2r=p}} \binom{n}{t, s, 2r, n-p} \frac{2^r \Gamma(r + \frac{1}{2})}{\Gamma(\frac{1}{2})} c_1^r c_2^s (c_1 z)^t \partial^{n-p}.$$

The preceding lemma allows us to give an explicit formula expressing the slash action of G on pseudodifferential operators $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ with all ∂ to the far right. Compare with [16, Equation (1.7)]. Note that setting $\lambda = \mu = \kappa = 0$, $z = 0$, and $m_r = m_l = 0$ in the corollary below gives the action of the modular group. We interpret 0^0 as 1 in the next formula.

Corollary 4.8. *For any $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, (\lambda, \mu), \kappa$ in G and $m_l, m_r \in \mathbb{C}$ with $m = m_r + m_l$,*

$$\phi \partial^n |_{k, (m_l, m_r)} g = \sum_{0 \leq p} \sum_{\substack{0 \leq t, s, r \\ t+s+2r=p}} c_g(t, s, r, n) \frac{\alpha_m(g, (\tau, z))}{(c\tau + d)^{r+t+s+k-n}} z^t (\phi \circ g) \partial^{n-p}$$

for $n \in \mathbb{Z}$ and $\phi = \phi(z)$ in $C^\infty(\mathbb{H}_J)$, where

$$c_g(t, s, r, n) := \binom{n}{t, s, 2r, n-p} \frac{2^r \Gamma(r + \frac{1}{2})}{\Gamma(\frac{1}{2})} (4\pi i m_r)^{r+t+s} (-c)^{r+t} (d\lambda - c\mu)^s \in \mathbb{C}.$$

Proof. To express $\phi \partial^n | g = \beta^{k-n}(g) \alpha_{m_l}(g) (\phi \circ g) \partial^n \alpha_{m_r}(g)$ with ∂ to the far right, we determine the commutator of ∂^n and the Jacobi cocycle $\alpha_{m_r} := \alpha_{m_r}(g, (\tau, z))$. Note that

$$\frac{\partial}{\partial z} \alpha_{m_r} = \alpha_{m_r} \cdot (c_1 z + c_2)$$

for some $c_1, c_2 \in \mathbb{C}$ depending on g, τ , and m_r ; in fact,

$$c_1 = 4\pi i m_r \frac{-c}{c\tau + d} \quad \text{and} \quad c_2 = 4\pi i m_r \frac{d\lambda - c\mu}{c\tau + d}.$$

Thus in the algebra $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ (with the noncommutative multiplication of [Eq. \(3.3\)](#)),

$$\partial \cdot \alpha_{m_r} = \alpha_{m_r} \cdot \partial + \frac{\partial}{\partial z} (\alpha_{m_r}) = \alpha_{m_r} \cdot (\partial + c_1 z + c_2)$$

and

$$\partial^n \cdot \alpha_{m_r} = \alpha_{m_r} \cdot (\partial + c_1 z + c_2)^n.$$

Lemma 4.7 gives the advertised formula using the fact that $(c\tau + d)$ and ∂ commute. $\square$

Recall the filtration $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_f)) = \bigcup_k \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_f))_k$ (see [Subsection 3.4](#)) with subscript k indicating the order, i.e., highest power k on ∂ that may appear with nonzero coefficient.

Corollary 4.9. Fix $k \in \mathbb{Z}$ and $m_l, m_r \in \mathbb{C}$ with $m = m_l + m_r$. For any ϕ in $C^\infty(\mathbb{H}_f)$,

$$\phi \partial^{-k} |_{0, (m_l, m_r)} g = (\phi |_{k, m} g) \partial^{-k} \quad \text{modulo} \quad \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_f))_{-k-1}.$$

5. AN EQUIVARIANT SPLITTING OF FILTERED MODULES

Throughout this section, we fix a field $\mathbb{F}$ of characteristic zero and establish a general observation on filtered modules whose graded versions decompose into eigenspaces with distinct eigenvalues for some operator. We take R to be a commutative ring with unity in this section.

5.1. Filtered and associated graded modules. Recall that a module M over the (ungraded) ring R is $\mathbb{Z}$ -filtered when $M = \bigcup_{i \in \mathbb{Z}} M_i$ with each M_i an R -submodule of M with $M_i \subset M_{i+1}$ for all i :

$$\dots \subset M_{i-2} \subset M_{i-1} \subset M_i \subset M_{i+1} \subset \dots$$

The associated graded R -module is

$$\text{gr } M = \bigoplus_{i \in \mathbb{Z}} (\text{gr } M)_i \quad \text{for} \quad (\text{gr } M)_i = M_i / M_{i-1}, \quad i \in \mathbb{Z}$$

with R -action given by $r(m + M_{i-1}) = rm + M_{i-1}$ for r in R and m in M . We write $\pi_i : M_i \rightarrow (\text{gr } M)_i$ for the canonical projection onto the i -th component of the graded module.

5.2. Complete Filtrations. For a $\mathbb{Z}$ -filtered R -module $M = \bigcup_i M_i$, we consider the usual projective limit, $\text{proj } \lim_i M/M_i \subset \prod_i (m + M_i)$, with transition maps $M/M_j \rightarrow M/M_i$, $m + M_j \mapsto m + M_i$, for $M_j \subset M_i$. We say the filtration is *complete* over R if the map

$$(5.3) \quad M \longrightarrow \text{proj } \lim_i (M/M_i), \quad m \longmapsto \prod_i (m + M_i)$$

is an R -module isomorphism (see [\[2\]](#) and [\[39\]](#)). We use the next example in the proof of [Theorem 7.8](#).

Example 5.4. Consider Laurent series in the variable t^{-1} for t a formal symbol:

$$M = R((t^{-1})) = \left\{ \sum_{-\infty < n < \infty} r_n t^n : r_n \in R \right\}.$$

The natural $\mathbb{Z}$ -filtration is complete over R with i -th filtered component the R -module

$$M_i = \left\{ \sum_{-\infty < n \leq i} r_n t^n : r_n \in R \right\} \quad \text{for } i \in \mathbb{Z}$$

(so $M_i \subset M_{i+1}$ for all i). Indeed, $M = \bigcup_i M_i$, and the product of projection maps

$$f : M \longrightarrow \prod_i M/M_i, \quad m \longmapsto \prod_i m + M_i \quad \text{given by} \quad \sum_{n \leq k} r_n t^n \longmapsto \prod_i \left(\sum_{n \leq k} r_n t^n + M_i \right),$$

defines an R -module homomorphism to the projective limit, $f : M \rightarrow \text{proj } \lim_i M/M_i$, which is injective as $\bigcap_i M_i = 0$. In fact, if M is a module over another ring S preserving the filtration, i.e., with each M_i an S -module, then each M/M_i is an S -module and f is also an S -module homomorphism,

$$f(sm) = \prod_i (sm + M_i) = s \prod_i (m + M_i) = sf(m).$$

Hence the filtration on M is complete over S as well. Note that for any fixed N in $\mathbb{Z}$, the induced filtration on $M_N = \bigcup_{i \leq N} M_i$ is also complete (one may set $(M_N)_i = M_i$ for $i \leq N$ and $(M_N)_i = M$ for $i > N$).

In the next two propositions, we use the term “eigenvalues” loosely and include scalars by which x may act on spaces even when those spaces are zero.

Proposition 5.5. *Let $M = \bigcup_i M_i$ be a $\mathbb{Z}$ -filtered $\mathbb{F}[x]$ -module with filtration complete. Suppose x acts by a scalar λ_i in $\mathbb{F}$ on each component $(\text{gr } M)_i$ of the associated graded module with the eigenvalues λ_i for $i \in \mathbb{Z}$ mutually distinct. Then for every i , there is a splitting σ_i to π_i , i.e., an $\mathbb{F}[x]$ -module homomorphism $\sigma_i : (\text{gr } M)_i \rightarrow M_i$ with $\pi_i \circ \sigma_i = \text{Id}$:*

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \xrightarrow[\pi_i]{\sigma_i} (\text{gr } M)_i \longrightarrow 0.$$

Proof. Fix some integer ℓ and without loss of generality assume $\lambda_\ell = 0$ (else replace x by $x - \lambda_\ell$). For each $j \neq \ell$, the composition of the map $(\text{Id} - \lambda_j^{-1}x) : M \rightarrow M$ with projection onto M/M_{j-1} defines an $\mathbb{F}[x]$ -module homomorphism whose kernel contains M_j since x acts on M_j/M_{j-1} by the eigenvalue λ_j (nonzero as the eigenvalues of x are distinct):

$$m \longmapsto m - \lambda_j^{-1}xm + M_{j-1} = m - \lambda_j^{-1}\lambda_j m + M_{j-1} = 0 \quad \text{for } m \in M_j.$$

We obtain an $\mathbb{F}[x]$ -module homomorphism

$$f_j : M/M_j \longrightarrow M/M_{j-1}, \quad m + M_j \longmapsto (\text{Id} - \lambda_j^{-1}x)m + M_{j-1} \quad \text{for each } j \neq \ell.$$

For each $z \in M_\ell/M_{\ell-1}$, define $z_j = 0$ for $j \geq \ell$, $z_{\ell-1} = z$, and

$$z_j = f_{j+1} \cdots f_{\ell-1}(z) \in M/M_j \quad \text{for } j < \ell - 1.$$

We claim $\prod_j z_j$ in $\prod_j M/M_j$ lies in the projective limit $\varprojlim_j M/M_j$. Indeed,

$$xz_j = xf_{j+1} \cdots f_{\ell-1}z = f_{j+1} \cdots f_{\ell-1}xz = f_{j+1} \cdots f_{\ell-1}\lambda_\ell z = 0 \quad \text{for } j < \ell - 1$$

as each f_i is $\mathbb{F}[x]$ -linear, so if $z_j = m + M_j$ in M/M_j for m in M and $j < \ell - 1$, then

$$z_{j-1} = f_j(z_j) = (\text{Id} - \lambda_j^{-1}x)m + M_{j-1}$$

is sent under the transition morphism $M/M_{j-1} \rightarrow M/M_j$ defining the projective limit to

$$(\text{Id} - \lambda_j^{-1}x)m + M_j = z_j - \lambda_j^{-1}xz_j = z_j - 0$$

with $z_{\ell-1}$ sent to $z_\ell = 0$.

We define $\sigma_\ell(z)$ as the preimage of $\prod_j z_j$ under the $\mathbb{F}[x]$ -isomorphism $M \xrightarrow{\cong} \varprojlim_j M/M_j$ of Eq. (5.3), $m \mapsto \prod_j m + M_j$. Then $0 = z_\ell = \sigma_\ell(z) + M_\ell$ so $\sigma_\ell(z)$ lies in M_ℓ and $\pi_\ell \sigma_\ell(z) = \sigma_\ell(z) + M_{\ell-1} = z_{\ell-1} = z$. Thus $\sigma_\ell : M_\ell/M_{\ell-1} \rightarrow M$ is an $\mathbb{F}[x]$ -module homomorphism (as it is given by the composition of $\mathbb{F}[x]$ -module maps) with $\pi_\ell \sigma_\ell$ the identity. $\square$

In the next corollary, we consider the group algebra $\mathbb{F}[G]$ of a group G and take $\mathbb{F}[G][x]$ to be ungraded (i.e., graded with every element of degree 0).

Proposition 5.6. *Suppose M is a $\mathbb{Z}$ -filtered $\mathbb{F}[G][x]$ -module for a group G with filtration complete over $\mathbb{F}[x]$. Say x acts by a scalar λ_i in $\mathbb{F}$ on each $(\text{gr } M)_i$ with the eigenvalues λ_i for $i \in \mathbb{Z}$ mutually distinct. Then for every i , there is a unique $\mathbb{F}[G][x]$ -module splitting σ_i to π_i ,*

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \xrightarrow[\pi_i]{\sigma_i} (\text{gr } M)_i \longrightarrow 0.$$

In particular, σ_i is G -equivariant, i.e., for all g in G , the following diagram is commutative:

$$\begin{array}{ccc} M_i & \xleftarrow{\sigma_i} & \text{gr } (M)_i \\ g \downarrow & & \downarrow g \\ M_i & \xleftarrow{\sigma_i} & \text{gr } (M)_i \end{array}.$$

Proof. Note that the action of G and the action of x both preserve the filtration with actions commuting: $xg = gx$ as maps on each M_i for all g in G . Fix i and take the $\mathbb{F}[x]$ -module splitting $\sigma_i : (\text{gr } M)_i \rightarrow M_i$ of **Proposition 5.5**. To show that σ_i is G -equivariant, we note that the actions of x and g in G commute with the projection map $\pi_i : M_i \rightarrow (\text{gr } M)_i$ with $\pi_i \circ \sigma_i = \text{Id}$, giving a commutative diagram for g in G

$$\begin{array}{ccc} M_i & \xleftarrow{\sigma_i} & \text{gr } (M)_i \\ g \downarrow & & \downarrow g \\ M_i & \xrightarrow{\pi_i} & \text{gr } (M)_i . \end{array}$$

Thus for any z in $(\text{gr } M)_i$, the element $(g\sigma_i - \sigma_i g)(z)$ in M_i projects to zero in $(\text{gr } M)_i$ under π_i . Suppose $(g\sigma_i - \sigma_i g)(z)$ were nonzero. Then it would lie in the filtered component M_ℓ of M for some $\ell < i$ minimal as the filtration is complete (so $\bigcap_i M_i = 0$). Then $\pi_\ell(g\sigma_i - \sigma_i g)(z)$ would be nonzero in $(\text{gr } M)_\ell$. But since the actions of x and g commute and the splitting σ_i is an $\mathbb{F}[x]$ -map, this would imply that

$$x(g\sigma_i - \sigma_i g)(z) = (g\sigma_i - \sigma_i g)(xz) = \lambda_i(g\sigma_i - \sigma_i g)(z)$$

and hence

$$\lambda_i \pi_\ell(g\sigma_i - \sigma_i g)(z) = \pi_\ell(x(g\sigma_i - \sigma_i g))(z) = x \pi_\ell(g\sigma_i - \sigma_i g)(z) = \lambda_\ell \pi_\ell(g\sigma_i - \sigma_i g)(z)$$

which is impossible as $\lambda_\ell \neq \lambda_i$. Thus $(g\sigma_i - \sigma_i g)(z) = 0$ and each σ_i is G -equivariant.

A similar argument verifies that σ_i is unique: If σ'_i is another $\mathbb{F}[G][x]$ -splitting, then $\pi_i(\sigma_i - \sigma'_i)(z) = 0$ for nonzero z in $(\text{gr } M)_i$, and $(\sigma_i - \sigma'_i)(z)$ lies in M_ℓ for some $\ell < i$ with ℓ minimal. Then $x(\sigma_i - \sigma'_i)(z)$ is just $(\sigma_i - \sigma'_i)(xz) = \lambda_i(\sigma_i - \sigma'_i)(z)$ and thus $(\sigma_i - \sigma'_i)(z)$ must vanish as

$$\lambda_i \pi_\ell(\sigma_i - \sigma'_i)(z) = \pi_\ell(x(\sigma_i - \sigma'_i))(z) = x \pi_\ell(\sigma_i - \sigma'_i)(z) = \lambda_\ell \pi_\ell(\sigma_i - \sigma'_i)(z). \quad \square$$

Remark 5.7. **Proposition 5.5** and **Proposition 5.6** both extend to modules with filtration bounded above. Indeed, if the eigenvalues are distinct for $i < N$, then there is a splitting map for $i < N$ as in the conclusion of the two propositions.

6. RAISING AND LOWERING OPERATORS AND A CASIMIR OPERATOR

We will construct a linear map

$$\Upsilon : \mathcal{O}(\mathbb{H}_J) \longrightarrow \Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))$$

explicitly in the next section using the action of a Casimir operator on $\Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))$ defined in this section. We first introduce raising and lowering operators on $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ and then define our Casimir operator as a composition of these operators. In **Appendix A**, we use analysis of the complexified Lie algebra $\mathfrak{g}$ of G to explain our construction of these operators and of the Casimir operator $\mathcal{C}_{k,(m_l,m_r)}$ on $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$.

6.1. A Casimir on $\mathbb{H}_J$. Berndt and Schmidt [3] identified a Casimir operator on $\mathbb{H}_J$ (see also Pitale [35]). We consider the Casimir operator $\mathcal{C}_{k,m} : C^\infty(\mathbb{H}_J) \rightarrow C^\infty(\mathbb{H}_J)$ constructed by Conley and Raum [18, Theorem 2.4] for $k \in \mathbb{Z}$ and $0 \neq m \in \mathbb{C}$ (see **Remark 6.11**) and restrict to holomorphic functions:

$$\text{the Casimir operator } \mathcal{C}_{k,m} : \mathcal{O}(\mathbb{H}_J) \rightarrow \mathcal{O}(\mathbb{H}_J) \quad \text{acts by the scalar } 2\pi i m(k^2 - 3k).$$

We also use the standard heat operator from [24, p. 32]:

$$\text{the heat operator } \mathbb{L}_m : \mathcal{O}(\mathbb{H}_J) \rightarrow \mathcal{O}(\mathbb{H}_J) \quad \text{is defined as } \mathbb{L}_m = 8\pi i m \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2}.$$

We extend $\mathbb{L}_m$ to $m = 0$. See [1] for a Casimir operator when $m = 0$.

6.2. Raising and lowering operators. Berndt and Schmidt [3] use techniques from reductive groups to analyze the Jacobi group G , although it is not reductive, defining raising and lowering operators on $C^\infty(\mathbb{H}_J)$. We introduce related raising and lowering operators on the space of pseudodifferential forms. We write $\phi_\tau, \phi_z, \phi_{\bar{\tau}}, \phi_{\bar{z}}$ for the partial derivative of ϕ in $C^\infty(\mathbb{H}_J)$ with respect to $\tau, z, \bar{\tau}, \bar{z}$, respectively, where for $\tau = x + iy$ and $z = u + iv$,

$$\frac{\partial}{\partial \tau} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{\tau}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).$$

We justify the following definition of covariant differential operators in [Appendix A](#) (see [Proposition A.8](#)). Here, the superscript of J indicates origin with respect to the elliptic variable z as opposed to the modular variable τ . We scale the operators to match generators of the corresponding Lie algebra $\mathfrak{g}$ from [18].

Definition 6.3. Fix $k \in \mathbb{Z}$ and $m_l, m_r \in \mathbb{C}$. Define

$$\begin{aligned} \text{raising operators} \quad R_{k,(m_l,m_r)}, \quad R_{k,(m_l,m_r)}^J &: \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)) \rightarrow \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)), \\ \text{lowering operators} \quad L_{k,(m_l,m_r)}, \quad L_{k,(m_l,m_r)}^J &: \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)) \rightarrow \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)), \quad \text{and} \\ \text{constant operators} \quad C_{k,(m_l,m_r)}, \quad C_{k,(m_l,m_r)}^J &: \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)) \rightarrow \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)) \end{aligned}$$

by

$$\begin{aligned} R(\phi \partial^n) &= (2i(\phi_\tau + \frac{v}{y} \phi_z + 2i\pi m_r \frac{v^2}{y^2} \phi) + \frac{k-n}{y} \phi) \partial^n + 4\pi i m_r \frac{nv}{y^2} \phi \partial^{n-1} + \pi m_r \frac{n(n-1)}{y^2} \phi \partial^{n-2}, \\ L(\phi \partial^n) &= -2iy(y\phi_{\bar{\tau}} + v\phi_{\bar{z}}) \partial^n - \pi m_r n(n-1) \phi \partial^{n-2}, \\ R^J(\phi \partial^n) &= (i\phi_z - 4\pi m_r \frac{v}{y} \phi) \partial^n + 2\pi i m_r \frac{n}{y} \phi \partial^{n-1}, \\ L^J(\phi \partial^n) &= -iy\phi_{\bar{z}} \partial^n + 2\pi i m_r n \phi \partial^{n-1}, \\ C(\phi \partial^n) &= (k-n) \phi \partial^n + 4\pi i m_r \frac{nv}{y} \phi \partial^{n-1} + 2\pi m_r \frac{n(n-1)}{y} \phi \partial^{n-2}, \\ C^J(\phi \partial^n) &= 2\pi i m \phi \partial^n \end{aligned}$$

for ϕ in $C^\infty(\mathbb{H}_J)$ and n in $\mathbb{Z}$, where $m = m_l + m_r$, with subscripts $k, (m_l, m_r)$ on operators suppressed.

The raising, lowering, and constant operators are covariant maps:

Proposition 6.4. For each $k \in \mathbb{Z}$ and $m_l, m_r \in \mathbb{C}$, the operators

$$R := R_{k,(m_l,m_r)}, \quad L := L_{k,(m_l,m_r)}, \quad R^J := R_{k,(m_l,m_r)}^J, \quad \text{and} \quad L^J := L_{k,(m_l,m_r)}^J$$

are covariant, and the operators $C := C_{k,(m_l,m_r)}$ and $C^J := C_{k,(m_l,m_r)}^J$ are invariant:

$$\begin{aligned} R(\psi|_{k,(m_l,m_r)} g) &= R(\psi)|_{k+2,(m_l,m_r)} g, & L(\psi|_{k,(m_l,m_r)} g) &= L(\psi)|_{k-2,(m_l,m_r)} g, \\ R^J(\psi|_{k,(m_l,m_r)} g) &= R^J(\psi)|_{k+1,(m_l,m_r)} g, & L^J(\psi|_{k,(m_l,m_r)} g) &= L^J(\psi)|_{k-1,(m_l,m_r)} g, \\ C(\psi|_{k,(m_l,m_r)} g) &= C(\psi)|_{k,(m_l,m_r)} g, & C^J(\psi|_{k,(m_l,m_r)} g) &= C^J(\psi)|_{k,(m_l,m_r)} g \end{aligned}$$

for all $g \in G$ and $\psi \in \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$.

Proof. Direct computation verifies the claim using generators for G , for example,

$$\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, (0, \mu), 0 \right), \quad \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, (0, 0), 0 \right) \quad \text{for } b, \mu \text{ in } \mathbb{R}.$$

Alternatively, one may adapt the more theoretical approach of [18] for finding covariant operators to the setting of infinite-dimensional K -vector bundles for $K = \text{stab}_G(i, 0)$ using the fact that $G/K \cong \mathbb{H}_J$ is a reductive coset space although G is not reductive, see [Appendix A](#). $\square$

Remark 6.5. **Proposition 6.4** may be rephrased succinctly in terms of *raising or lowering* the weight k giving the slash group action $|_{k,(m_l,m_r)}$ on $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ (compare with the operators of [3]):

$$\begin{array}{ll} R & \text{raises by 2,} \\ R^J & \text{raises by 1,} \\ C & \text{preserves weight,} \end{array} \quad \begin{array}{ll} L & \text{lowers by 2,} \\ L^J & \text{lowers by 1, and} \\ C^J & \text{preserves weight.} \end{array}$$

6.6. A Casimir operator. We construct the Casimir operator on $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ as a composition of degree 1 operators. These arise from a convenient basis for the complexified Lie algebra $\mathfrak{g}$ of G and a Casimir element in the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ identified in [18]; see Appendix A.

Definition 6.7. For $k \in \mathbb{Z}$ and $m_l, m_r \in \mathbb{C}$ with $m = m_l + m_r \neq 0$, define a *Casimir operator* ($\mathbb{C}$ -linear map)

$$\mathcal{C}_{k,(m_l,m_r)} : \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)) \longrightarrow \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$$

as this composition (applied left to right) of raising, lowering, and constant covariant operators:

$$\mathcal{C}_{k,(m_l,m_r)} = 4C^J R L + 2i(R^J)^2 L - 2iR(L^J)^2 + 2iR^J L^J (C - 2) - 3C^J C + C^J C^2.$$

Proposition 6.4 implies that the Casimir operator on $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ is equivariant with respect to the slash action:

Theorem 6.8. Fix $k \in \mathbb{Z}$ and $m_l, m_r \in \mathbb{C}$ with $m = m_l + m_r \neq 0$. The Casimir operator $\mathcal{C}_{k,(m_l,m_r)}$ is G -equivariant with respect to the slash action $|_{k,(m_l,m_r)}$ on $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$:

$$\mathcal{C}_{k,(m_l,m_r)}(\psi |_{k,(m_l,m_r)} g) = \mathcal{C}_{k,(m_l,m_r)}(\psi) |_{k,(m_l,m_r)} g$$

for all g in G and ψ in $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$.

6.9. Casimir operator explicitly. We verify that the action of the Casimir operator on $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ restricts to an action on $\Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))$ using the following formula for $\mathcal{C}_{k,(m_l,m_r)}$ obtained by straightforward calculation using **Definition 6.3** and **Definition 6.7**.

Proposition 6.10. Fix $k \in \mathbb{Z}$ and $m_l, m_r \in \mathbb{C}$ with $m = m_l + m_r \neq 0$. The Casimir operator $\mathcal{C}_{k,(m_l,m_r)}$ acting on the space of pseudodifferential operators $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ is

$$\mathcal{C}_{k,(m_l,m_r)}(\phi \partial^n) = \mathcal{C}_{k,(m_l,m_r)}^{(0)}(\phi) \partial^n + \mathcal{C}_{k,(m_l,m_r)}^{(1)}(\phi) \partial^{n-1} + \mathcal{C}_{k,(m_l,m_r)}^{(2)}(\phi) \partial^{n-2},$$

where

$$\begin{aligned} \mathcal{C}_{k,(m_l,m_r)}^{(0)}(\phi) &= 2\pi i m ((k-n)^2 - 3(k-n)) \phi + 32\pi i m y^2 \phi_{\tau\bar{\tau}} - 8\pi m (1-2k+2n) y \phi_{\bar{\tau}} \\ &\quad + 32\pi i m y v \phi_{\tau\bar{z}} - 4y^2 (\phi_{\tau\bar{z}\bar{z}} + \phi_{\bar{\tau}zz}) - 8\pi m (1-k+n) v \phi_{\bar{z}} - 8\pi i m v^2 \phi_{\bar{z}\bar{z}} \\ &\quad + 2i(k-n)y(\phi_{z\bar{z}} + \phi_{\bar{z}\bar{z}}) - 4yv(\phi_{z\bar{z}\bar{z}} + \phi_{\bar{z}zz}), \\ \mathcal{C}_{k,(m_l,m_r)}^{(1)}(\phi) &= 4\pi i m_r n(1-k+n)(\phi_z + \phi_{\bar{z}}) + 16\pi m_r n y (\phi_{\tau\bar{z}} - \phi_{\bar{\tau}z}) - 8\pi m_r n v (\phi_{z\bar{z}} + \phi_{\bar{z}\bar{z}}), \\ \mathcal{C}_{k,(m_l,m_r)}^{(2)}(\phi) &= 16\pi^2 m_l m_r n(n-1)(\phi_\tau + \phi_{\bar{\tau}}) + 2\pi i m_r n(n-1)(\phi_{zz} + 2\phi_{z\bar{z}} + \phi_{\bar{z}\bar{z}}). \end{aligned}$$

Remark 6.11. Observe that the zero-graded piece $\mathcal{C}_{k,(m_l,m_r)}^{(0)}$ as an operator on $C^\infty(\mathbb{H}_J)$ depending on n coincides with the Casimir operator $\mathcal{C}_{k-n,m}$ acting on $C^\infty(\mathbb{H}_J)$ (with slash action $|_{k-n,m}$) given in [18].

We would expect the Casimir operator $\mathcal{C}_{k,(m_l,m_r)}$ to take the space of holomorphic pseudodifferential operators to itself if G were reductive. As G is non-reductive, we instead use **Proposition 6.10**:

Corollary 6.12. Fix $k \in \mathbb{Z}$ and $m_l, m_r \in \mathbb{C}$ with $m = m_l + m_r \neq 0$. The Casimir operator $\mathcal{C}_{k,(m_l,m_r)}$ on pseudo-differential operators $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ restricts to a Casimir operator

$$\mathcal{C}_{k,(m_l,m_r)} : \Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J)) \longrightarrow \Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))$$

given by

$$\begin{aligned} \mathcal{C}_{k,(m_l,m_r)}(\phi \partial^n) &= 2\pi i m ((k-n)^2 - 3(k-n)) \phi \partial^n \\ &\quad + 4\pi i m_r n(1-k+n) \frac{\partial}{\partial z}(\phi) \partial^{n-1} \\ &\quad - 2\pi i m_r n(n-1) \mathbb{L}_{m_l}(\phi) \partial^{n-2} \end{aligned}$$

for $\phi \in \mathcal{O}(\mathbb{H}_J)$. In particular, for $k = 0$,

$$\mathcal{C}_{0,(m_l,m_r)}(\phi \partial^n) = 2\pi i m n(n+3) \phi \partial^n + 4\pi i m_r n(n+1) \frac{\partial}{\partial z}(\phi) \partial^{n-1} - 2\pi i m_r n(n-1) \mathbb{L}_{m_l}(\phi) \partial^{n-2}.$$

7. EQUIVARIANT SPLITTING FOR PSEUDODIFFERENTIAL OPERATORS

In this section, we identify a family of linear maps from the space of pseudodifferential operators to the holomorphic functions on the Jacobi upper half place,

$$\Upsilon : \Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J)) \longrightarrow \mathcal{O}(\mathbb{H}_J),$$

equivariant under the action of the real Jacobi group G . If this acting Lie group G were reductive, general theory would predict existence of a map Υ splitting projection onto the top coefficient on each filtered component. Since G is not reductive, we use other arguments to prove that a splitting map exists. We identified a Casimir operator $\mathcal{C}_{k,(m_l,m_r)}$ acting on $\Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))$ in [Section 6](#). Here, we take advantage of the fact that the eigenvalues of the induced action on the graded pieces are distinct and apply [Proposition 5.6](#) to the weight k slash action on pseudodifferential operators.

To distinguish various actions of G , we write $(M, |_{k,m})$ for any G -module M defined by a right action of G by a slash operator $|_{k,m}$, and we write $(M, |_{k,m})^G$ or M^G for its submodule of G -invariants. For a fixed k in $\mathbb{Z}$, we abbreviate π for the map projecting each operator in the component of filtered degree $-k$ to its top coefficient, i.e., the coefficient of ∂^{-k} , see [Subsection 3.4](#):

$$(7.1) \quad \pi = \pi_{-k} : \Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))_{-k} \longrightarrow \mathcal{O}(\mathbb{H}_J), \quad \sum_{n=0}^{\infty} \phi_n \partial^{-n-k} \longmapsto \phi_0.$$

7.2. Exact sequence. The natural filtration on $\Psi = \Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))$ (see [Subsection 3.4](#)) induces an exact sequence of G -modules for each k in $\mathbb{Z}$ and m_l, m_r in $\mathbb{C}$ with $m = m_l + m_r$:

$$(7.3) \quad 0 \longrightarrow (\Psi_{-k-1}, |_{0,(m_l,m_r)}) \longrightarrow (\Psi_{-k}, |_{0,(m_l,m_r)}) \xrightarrow{\pi} (\mathcal{O}(\mathbb{H}_J), |_{k,m}) \longrightarrow 0.$$

We construct an explicit G -equivariant splitting map for each k and $m \neq 0$,

$$\Upsilon = \Upsilon_{k,(m_l,m_r)} : \mathcal{O}(\mathbb{H}_J) \longrightarrow \Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))_{-k},$$

i.e., an equivariant map with $\pi \circ \Upsilon = \text{Id}$, the identity, so that taking G' -invariants of each term in the exact sequence for a suitable subgroup G' of G (see [Section 8](#)) gives an exact sequence of invariants (recalling that the invariant-functor is generally only left-exact):

$$(7.4) \quad 0 \longrightarrow (\Psi_{-k-1}, |_{0,(m_l,m_r)})^{G'} \longrightarrow (\Psi_{-k}, |_{0,(m_l,m_r)})^{G'} \xrightarrow{\pi} (\mathcal{O}(\mathbb{H}_J), |_{k,m})^{G'} \longrightarrow 0.$$

7.5. Recursion on coefficients. We show the existence of the G -equivariant splitting map Y of exact sequence (7.3) in **Theorem 7.8** below; we will use the next lemma to explicitly describe this map in that same result.

Lemma 7.6. Fix $2 \leq k \in \mathbb{Z}$ and $m_l, m_r \in \mathbb{C}$ with $m = m_l + m_r \neq 0$. Suppose a $\mathbb{C}$ -linear function $Y : \mathcal{O}(\mathbb{H}_J) \rightarrow \Psi\text{DO}(\mathfrak{d}, \mathcal{O}(\mathbb{H}_J))_{-k}$ preserves the leading coefficient and commutes with Casimir actions, i.e., for all $\phi \in \mathcal{O}(\mathbb{H}_J)$,

$$Y(\phi) = \phi \mathfrak{d}^{-k} + \sum_{0 < n} \phi_n \mathfrak{d}^{-k-n} \quad \text{with } \phi_n \in \mathcal{O}(\mathbb{H}_J) \quad \text{and} \\ Y(\mathcal{C}_{k,m}(\phi)) = \mathcal{C}_{0,(m_l,m_r)}(Y(\phi)).$$

Then the coefficients of $Y(\phi)$ satisfy the recursion $\phi_{-1} = 0$, $\phi_0 = \phi$, and

$$\phi_n = \frac{m_r(k+n-1)(k+n-2)}{mn(3-n-2k)} \left(2 \frac{\partial}{\partial \bar{z}} (\phi_{n-1}) - \mathbb{L}_{m_l}(\phi_{n-2}) \right) \quad \text{for } n > 0.$$

Proof. Recall that $\mathcal{C}_{k,m}$ acts by a scalar on $\mathcal{O}(\mathbb{H}_J)$ (see **Subsection 6.1**),

$$2\pi i m(k^2 - 3k) Y(\phi) = Y(\mathcal{C}_{k,m}(\phi)) = \mathcal{C}_{0,(m_l,m_r)}(Y(\phi)),$$

giving an eigenvector equation. We equate the coefficients of $\mathfrak{d}^{-k-n}$ using **Corollary 6.12**. On the left side, this coefficient is just $2\pi i m(k^2 - 3k)\phi_n$, whereas on the right side, this coefficient is

$$-2\pi i m_r(k+n-2)(k+n-1) \mathbb{L}_{m_l}(\phi_{n-2}) + 4\pi i m_r(k+n-1)(k+n-2) \frac{\partial}{\partial \bar{z}} \phi_{n-1} + 2\pi i m(k+n)(k+n-3)\phi_n$$

giving the recursive formula in the lemma. $\square$

7.7. Using the Casimir to find the equivariant splitting map. We describe the G -equivariant splitting map Y recursively using the explicit formula for $\mathcal{C}_{k,(m_l,m_r)}$ in **Section 6** and the heat operator $\mathbb{L}_m = 8\pi i m \frac{\partial}{\partial \bar{\tau}} - \frac{\partial^2}{\partial \bar{z}^2}$ before giving a closed formula in **Remark 7.9**. We restrict to weights $k \geq 2$ in order to secure distinct eigenvalues needed to invoke **Proposition 5.6**.

Theorem 7.8. For all $2 \leq k \in \mathbb{Z}$ and $m_l, m_r \in \mathbb{C}$ with $m = m_l + m_r \neq 0$, there is a G -equivariant $\mathbb{C}$ -linear map

$$Y = Y_{k,(m_l,m_r)} : (\mathcal{O}(\mathbb{H}_J), |_{k,m}) \longrightarrow (\Psi\text{DO}(\mathfrak{d}, \mathcal{O}(\mathbb{H}_J))_{-k}, |_{0,(m_l,m_r)})$$

that commutes with the Casimir operators and preserves the leading coefficient, i.e.,

- 1) $Y(\phi |_{k,m} g) = Y(\phi) |_{0,(m_l,m_r)} g \quad \text{for all } g \in G,$
- 2) $Y(\mathcal{C}_{k,m}(\phi)) = \mathcal{C}_{0,(m_l,m_r)}(Y(\phi)), \quad \text{and}$
- 3) $Y(\phi) = \phi \mathfrak{d}^{-k} + \sum_{0 < n} \phi_n \mathfrak{d}^{-k-n} \quad \text{for some } \phi_n \in \mathcal{O}(\mathbb{H}_J),$

for all ϕ in $\mathcal{O}(\mathbb{H}_J)$. Furthermore, $Y(\phi)$ is given in terms of the heat operator:

$$\phi_n = \sum_{i \in \mathbb{Z}: 0 \leq i \leq \frac{n}{2}} c_{n,i} \left(\left(\frac{\partial}{\partial \bar{z}} \right)^{n-2i} \mathbb{L}_{m_l}^i \right) (\phi) \quad \text{for } n > 0$$

for $c_{n,i}$ defined recursively by $c_{0,0} = 1$, $c_{*, -1} = 0$, $c_{n,i} = 0$ for $i > n/2$, and

$$c_{n,i} = \frac{m_r(k+n-1)(k+n-2)}{mn(3-n-2k)} (2c_{n-1,i} - c_{n-2,i-1}) \quad \text{for } i \leq n/2.$$

Remark 7.9. We may expand the equivariant splitting map Y as

$$Y(\phi) = \phi \mathfrak{d}^{-k} - \frac{m_r k}{m} \frac{\partial}{\partial \bar{z}} (\phi) \mathfrak{d}^{-k-1} + \frac{m_r k(k+1)}{2m^2(1-2k)} \left((m - 2m_r k) \frac{\partial^2}{\partial \bar{z}^2} (\phi) - (8\pi i m m_l) \frac{\partial}{\partial \bar{\tau}} (\phi) \right) \mathfrak{d}^{-k-2} + \dots$$

Proof of Theorem 7.8. We regard $M = \Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))_{-2}$ as a filtered $\mathcal{O}(\mathbb{H}_J)$ -module: $M = \bigcup_{i \leq -2} M_i$ where $M_i = \Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))_i$. The associated graded module is $\text{gr } M = \bigoplus_{i \leq -2} (\text{gr } M)_i = \bigoplus_{i \leq -2} M_i / M_{i-1}$ as in [Subsection 3.4](#). We regard M as a $\mathbb{C}[x]$ -module by defining an action of x on each M_k by $xz = \mathcal{C}_{0,(m_l,m_r)}(z)$ (extended via composition). Then $\text{gr } M$ inherits both the $\mathbb{C}[x]$ -module structure and the G -module structure on M as both x and g in G preserve the filtration. The Casimir operator $\mathcal{C}_{0,(m_l,m_r)}$ is G -equivariant by [Theorem 6.8](#) so the action of G and x commute, and hence M is a $\mathbb{C}[G][x]$ -module.

Fix $k \geq 2$ and identify $(\text{gr } M)_{-k}$ with $\mathcal{O}(\mathbb{H}_J)$ via $\phi \partial^{-k} + M_{-k-1} \mapsto \phi$. The induced action of $\mathbb{C}[x]$ on $\mathcal{O}(\mathbb{H}_J)$ under this identification is given by

$$x\phi = 2\pi i m(k^2 - 3k)\phi = \mathcal{C}_{k,m}(\phi) \quad \text{for } \phi \in \mathcal{O}(\mathbb{H}_J)$$

by [Corollary 6.12](#) (see [Subsection 6.1](#)), so $(\text{gr } M)_{-k}$ is an eigenspace for the action of x with eigenvalue $2\pi i m(k^2 - 3k)$. Furthermore, we observe that the induced action of G on $\mathcal{O}(\mathbb{H}_J)$ under this identification (via the slash operator $|_{0,(m_l,m_r)}$ on M) is given by the slash operator $|_{k,m}$ by [Corollary 4.9](#).

The filtration on M is complete over $\mathbb{C}[x]$, see [Subsection 5.2](#) and [Example 5.4](#) with $R = \mathcal{O}(\mathbb{H}_J)$, $t = \partial$, and $S = \mathbb{C}[x]$. Then as the eigenvalues of x on the spaces $(\text{gr } M)_i$ are distinct for distinct $i \leq -2$, [Proposition 5.6](#) (see [Remark 5.7](#)) implies the existence of a unique $\mathbb{C}[G][x]$ -module homomorphism splitting the projection $\pi_{-k} : \Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J)) \rightarrow \mathcal{O}(\mathbb{H}_J)$ onto the top coefficient, i.e., a G -equivariant map

$$\Upsilon = \Upsilon_{k,(m_l,m_r)} := \Upsilon_{k,(m_l,m_r)} : \mathcal{O}(\mathbb{H}_J) \longrightarrow \Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))_{-k}$$

with $\pi_{-k}\Upsilon$ the identity, for each $k \geq 2$. The recursion in [Lemma 7.6](#) gives the advertised form for Υ . $\square$

Remark 7.10. For all $2 \leq k \in \mathbb{Z}$ and $m_l, m_r \in \mathbb{C}$ with $0 \neq m = m_l + m_r$, the splitting map,

$$\Upsilon : (\mathcal{O}(\mathbb{H}_J), |_{k,m}) \longrightarrow (\Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))_{-k}, |_{0,(m_l,m_r)}),$$

of [Theorem 7.8](#) gives (see the proof) the following commutative diagrams (for all left arrows or all right arrows) for each g in the Jacobi group G , where π again is the projection onto the coefficient of ∂^{-k} :

$$\begin{array}{ccc} \mathcal{O}(\mathbb{H}_J) & \xrightleftharpoons[\pi]{\Upsilon} & \Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))_{-k} \\ \mathcal{C}_{k,m} \downarrow & & \downarrow \mathcal{C}_{0,(m_l,m_r)} \\ \mathcal{O}(\mathbb{H}_J) & \xrightleftharpoons[\pi]{\Upsilon} & \Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))_{-k} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{O}(\mathbb{H}_J) & \xrightleftharpoons[\pi]{\Upsilon} & \Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))_{-k} \\ g \downarrow & & \downarrow g \\ \mathcal{O}(\mathbb{H}_J) & \xrightleftharpoons[\pi]{\Upsilon} & \Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))_{-k} \end{array}$$

Remark 7.11. Setting $m_r = 0$ recovers the traditional landscape of Jacobi forms. Indeed, when $m_r = 0$, we set $m = m_l$, and the action of G on $\Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))$ corresponds to an action of G on $\mathcal{O}(\mathbb{H}_J)$ by the slash operator of index m and weight given by the negative power of ∂ :

$$\phi \partial^{-k} | g = (\phi |_{k,m} g) \partial^{-k}.$$

Thus ∂ serves merely as a placeholder in this case: its power indicates the weight of the slash operator by which G should act on the coefficient. The recursion in [Lemma 7.6](#) implies that all the coefficients of $\Upsilon_{k,(m_l,0)}$ but one vanish and

$$\Upsilon_{k,(m_l,0)}(\phi) = \phi \partial^{-k} \quad \text{for all } \phi \in \mathcal{O}(\mathbb{H}_J).$$

Thus the splitting map in this case is simply the identity map augmented by notation to indicate a choice of weight for the action of G on $\mathcal{O}(\mathbb{H}_J)$.

8. APPLICATIONS TO JACOBI FORMS

We show in this section that the formal product of Jacobi forms over all weights with a fixed index is in bijection with a set of invariant pseudodifferential operators. In this section, we will restrict the indices m to integers so that the slash actions of G induce slash actions of Γ on both $\mathcal{O}(\mathbb{H}_J)$ and $\Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))$ (since the integral points of the center of G act trivially).

8.1. Jacobi forms. A *Jacobi form* is an analogue of a modular form for the Jacobi upper half plane instead of the Poincaré upper half plane, i.e., a complex-valued function on the Jacobi upper half plane $\mathbb{H}_J$ with good growth behavior which is invariant under the action of the Jacobi group $\Gamma = \text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ via the slash-operator.

Definition 8.2. A (holomorphic) *Jacobi form* of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{Z}$ is a holomorphic function $\phi : \mathbb{H}_J \rightarrow \mathbb{C}$

- (i) that satisfies $\phi|_{k,m} \gamma = \phi$ for all $\gamma \in \Gamma$, and
- (ii) has growth given by $e^{2\pi i m \lambda^2 \tau} \phi(\tau, \lambda \tau + \mu) = O(1)$ for all $\lambda, \mu \in \mathbb{Q}$.

We denote the space of (holomorphic) Jacobi forms of fixed weight k and fixed index m by $J_{k,m}$.

Recall that $J_{k,m} = 0$ for $k < 0$ or $m < 0$ and $J_{k,m}$ recovers the space of elliptic modular forms for $m = 0$. Under the usual multiplication of complex-valued functions, the space of Jacobi forms is a ring bigraded by weight and index: If ϕ and ϕ' are Jacobi forms of respective weights k and k' and respective indices m and m' , then $\phi\phi'$ is a Jacobi form of weight $k + k'$, and index $m + m'$. We set J to be the bigraded ring over all weights k and all Jacobi indices m ,

$$J_{\bullet,\bullet} := \bigoplus_{k,m} J_{k,m}.$$

8.3. Jacobi pseudodifferential operators. We define Jacobi pseudodifferential operators as those invariant under the slash action with an eye to the growth condition:

Definition 8.4. The space of *Jacobi pseudodifferential operators* for $m_l, m_r \in \mathbb{C}$ with $m = m_l + m_r \in \mathbb{Z}$ using the slash action $|_{0,(m_l,m_r)}$ of Γ is

$$J\Psi_{(m_l,m_r)} := \left\{ \sum_{k \geq 0} \phi_k \partial^{-k} \in \left(\Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J)) \right)^\Gamma : e^{2\pi i m \lambda^2 \tau} \phi_n(\tau, \lambda \tau + \mu) = O(1) \text{ for all } \lambda, \mu \in \mathbb{Q} \right\}.$$

Jacobi pseudodifferential operators inherit a filtration from pseudodifferential operators with filtered component of degree $-k$ given by

$$(8.5) \quad \left(J\Psi_{(m_l,m_r)} \right)_{-k} := J\Psi_{(m_l,m_r)} \cap \Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))_{-k}.$$

As expected, the map $\pi_{-k} : \sum_{n \geq 0} \phi_n \partial^{-k-n} \mapsto \phi_0$ projecting any pseudodifferential operator of filtered degree $-k$ to its top coefficient takes Jacobi pseudodifferential operators to Jacobi forms, see [Remark 7.10](#):

$$\pi_{-k} \left(\left(J\Psi_{(m_l,m_r)} \right)_{-k} \right) \subset J_{k,m_l+m_r} \quad \text{for all } k \in \mathbb{Z}, \text{ and } m_l, m_r \in \mathbb{C} \text{ with } m = m_l + m_r \in \mathbb{Z}.$$

The noncommutative product of pseudodifferential operators invariant under the Jacobi group Γ is again invariant when we adjust the action appropriately, i.e., when the subscripts recording the actions align: For any $\mathbf{t}$ in $\mathbb{C}$, by [Proposition 4.4](#) (see [Definition 8.4](#)),

$$(8.6) \quad \psi \in J\Psi_{(m-\mathbf{t},\mathbf{t})} \quad \text{and} \quad \psi' \in J\Psi_{(-\mathbf{t},\mathbf{t}+m')} \quad \text{implies} \quad \psi \cdot \psi' \in J\Psi_{(m-\mathbf{t},\mathbf{t}+m')}.$$

Notice here that the growth condition in [Eq. \(8.5\)](#) is respected by the product formula in [Subsection 3.2](#). Thus the natural noncommutative multiplication map on pseudodifferential operators restricts for each fixed m, m' in $\mathbb{Z}$ and $\mathbf{t}$ in $\mathbb{C}$ to a multiplication map

$$J\Psi_{(m-\mathbf{t},\mathbf{t})} \otimes J\Psi_{(-\mathbf{t},\mathbf{t}+m')} \longrightarrow J\Psi_{(m-\mathbf{t},\mathbf{t}+m')}.$$

8.7. Splitting map for Jacobi pseudodifferential operators. As a consequence of the theory developed in the last section, we obtain a splitting map of Γ -invariants for m in $\mathbb{Z}$. Note that we restrict to index $m \neq 0$. When $m = 0$, the top coefficient ϕ_0 of a Jacobi pseudodifferential operator is a Jacobi form which is constant in the elliptic variable z , and we fall into the classical setting of modular forms. In addition, our splitting map Υ is not defined for $m = 0$, so it does not give a decomposition of $\mathrm{J}\Psi_{(m_l, m_r)}$ directly.

Corollary 8.8. Fix $2 \leq k \in \mathbb{Z}$ and $m_l, m_r \in \mathbb{C}$ with $0 \neq m = m_l + m_r \in \mathbb{Z}$. The $\mathbb{C}$ -linear map

$$\Upsilon = \Upsilon_{k, (m_l, m_r)} : (\mathcal{O}(\mathbb{H}_J), |_{k, m}) \longrightarrow (\Psi\mathrm{DO}(\partial, \mathcal{O}(\mathbb{H}_J))_{-k}, |_{0, (m_l, m_r)})$$

of [Theorem 7.8](#) restricts to a $\mathbb{C}$ -linear map of Γ -equivariants

$$\begin{array}{ccc} (\mathcal{O}(\mathbb{H}_J))^\Gamma & \xrightarrow{\Upsilon} & (\Psi\mathrm{DO}(\partial, \mathcal{O}(\mathbb{H}_J))_{-k})^\Gamma \\ & \searrow \pi & \end{array}$$

with $\pi \circ \Upsilon = \mathrm{Id}$, the identity map, for $\pi = \pi_{-k}$ and Γ acting on the domain by the slash action $|_{k, m}$ and acting on the codomain by $|_{0, (m_l, m_r)}$.

Now we consider the growth condition.

Corollary 8.9. For any $2 \leq k \in \mathbb{Z}$ and $m_l, m_r \in \mathbb{C}$ with $0 \neq m = m_l + m_r \in \mathbb{Z}$, the map $\Upsilon_{k, (m_l, m_r)}$ of [Theorem 7.8](#) restricted to $\mathrm{J}_{k, m}$ induces a $\mathbb{C}$ -linear splitting

$$\Upsilon = \Upsilon_{k, (m_l, m_r)} : \mathrm{J}_{k, m} \xrightarrow{\Upsilon} (\mathrm{J}\Psi_{(m_l, m_r)})_{-k}$$

$\nwarrow \pi$

to the projection map $\pi = \pi_{-k}$ restricted to $(\mathrm{J}\Psi_{(m_l, m_r)})_{-k}$, i.e., $\pi \circ \Upsilon = \mathrm{Id}$ on $\mathrm{J}_{k, m}$.

Proof. By [Theorem 7.8](#), it suffices to check that $\Upsilon_{k, (m_l, m_r)}(\phi)$ lies in $\mathrm{J}\Psi_{(m_l, m_r)}$ for ϕ in $\mathrm{J}_{k, m}$, for fixed k and m . Holomorphicity and invariance follow from [Corollary 8.8](#), so we focus on the growth condition. We use the recursive formula for $\Upsilon_{k, (m_l, m_r)}$ from [Theorem 7.8](#) and check that $(\frac{\partial}{\partial z})^{n-2i} \mathbb{L}_m^i \phi$ satisfies the growth condition. By the theta-decomposition of Jacobi forms, this amounts to checking the growth condition for quasi-modular forms, where it is obvious. $\square$

8.10. Modeling Jacobi forms as power series. To formulate applications to Jacobi forms, we use a formal variable X with exponent indicating weight of a Jacobi form. Define

$$\mathcal{O}(\mathbb{H}_J)[[X^{-1}]] = \left\{ \sum_{k \geq 0} \phi_k X^{-k} : \phi_k \in \mathcal{O}(\mathbb{H}_J) \right\}$$

with an slash action of G for fixed $m \in \mathbb{C}$ defined by

$$(8.11) \quad \left(\sum_k \phi_k X^{-k} \right) |_{\bullet, m} g = \sum_k (\phi_k |_{k, m} g) X^{-k} \quad \text{for } \phi_k \in \mathcal{O}(\mathbb{H}_J), \ g \in G.$$

We take the natural filtration on $M = \mathcal{O}(\mathbb{H}_J)[[X^{-1}]]$ and set

$$M_{-2} = \left\{ \sum_{k \geq 2} \phi_k X^{-k} : \phi_k \in \mathcal{O}(\mathbb{H}_J) \right\},$$

the natural filtration on $\Psi\mathrm{DO}(\partial, \mathcal{O}(\mathbb{H}_J))$ as in [Subsection 3.4](#), and use the splitting maps $\Upsilon_{k, (m_l, m_r)}$ from [Theorem 7.8](#).

Theorem 8.12. For any $m_l, m_r \in \mathbb{C}$ with $m = m_l + m_r \neq 0$, the map

$$\begin{aligned} \Upsilon_{(m_l, m_r)} : (\mathcal{O}(\mathbb{H}_J)[[X^{-1}]]_{-2}, |_{\bullet, m}) &\longrightarrow (\Psi\mathrm{DO}(\partial, \mathcal{O}(\mathbb{H}_J))_{-2}, |_{0, (m_l, m_r)}), \\ \sum_{k \geq 2} \phi_k X^{-k} &\longmapsto \sum_{k \geq 2} \Upsilon_{k, (m_l, m_r)}(\phi_k) \end{aligned}$$

is a G -equivariant filtered $\mathbb{C}$ -linear isomorphism with slash action $|_{\bullet, m}$ as given in [Eq. \(8.11\)](#).

Proof. For g in G , **Theorem 7.8** implies that $Y = Y_{(m_l, m_r)}$ takes $(\sum_k \phi_k X^{-k})|_{\bullet, m} g = \sum_k (\phi_k|_{k, m} g) X^{-k}$ to

$$\begin{aligned} \sum_k Y_{k, (m_l, m_r)} (\phi_k|_{k, m} g) &= \sum_k \left(Y_{k, (m_l, m_r)} (\phi_k)|_{0, (m_l, m_r)} g \right) \\ &= \left(\sum_k Y_{k, (m_l, m_r)} (\phi_k) \right)|_{0, (m_l, m_r)} g \\ &= Y \left(\sum_k \phi_k X^{-k} \right)|_{0, (m_l, m_r)} g, \end{aligned}$$

where the second equality holds as there are only finitely many terms in the sum with nonzero coefficient of a fixed ∂^{-n} and the action of g respects the filtration.

We construct a G -invariant inverse map $\Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J)) \rightarrow \mathcal{O}(\mathbb{H}_J)[[X^{-1}]]$,

$$\psi = \sum_{n \geq 2} \phi_n \partial^{-n} \longmapsto \sum_{n \geq 2} \phi'_n X^{-n},$$

by setting $\phi'_2 := \phi_2$, subtracting $Y_{2, (m_l, m_r)}(\phi_2)$ from ψ to cancel top coefficients, and then using the new top coefficient as the coefficient ϕ'_3 of X^{-3} , etc: We define recursively

$$\phi'_k := \pi_{-k} \left(\psi - \sum_{n=2}^{k-1} Y_{n, (m_l, m_r)}(\phi'_n) \right) \quad \text{for } k \geq 2,$$

where π_{-k} again is the projection onto the (top) coefficient of ∂^{-k} as in (7.1) and (7.3). One may check that this map is both left and right inverse to $Y_{(m_l, m_r)}$ (by inducting on filtered degree k and working modulo $\Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))_{-k}$). $\square$

Corollary 8.13. *For any $m_l, m_r \in \mathbb{C}$ with $0 \neq m = m_l + m_r \in \mathbb{Z}$, the map of **Theorem 8.12** restricts to a $\mathbb{C}$ -vector space isomorphism of invariants:*

$$Y_{(m_l, m_r)} : \left(\mathcal{O}(\mathbb{H}_J)[[X^{-1}]]_{-2}, |_{\bullet, m} \right)^\Gamma \xrightarrow{\cong} \left(\Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))_{-2}, |_{0, (m_l, m_r)} \right)^\Gamma.$$

We now observe that this last map is compatible with the growth conditions defining Jacobi forms and Jacobi pseudodifferential operators.

Corollary 8.14. *For any $m_l, m_r \in \mathbb{C}$ with $0 \neq m = m_l + m_r \in \mathbb{Z}$, the map $Y_{(m_l, m_r)}$ of **Theorem 8.12** induces a linear isomorphism from the formal direct product over k of all Jacobi forms of index m and weight k to the space of Jacobi pseudodifferential operators:*

$$Y_{(m_l, m_r)} : \prod_{k \geq 2} J_{k, m} \xrightarrow{\cong} (J\Psi_{(m_l, m_r)})_{-2}, \quad \prod_{k \geq 2} \phi_k \longmapsto Y_{(m_l, m_r)} \left(\sum_{k \geq 2} \phi_k X^{-k} \right) = \sum_{k \geq 2} Y_{k, (m_l, m_r)}(\phi_k)$$

Proof. We identify $\prod_{k \geq 2} J_{k, m}$ with the subspace of $(\mathcal{O}(\mathbb{H}_J)[[X^{-1}]]_{-2})^\Gamma$ under the slash action $|_{\bullet, m}$ satisfying the necessary growth condition of **Definition 8.2** in the obvious way, i.e., via $\prod_{k \geq 2} \phi_k \mapsto \sum_{k \geq 2} \phi_k X^{-k}$. The map $Y_{(m_l, m_r)}$ of **Corollary 8.13** restricts to an injective linear map taking $\prod_{k \geq 2} J_{k, m}$ to $(J\Psi_{(m_l, m_r)})_{-2}$ by **Corollary 8.9** and **Corollary 8.13**. One may show that the inverse map constructed in the proof of **Theorem 8.12** is compatible with the growth condition and therefore restricts to a map that takes $(J\Psi_{(m_l, m_r)})_{-2}$ to $\prod_{k \geq 2} J_{k, m}$, so $Y_{(m_l, m_r)}$ restricts to an isomorphism as advertised. $\square$

Note that the linear isomorphism of **Corollary 8.14** is not a ring homomorphism under multiplication of Jacobi forms and multiplication of pseudodifferential operators. In the next section, we define the Rankin-Cohen brackets as a measure of the failure of the map Y to respect these multiplications, or rather, as the mechanism to transport the noncommutative multiplication of pseudodifferential operators to Jacobi forms.

9. RANKIN-COHEN BRACKETS FOR JACOBI FORMS

We now define distinguished families of Rankin-Cohen brackets for Jacobi forms using the correspondence with invariant pseudodifferential operators established in the previous section. These Rankin-Cohen brackets record the noncommutative multiplication of pseudodifferential operators transported to the product space of Jacobi forms. Indeed, we define the Rankin-Cohen bracket as the pull-back of the noncommutative multiplication under the map Υ from Jacobi forms to pseudodifferential operators as in [Corollary 8.14](#).

9.1. Rankin-Cohen brackets. We say a $\mathbb{C}$ -bilinear function $\mathcal{O}(\mathbb{H}_J) \times \mathcal{O}(\mathbb{H}_J) \rightarrow \mathcal{O}(\mathbb{H}_J)$ is a ν -th *Rankin-Cohen bracket on Jacobi forms* (see [\[6\]](#)) for fixed $k, k', m, m' \geq 0$ if it raises weight by ν and is a holomorphic differential operator, i.e., has the form

$$f : (\phi, \phi') \mapsto \sum_{a,b,c,d \geq 0} c_{a,b,c,d} \frac{\partial^a}{\partial \tau^a} \frac{\partial^b}{\partial z^b} (\phi) \frac{\partial^c}{\partial \tau^c} \frac{\partial^d}{\partial z^d} (\phi')$$

with the sum finite, $c_{a,b,c,d}$ in $\mathbb{C}$, and

$$f(\phi, \phi')|_{k+k'+\nu, m+m'} g = f(\phi|_{k,m} g, \phi'|_{k',m'} g) \quad \text{for all } g \in G.$$

Every ν -th Rankin-Cohen bracket restricts to a $\mathbb{C}$ -bilinear function

$$J_{k,m} \times J_{k',m'} \longrightarrow J_{k+k'+\nu, m+m'}.$$

Böcherer [\[6\]](#) showed that the $\mathbb{C}$ -dimension of the space of ν -th Rankin-Cohen brackets for fixed weights k, k' and indices $m, m' > 0$ is generically $1 + \lfloor \nu/2 \rfloor$. Choie and Eholzer [\[12\]](#) found a basis for this space by leveraging the heat operator, effectively tapping into elliptic modular forms via the theta decomposition, but no particularly privileged basis has been identified. This is in stark contrast to the case of Rankin-Cohen brackets of elliptic modular forms: there the space of ν -th Rankin-Cohen brackets is 1-dimensional for ν even and 0 for ν odd. Thus for elliptic modular forms, fixing a choice of Rankin-Cohen bracket as ν ranges amounts to choosing some coefficients in $\mathbb{C}$.

For Jacobi forms, one seeks families of Rankin-Cohen brackets that capture key properties of the Jacobi setting. Choie [\[9\]](#) identified a one-dimensional subspace of Rankin-Cohen brackets for each ν arising as polynomials in the heat operator (see [\[6\]](#)). In the next theorem, we define families of Rankin-Cohen brackets of Jacobi forms parametrized by a parameter $\mathbf{t}$ in $\mathbb{C}$ with geometric origin. We use the noncommutative multiplication of pseudodifferential operators given in terms of the maps

$$\Upsilon_{(m_l, m_r)} : \prod_{k \geq 2} J_{k,m} \xrightarrow{\cong} (J\Psi_{(m_l, m_r)})_{-2}$$

of [Corollary 8.14](#). We convert two Jacobi forms into two operators using Υ , multiply the operators, and then apply Υ^{-1} to convert back into a family of Jacobi forms. The ν -th element of this family then will define the ν -th Rankin-Cohen bracket. In the next theorem, we embed each space $J_{k,m}$ of Jacobi forms for fixed m and $k \geq 2$ into the product $\prod_{k' \geq 2} J_{k',m}$ as the k -th component with other components zero.

Theorem 9.2. Fix a complex parameter $\mathbf{t}$. For any two Jacobi forms ϕ and ϕ' of respective weights $k, k' \geq 2$ and indices $m, m' > 0$, define a family of Rankin-Cohen brackets by

$$\prod_{i \geq k+k'} [\phi, \phi']_{(i-k-k')}^{\mathbf{t}} = \Upsilon_{(m-\mathbf{t}, \mathbf{t}+m')}^{-1} \left(\Upsilon_{(m-\mathbf{t}, \mathbf{t})}(\phi) \cdot \Upsilon_{(-\mathbf{t}, \mathbf{t}+m')}(\phi') \right),$$

i.e., $[\phi, \phi']_{(\nu)}^{\mathbf{t}}$ is the $(\nu + k + k')$ -th coordinate of the image under Υ^{-1} for $\nu \geq 0$. Then each $[\phi, \phi']_{(\nu)}^{\mathbf{t}}$ is a Jacobi form of weight $k + k' + \nu$ and index $m + m'$, and

$$\prod_{\nu \geq 0} [\ , \]_{(\nu)}^{\mathbf{t}} : J_{k,m} \otimes J_{k',m'} \longrightarrow \prod_{\nu \geq k+k'} J_{k+k'+\nu, m+m'}$$

defines $\mathbb{C}$ -linear function for $k, k' \geq 2$ and $m, m' > 0$.

Proof. We use the multiplication Eq. (8.6) on pseudodifferential Jacobi forms. Corollaries 8.9 and 8.14 imply that for any weights $k, k' \geq 2$, indices $m, m' > 0$, and parameter $\mathbf{t} \in \mathbb{Z}$, the following composition is well-defined:

$$\begin{aligned} J_{k,m} \otimes J_{k',m'} &\xrightarrow{\Upsilon_{k,(m-\mathbf{t},\mathbf{t})} \otimes \Upsilon_{k',(-\mathbf{t},\mathbf{t}+m')}} (J\Psi_{(m-\mathbf{t},\mathbf{t})})_{-k} \otimes (J\Psi_{(-\mathbf{t},\mathbf{t}+m')})_{-k'} \\ &\xrightarrow{\text{multiplication}} (J\Psi_{(m-\mathbf{t},\mathbf{t}+m')})_{-k-k'} \\ &\xrightarrow{(\Upsilon_{(m-\mathbf{t},\mathbf{t}+m')})^{-1}} \prod_{v \geq 0} J_{k+k'+v, m+m'}. \end{aligned}$$

□

We next consider a subvariety of the space of all Rankin-Cohen brackets as considered by Böcherer [6].

Corollary 9.3. *For each fixed $v \geq 0$, the family of Rankin-Cohen brackets $[\cdot, \cdot]_{(v)}^{\mathbf{t}}$ defined in Theorem 9.2 as $\mathbf{t}$ ranges over complex values defines a subvariety of lines of expected dimension 1 in the space of all Rankin-Cohen brackets for Jacobi forms of fixed respective weights $k, k' \geq 2$ and indices $m, m' > 0$.*

Proof. First note that the map $\Upsilon_{k,(m_1,m_2)}$ of Corollary 8.9 takes each Jacobi form ϕ to a pseudodifferential operator whose coefficient of each ∂^n is a finite sum of a finite number of compositions of partial derivatives $\partial/\partial\tau$ and $\partial/\partial z$ applied to ϕ with coefficients in $\mathbb{Q}$, see Remark 7.9. The v -th Rankin-Cohen bracket of Theorem 9.2 is defined by applying the maps $\Upsilon_{k,(m-\mathbf{t},\mathbf{t})}$ and $\Upsilon_{k',(-\mathbf{t},\mathbf{t}+m')}$ of Corollary 8.9 to two input Jacobi forms ϕ and ϕ' respectively, multiplying the resulting pseudodifferential operators, and pulling back via the map $\Upsilon_{m-\mathbf{t},\mathbf{t}+m'}^{-1}$ of Corollary 8.14. We can thus express the coefficient of each ∂^n in each pseudodifferential operator as a finite sum of mixed partial derivatives with each coefficient in $\mathbb{Q}[\mathbf{t}]$, and the same can be said for their product. Pulling back via $\Upsilon_{m-\mathbf{t},\mathbf{t}+m'}^{-1}$ gives a Jacobi form for each v expressed as a finite sum of mixed partial derivatives in $\partial/\partial\tau$ and $\partial/\partial z$ of the original Jacobi forms with coefficients in $\mathbb{Q}[\mathbf{t}]$ by the proof of Theorem 8.12, and we obtain the advertised subvariety. □

9.4. Iterative approach to Rankin-Cohen brackets. The proof of Theorem 8.12 gives a recursive construction for the Rankin-Cohen bracket of two Jacobi forms which we review now. Suppose Jacobi forms ϕ, ϕ' have respective weights $k, k' \geq 2$ and indices $m, m' > 0$. The zero-th Rankin-Cohen bracket is just their product:

$$[\phi, \phi']_{(0)}^{\mathbf{t}} = \phi \cdot \phi' \quad \text{is a Jacobi form of weight } k + k' \text{ and index } m + m'.$$

We observe that the top terms cancel in $\Upsilon(\phi)\Upsilon(\phi')$ and $\Upsilon(\phi\phi')$ and the first Rankin-Cohen bracket $[\phi, \phi']_{(1)}^{\mathbf{t}}$ is the new top coefficient of their difference. Explicitly, both

$$\Upsilon_{k,(m-\mathbf{t},\mathbf{t})}(\phi) \cdot \Upsilon_{k',(-\mathbf{t},\mathbf{t}+m')}(\phi') \quad \text{and} \quad \Upsilon_{k+k', (m-\mathbf{t},\mathbf{t}+m')}(\phi\phi')$$

are $(\phi\phi')\partial^{-k-k'}$ modulo lower order terms, i.e., modulo $\Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_j))_{-k-k'-1}$, hence

$$(9.5) \quad \Upsilon(\phi) \cdot \Upsilon(\phi') - \Upsilon(\phi\phi') := \Upsilon_{k,(m-\mathbf{t},\mathbf{t})}(\phi) \cdot \Upsilon_{k',(-\mathbf{t},\mathbf{t}+m')}(\phi') - \Upsilon_{k+k', (m-\mathbf{t},\mathbf{t}+m')}(\phi\phi')$$

is just $\phi''\partial^{-k-k'-1}$ modulo $\Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_j))_{-k-k'-2}$ for some unique $\phi'' := [\phi, \phi']_{(1)}^{\mathbf{t}}$ in $\mathcal{O}(\mathbb{H}_j)$:

$$[\phi, \phi']_{(1)}^{\mathbf{t}} := \pi_{-k-k'-1} \left(\Upsilon_{k,(m-\mathbf{t},\mathbf{t})}(\phi) \cdot \Upsilon_{k',(-\mathbf{t},\mathbf{t}+m')}(\phi') - \Upsilon_{k+k', (m-\mathbf{t},\mathbf{t}+m')}(\phi\phi') \right).$$

Using Υ , we lift $[\phi, \phi']_{(1)}^{\mathbf{t}}$ to a pseudodifferential operator whose top term cancels with that of Eq. (9.5) and set $[\phi, \phi']_{(2)}^{\mathbf{t}}$ to be the new top coefficient of the difference. We continue inductively, subtracting off $\Upsilon_{k+k'+i, (m-\mathbf{t},\mathbf{t}+m')}([\phi, \phi']_{(i)}^{\mathbf{t}})$ at each step of the recursion, using the projection map π to pick off the top coefficient at each step. The proof of Theorem 8.12 thus gives the next corollary.

Corollary 9.6. For Jacobi forms ϕ, ϕ' , the Rankin-Cohen bracket may be given iteratively by

$$[\phi, \phi']_{(v)} := \pi \left(\Upsilon(\phi) \cdot \Upsilon(\phi') - \sum_{j=0}^{v-1} \Upsilon([\phi, \phi']_{(j)}) \right),$$

i.e., for ϕ, ϕ' of respective weights $k, k' \geq 2$ in $\mathbb{Z}$ and indices $m, m' > 0$ in $\mathbb{Z}$, and any parameter $\mathbf{t}$ in $\mathbb{C}$,

$$[\phi, \phi']_{(v)}^{\mathbf{t}} := \pi \left(\Upsilon_{k, (m-\mathbf{t}, \mathbf{t})}(\phi) \cdot \Upsilon_{k', (-\mathbf{t}, \mathbf{t}+m')}(\phi') - \sum_{j=0}^{v-1} \Upsilon_{k+k'+j, (m-\mathbf{t}, \mathbf{t}+m')}([\phi, \phi']_{(j)}^{\mathbf{t}}) \right)$$

for π the projection onto the coefficient of $\partial^{-k-k'-v}$.

Remark 9.7. We may rephrase [Theorem 9.2](#) using this iterative approach: For Jacobi forms ϕ, ϕ' ,

$$\Upsilon(\phi) \Upsilon(\phi') = \sum_{v \geq 0} \Upsilon([\phi, \phi']_{(v)}),$$

i.e., for ϕ, ϕ' of respective weights $k, k' \geq 2$ and indices $m, m' > 0$ and for any parameter $\mathbf{t} \in \mathbb{C}$,

$$\Upsilon_{k, (m-\mathbf{t}, \mathbf{t})}(\phi) \cdot \Upsilon_{k', (-\mathbf{t}, \mathbf{t}+m')}(\phi') = \sum_{v \geq 0} \Upsilon_{k+k'+v, (m-\mathbf{t}, \mathbf{t}+m')}([\phi, \phi']_{(v)}^{\mathbf{t}}).$$

9.8. Explicit formulas for the initial brackets. To give the first and second brackets explicitly, we use the constants

$$c_{n, k, (m_l, m_r)} := \frac{2m_r(k+n-1)(k+n-2)}{(m_l + m_r)n(3-n-2k)}$$

and abbreviate

$$a_n = c_{n, k, (m-\mathbf{t}, \mathbf{t})}, \quad a'_n = c_{n, k', (-\mathbf{t}, \mathbf{t}+m')}, \quad b_n = c_{n, k+k', (m-\mathbf{t}, \mathbf{t}+m')}, \quad d = c_{1, k+k'+1, (m-\mathbf{t}, \mathbf{t}+m')}.$$

Then for Jacobi forms ϕ and ϕ' of respective weights $k, k' \geq 2$ and indices $m, m' > 0$, the first bracket is

$$\begin{aligned} [\phi, \phi']_{(1)}^{\mathbf{t}} &= (a'_1 - b_1 - k) \phi \frac{\partial \phi'}{\partial z} + (a_1 - b_1) \phi' \frac{\partial \phi}{\partial z} \\ &= -\frac{km'(m-\mathbf{t}) + k'm(m'+\mathbf{t})}{m'(m+m')} \phi \frac{\partial \phi'}{\partial z} + \frac{km'(m-\mathbf{t}) + k'm(m'+\mathbf{t})}{m(m+m')} \phi' \frac{\partial \phi}{\partial z} \\ &= -\frac{km'(m-\mathbf{t}) + k'm(m'+\mathbf{t})}{mm'(m+m')} (m \phi \frac{\partial \phi'}{\partial z} + m' \phi' \frac{\partial \phi}{\partial z}). \end{aligned}$$

The second bracket is

$$\begin{aligned} [\phi, \phi']_{(2)}^{\mathbf{t}} &= \left(\frac{1}{2}k(k+1) - ka'_1 + a'_1a'_2 - b_1b_2 - d(a'_1 - b_1 - k) \right) \phi \frac{\partial^2 \phi'}{\partial z^2} \\ &\quad + \left(a_2(a_1 + \frac{1}{2}) - b_2(b_1 + \frac{1}{2}) - d(a_1 - b_1) \right) \phi' \frac{\partial^2 \phi}{\partial z^2} \\ &\quad + \left(a_1a'_1 - a_1(k+1) - b_2(2b_1+1) - d(a_1 + a'_1 - 2b_1 - k) \right) \frac{\partial \phi}{\partial z} \frac{\partial \phi'}{\partial z} \\ &\quad + 4\pi i (\mathbf{t}a'_2 + (m-\mathbf{t})b_2) \phi \frac{\partial \phi'}{\partial \bar{z}} \\ &\quad + 4\pi i (\mathbf{t} - m)(a_2 - b_2) \phi' \frac{\partial \phi}{\partial \bar{z}}. \end{aligned}$$

In terms of the heat operator $\mathbb{L}_m = 8\pi i m \frac{\partial}{\partial \bar{z}} - \frac{\partial^2}{\partial z^2}$, this is just

$$\begin{aligned} [\phi, \phi']_{(2)}^{\mathbf{t}} &= \left(\frac{1}{2}k(k+1) - ka'_1 + a'_1a'_2 - b_1b_2 - d(a'_1 - b_1 - k) \right) \phi \frac{\partial^2 \phi'}{\partial z^2} \\ &\quad + \left(a_1a_2 - b_1b_2 - d(a_1 - b_1) \right) \phi' \frac{\partial^2 \phi}{\partial z^2} \\ &\quad + \left(a_1a'_1 - a_1(k+1) - 2b_1b_2 - d(a_1 + a'_1 - 2b_1 - k) \right) \frac{\partial \phi}{\partial z} \frac{\partial \phi'}{\partial z} \\ &\quad - \frac{1}{2}a'_2 \phi \mathbb{L}_{-\mathbf{t}}(\phi') - \frac{1}{2}a_2 \phi' \mathbb{L}_{m-\mathbf{t}}(\phi) + \frac{1}{2}b_2 \mathbb{L}_{m-\mathbf{t}}(\phi \phi'). \end{aligned}$$

APPENDIX A. JACOBI LIE ALGEBRA AND FINDING THE CASIMIR OPERATOR

We explain here the theory behind the construction of the Casimir operator $\mathcal{C}_{k,(m_l,m_r)}$ in **Definition 6.7**. We use the action of the Jacobi Lie algebra $\mathfrak{g}$ and identify covariant operators used to construct $\mathcal{C}_{k,(m_l,m_r)}$ from a Casimir element Ω in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ for the complexified Lie algebra $\mathfrak{g}$ of the real Jacobi Lie group G .

A.1. Lie algebra of the extended real Jacobi group. We identify $\mathfrak{g}$ with the tangent space of G at the identity so that

$$\mathfrak{g} = \{(M, X, \kappa) : M \in \mathfrak{sl}_2(\mathbb{C}), X \in \mathbb{C}^2, \kappa \in \mathbb{C}\}$$

with Lie bracket given by (see [7])

$$[(M, X, \kappa), (M', X', \kappa')] = (MM' - M'M, XM' - X'M, 2\det\begin{bmatrix} X \\ X' \end{bmatrix})$$

and exponential map given in Conley and Raum [18]. Here $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{H}_3$ where $\mathfrak{H}_3$ is the complexified Lie algebra of the real Heisenberg Lie group $H_3(\mathbb{R})$. (Note that Bringmann, Conley, and Richter [7] consider the Lie algebra of the extended complex Jacobi Lie group instead.) The Lie algebra $\mathfrak{g}$ has $\mathbb{C}$ -basis

$$\begin{aligned} E &= \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, (0, 0), 0\right), & F &= \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, (0, 0), 0\right), & H &= \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, (0, 0), 0\right), \\ e &= \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, (0, 1), 0\right), & f &= \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, (1, 0), 0\right), & Z &= \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, (0, 0), 1\right). \end{aligned}$$

A.2. Alternate basis of Lie algebra. We use an alternate $\mathbb{C}$ -basis of $\mathfrak{g}$ from [18] (see [3]) to define covariant operators (as these span the characters of the compact group $SO_2(\mathbb{R})$ embedded in the Jacobi group acting via the adjoint representation). Define

$$\tilde{E} = \frac{1}{2}(H + i(E + F)), \quad \tilde{e} = \frac{1}{2}(f + ie), \quad \tilde{H} = i(F - E), \quad \tilde{F} = \frac{1}{2}(H - i(E + F)), \quad \tilde{f} = \frac{1}{2}(f - ie), \quad \tilde{Z} = \frac{1}{2}iZ.$$

A.3. Lie algebra action. Suppose the connected Lie group G acts smoothly on the right on some complex vector space W with action denoted via a slash operator: $(w, g) \mapsto w|g$ for w in W , g in G . Recall that the *differential action* on W of the complexified Lie algebra $\mathfrak{g}$ of G given by

$$w|X = \left. \frac{d}{ds} \right|_{s=0} (w|\exp(sX)) \quad \text{for } w \in W, X \in \mathfrak{g}$$

defines a Lie algebra representation of $\mathfrak{g}$. We compute the explicit action of $\mathfrak{g}$ on $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_j))$ at the marked point $(i, 0)$ of $\mathbb{H}_j$ to simplify calculations. These generalize actions given in Berndt and Schmidt [3, Section 3.5] on $C^\infty(\mathbb{H}_j)$.

Lemma A.4. Fix some k, m_l, m_r in $\mathbb{C}$. At $(\tau, z) = (i, 0)$, for slash action $|_{k,(m_l,m_r)}$ denoted simply by $|$,

$$\begin{aligned} \phi \partial^n | E &= (\phi_\tau + \phi_{\bar{\tau}}) \partial^n, & \phi \partial^n | F &= \phi_\tau + \phi_{\bar{\tau}} + i(n-k) \phi \partial^n - 2\pi i m_r n(n-1) \phi \partial^{n-2}, \\ \phi \partial^n | e &= (\phi_z + \phi_{\bar{z}}) \partial^n, & \phi \partial^n | f &= (i\phi_z - i\phi_{\bar{z}}) \partial^n + 4\pi i m_r n \phi \partial^{n-1}, \\ \phi \partial^n | H &= (2i\phi_\tau - 2i\phi_{\bar{\tau}} - (n-k)\phi) \partial^n, & \phi \partial^n | Z &= 2\pi i(m_l + m_r) \phi \partial^n. \end{aligned}$$

Direct computation using **Lemma A.4** gives the following actions for the alternate basis of $\mathfrak{g}$.

Lemma A.5. Fix some k, m_l, m_r in $\mathbb{C}$. At $(\tau, z) = (i, 0)$, for slash action $|_{k,(m_l,m_r)}$ denoted simply by $|$,

$$\begin{aligned} \phi \partial^n | \tilde{E} &= (2i\phi_\tau + (k-n)\phi) \partial^n + \pi m_r n(n-1) \phi \partial^{n-2}, & \phi \partial^n | \tilde{F} &= -2i\phi_{\bar{\tau}} \partial^n - \pi m_r n(n-1) \phi \partial^{n-2}, \\ \phi \partial^n | \tilde{e} &= i\phi_z \partial^n + 2\pi i m_r n \phi \partial^{n-1}, & \phi \partial^n | \tilde{f} &= -i\phi_{\bar{z}} \partial^n + 2\pi i m_r n \phi \partial^{n-1}, \\ \phi \partial^n | \tilde{H} &= (k-n)\phi \partial^n + 2\pi m_r n(n-1) \phi \partial^{n-2}, & \phi \partial^n | \tilde{Z} &= -\pi(m_l + m_r) \phi \partial^n. \end{aligned}$$

A.6. Covariant differential operators. We follow the strategy of [18] (see also Conley and Dahal [17]) which extends Helgason's treatment (see [27]) of the reductive case to the nonreductive Jacobi group. We consider operators analogous to the covariant differential operators $\text{CDO}_{\gamma, \gamma'}$ constructed in [18] and [7]. For the next definition, we fix a choice of section $s : \mathbb{H}_J \rightarrow G$ to the map $G \rightarrow \mathbb{H}_J$, $g \mapsto g(i, 0)$, with $s(i, 0) = 1_G$ the identity, so $s(\tau, z)(i, 0) = (\tau, z)$. We use the adjoint action of g in G on X in $\mathfrak{g}$ denoted by $gXg^{-1} = \text{ad}_g(X) = \frac{d}{dt}|_{t=0} g \exp(tX) g^{-1}$.

Definition A.7. For any X in $\mathfrak{g}$, define an operator

$$\begin{aligned} \mathcal{D}_{k, (m_l, m_r)}(X) : \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)) &\rightarrow \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)) && \text{by} \\ \phi(\tau, z) \partial^n &\mapsto (\phi \partial^n |_{k, (m_l, m_r)} gXg^{-1})(\tau, z) \end{aligned}$$

for all $(\tau, z) \in \mathbb{H}_J$ and $\phi \in C^\infty(\mathbb{H}_J)$, for $g = s(\tau, z) \in G$ (so that $g(i, 0) = (\tau, z)$).

Proposition A.8. *The raising, lowering, and constant operators on $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ of Definition 6.3 arise from the alternate basis $\tilde{E}, \tilde{F}, \tilde{e}, \tilde{f}, \tilde{H}, \tilde{Z}$ of the Lie algebra $\mathfrak{g}$: For any k, m_l, m_r in $\mathbb{Z}$, with subscript $k, (m_l, m_r)$ suppressed,*

$$\begin{aligned} \mathcal{D}(\tilde{E}) &= R, & \mathcal{D}(\tilde{F}) &= L, \\ \mathcal{D}(\tilde{e}) &= R^J, & \mathcal{D}(\tilde{f}) &= L^J, \\ \mathcal{D}(\tilde{H}) &= C, & \mathcal{D}(-2i\tilde{Z}) &= C^J \end{aligned}$$

as operators $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)) \rightarrow \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$.

Proof. We use results from [18] describing covariant operators. Although the Jacobi Lie algebra $\mathfrak{g}$ is not reductive, $\mathbb{H}_J \cong G/K$ is a *reductive coset space* (as described by Helgason [26, 27]) for (noncompact) K the stabilizer in G of $(i, 0)$ in $\mathbb{H}_J$ since $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ with $\mathfrak{k}$ the complexified Lie algebra of K and $\mathfrak{m}$ a Lie subalgebra invariant under the action of $\text{Ad}_G(K)$: Here (see [18], [7], and [3]),

$$\begin{aligned} K &= \{(M, 0, \kappa) : M \in \text{SO}_2(\mathbb{R}), \kappa \in \mathbb{R}\}, \\ \mathfrak{k} &= \{(M, 0, \kappa) : M \in \mathfrak{so}_2(\mathbb{C}), \kappa \in \mathbb{C}\} = \mathbb{C}\text{-span}\{\tilde{H}, \tilde{Z}\}, \\ \mathfrak{m} &= \{(M, X, 0) : M \in \mathfrak{sl}_2(\mathbb{C}), M = M^t, X \in \mathbb{C}^2\} = \mathbb{C}\text{-span}\{\tilde{E}, \tilde{F}, \tilde{e}, \tilde{f}\}, \end{aligned}$$

with $\tilde{H}$ acting diagonally on $\mathfrak{m}$ with respect to an eigenbasis $\tilde{E}, \tilde{F}, \tilde{e}, \tilde{f}$ with respective $\tilde{H}$ -weights $2, -2, 1, -1$. We view $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ as $C^\infty(G) \otimes_K V$ for ring of Laurent series $V = \mathbb{C}((\partial^{-1}))$.

By Proposition 6.4, R is a covariant operator that raises the weight by 2, see Remark 6.5. The operator $\mathcal{D}(\tilde{E})$ also defines a covariant operator that raises the weight by 2 by Corollary 5.8 of [18], see the argument in Section 5.4 there. In other words, both operators lie in the vector space of equivariant maps

$$\left(\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)), |_{k, (m_l, m_r)} \right)^G \longrightarrow \left(\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)), |_{k+2, (m_l, m_r)} \right)^G.$$

By [18, Corollary 5.8], this space of covariant operators is 1-dimensional over $\mathbb{C}$, and thus R and $\mathcal{D}(\tilde{E})$ agree up to a constant. The operator $\mathcal{D}(\tilde{E})$ at the point $(\tau, z) = (i, 0)$ in $\mathbb{H}_J$ is just slashing by $\tilde{E}$ since we take the adjoint action of $g = 1_G$ on $\tilde{E}$. Upon inspection using Lemma A.5, we see that $(- | \tilde{E})$ and R agree at the point $(i, 0)$, hence this constant is 1 and $R = \mathcal{D}(\tilde{E})$. Analogous reasoning applies to each of the other indicated operators using the representation theory arguments of [18] and [7]. $\square$

Remark A.9. In practice, to compute the covariant operators $\mathcal{D}(X) = \mathcal{D}_{k, (m_l, m_r)}(X)$ for Lie algebra elements $X = \tilde{E}, \tilde{F}, \tilde{H}, \tilde{e}, \tilde{f}, \tilde{Z}$ that give the raising and lowering operators R, L, C, L^J, R^J, C^J , we use the section $s : \mathbb{H}_J \rightarrow G$ given by

$$(\tau, z) \mapsto g = \frac{1}{\sqrt{y}} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, (v, u, 0) \in G,$$

for $\tau = x + iy$, $z = u + iv$, so that $g(i, 0) = (\tau, z)$. We compute $\left((\phi \partial_z^n | g) | X\right)(i, 0)$ for fixed (τ, z) and then find the functions $\phi'_j : \mathbb{H}_J \rightarrow \mathbb{C}$ (depending on X) for which

$$\left((\phi \partial_z^n | g) | X\right)(i, 0) = \left(\sum_j \phi'_j \partial_z^{n-j}\right)(i, 0).$$

After slashing with g^{-1} , we may evaluate at (τ, z) to obtain $(\mathcal{D}(X) \phi \partial_z^n)(\tau, z) = (\sum_j \phi'_j \partial_z^{n-j} | g^{-1})(\tau, z)$.

A.10. Casimir element. Our proof of the existence of a splitting map Υ as in [Theorem 7.8](#) relied upon construction of an operator on $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ that acts by a scalar on each graded piece. Hence we look at the center of the universal enveloping algebra with an eye to Schur's Lemma. We explain here how we use the Casimir element for the extended complex Jacobi Lie algebra $\mathfrak{g}$ to find $\mathcal{C}_{k, (m_l, m_r)}$.

The center, $Z(\mathcal{U}(\mathfrak{g}))$, of the universal enveloping algebra of $\mathfrak{g}$ is generated by two elements: Z and

$$\Omega := ZH^2 - 3ZH + 4ZEF - Hef + 2ef + Ef^2 - e^2F$$

(see [\[18\]](#)). The element Z merely acts by a scalar multiple of the index on $\mathcal{O}(\mathbb{H}_J)$ and $\Psi\text{DO}(\partial, \mathcal{O}(\mathbb{H}_J))$, see [Lemma A.4](#). Hence we regard Ω as the *Casimir element* and rewrite it in terms of the alternate basis of $\mathfrak{g}$ (see [Appendix A.2](#)) using a Gröbner basis computed with the help of software extension Plural [\[34\]](#) of Singular [\[38\]](#):

$$(A.11) \quad \Omega = -8i\tilde{Z}\tilde{E}\tilde{F} + 2i(\tilde{e}^2\tilde{F} - \tilde{E}\tilde{f}^2) + 2i\tilde{e}\tilde{f}(\tilde{H} - 2) + 6i\tilde{Z}\tilde{H} - 2i\tilde{Z}\tilde{H}^2.$$

(Compare to Proposition 2.7 and the remark before the proof of Proposition 2.6 in [\[18\]](#).)

A.12. Casimir operator. As $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ is a Lie algebra representation, we may use the Poincaré-Birkhoff-Witt Theorem to extend the function

$$\mathfrak{g} \times \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)) \longrightarrow \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)), \quad (X, \phi \partial^n) \longmapsto \mathcal{D}_{k, (m_l, m_r)}(X)(\phi \partial^n)$$

to a linear map

$$\mathcal{U}(\mathfrak{g}) \times \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)) \longrightarrow \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)),$$

i.e., a linear map from $\mathcal{U}(\mathfrak{g})$ to the ring of operators on $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ under composition. We just choose an ordering of the tilde basis of $\mathfrak{g}$, say $\tilde{Z} < \tilde{E} < \tilde{e} < \tilde{F} < \tilde{f} < \tilde{H}$, and fix a basis for $\mathcal{U}(\mathfrak{g})$ of monomials in these elements in this order. The Casimir element Ω then maps to a *Casimir operator*

$$\mathcal{C}_{k, (m_l, m_r)} := \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J)) \longrightarrow \Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$$

given as the composition of order 1 operators (applied left to right):

$$\begin{aligned} \mathcal{C}_{k, (m_l, m_r)} = & -8i\mathcal{D}(\tilde{Z})\mathcal{D}(\tilde{E})\mathcal{D}(\tilde{F}) + 2i\mathcal{D}(\tilde{e})^2\mathcal{D}(\tilde{F}) - 2i\mathcal{D}(\tilde{E})\mathcal{D}(\tilde{f})^2 \\ & + 2i\mathcal{D}(\tilde{e})\mathcal{D}(\tilde{f})(\mathcal{D}(\tilde{H}) - 2) + 6i\mathcal{D}(\tilde{Z})\mathcal{D}(\tilde{H}) - 2i\mathcal{D}(\tilde{Z})\mathcal{D}(\tilde{H})^2, \end{aligned}$$

giving [Definition 6.7](#) by [Proposition A.8](#). One may employ an argument using a vector bundle over the homogeneous space G/K for K the stabilizer in G of $(i, 0)$ in $\mathbb{H}_J$ reminiscent of that in [\[18, 17\]](#) to argue that the operator $\mathcal{C} = \mathcal{C}_{k, (m_l, m_r)}$ is G -equivariant for all k in $\mathbb{Z}$ and m_l, m_r in $\mathbb{C}$, see [Theorem 6.8](#). We have omitted this lengthy argument since the covariance of the order 1 operators can be checked directly.

Remark A.13. Our Casimir operator is not the usual Casimir operator determined by the Casimir element in historic Lie theory terms. Traditionally, the Casimir operator would be found using Helgason's theory of differential operators (see [\[26, Theorem 10\]](#)) which gives a linear map from $\mathcal{U}(\mathfrak{g})$ to the set of operators on $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$ using *symmetrization* combined with partial differentiation,

$$\phi \partial^n | [Y_1 \cdots Y_l] = \frac{d}{ds_1} \Big|_{s_1=0} \cdots \frac{d}{ds_l} \Big|_{s_l=0} \left(\phi \partial^n | e^{s_1 Y_1 + \cdots + s_l Y_l} \right) \quad \text{for } Y_i \in \mathfrak{g}.$$

We instead use a *composition* of operators of degree 1, in parallel to a customary approach in modular forms.

APPENDIX B. NONCOMMUTATIVE MULTIPLICATION OF PSEUDODIFFERENTIAL OPERATORS

We include the advertised closed formula in [Section 4](#) for recording the noncommutative multiplication of pseudodifferential operators.

Lemma 4.7. In the noncommutative ring $\Psi\text{DO}(\partial, C^\infty(\mathbb{H}_J))$, for all $c_1, c_2 \in \mathbb{C}$ and $n \in \mathbb{Z}$,

$$(\partial + c_1 z + c_2)^n = \sum_{0 \leq p} \sum_{\substack{0 \leq t, s, r \\ t+s+2r=p}} 2^r \binom{n}{t, s, 2r, n-p} \frac{\Gamma(r + \frac{1}{2})}{\Gamma(\frac{1}{2})} c_1^r c_2^s (c_1 z)^t \partial^{n-p}.$$

Proof. We first show the formula for $n \geq 0$ via a generating series in n using the noncommutative multiplication in the ring of regular differential operators (see [Eq. \(3.3\)](#)) to move all ∂ to the far right:

$$(\partial + c_1 z + c_2)^n = \sum_{\substack{0 \leq j, t, s, r \\ j+t+s+2r=n}} c_{j,t,s,r} c_1^r c_2^s (c_1 z)^t \partial^j$$

for some complex scalars $c_{t,j,r,s}$. We encode these coefficients in the generating series

$$F(u, v, x, y) = \sum_{0 \leq j, t, s, r} c_{j,t,s,r} u^r v^s x^t y^j \quad \text{with commuting complex variables } u, v, x, y$$

with an initial condition $F(0, v, x, y) = (1 - v - x - y)^{-1}$, as contributions from $r = 0$ can be computed as in a commutative ring. We may regard F as a formal expression in noncommutative variables so that

$$F(c_1, c_2, c_1 z, \partial) = \sum_{0 \leq n} (\partial + c_1 z + c_2)^n$$

and expand

$$F(c_1, c_2, c_1 z, \partial) - 1 = \sum_{0 \leq n} (\partial + c_1 z + c_2)^{n+1} = (\partial + c_1 z + c_2) \sum_{0 \leq n} (\partial + c_1 z + c_2)^n$$

as

$$\begin{aligned} & (\partial + c_1 z + c_2) \sum_{\substack{0 \leq j, t, s, r \\ j+t+s+2r=n}} c_{j,t,s,r} c_1^r c_2^s (c_1 z)^t \partial^j \\ &= \sum_{\substack{0 \leq j, t, s, r \\ j+t+s+2r=n}} c_{j,t,s,r} (c_1^r c_2^s (c_1 z)^t \partial^{j+1} + t c_1^{r+1} c_2^s (c_1 z)^{t-1} \partial^j + c_1^r c_2^s (c_1 z)^{t+1} \partial^j + c_1^r c_2^{s+1} (c_1 z)^t \partial^j). \end{aligned}$$

We return to the commutative setting and use the Poincaré-Birkhoff-Witt property to rewrite this last expression as

$$yF(u, v, x, y) + xF(u, v, x, y) + vF(u, v, x, y) + u \frac{\partial}{\partial x} F(u, v, x, y)$$

formally evaluated at $(u, v, x, y) = (c_1, c_2, c_1 z, \partial)$. We conclude that

$$F(u, v, x, y) - 1 = (y + x + v + u \frac{\partial}{\partial x}) F(u, v, x, y)$$

and $(1 - x - y - v - u \frac{\partial}{\partial x}) F(u, v, x, y) = 1$.

To find F , we find a power series $\tilde{F}(u, w)$ in u with coefficients that are Laurent series in w solving the differential equation $(w + u \partial_w) \tilde{F}(u, w) = 1$ with initial condition $\tilde{F}(0, w) = w^{-1}$. We then obtain F by setting $F(u, v, x, y) = \tilde{F}(u, 1 - x - y - v)$. We write $\tilde{F}(u, w) = \sum_{0 \leq r} h_r(w) u^r$, and we notice that $h_0(w) = w^{-1}$ and $wh_r(w) + \partial_w h_{r-1}(w) = 0$ (for $r > 0$). We conclude that

$$h_r(w) = 2^r \frac{\Gamma(r + \frac{1}{2})}{\Gamma(\frac{1}{2})} w^{-1-2r}.$$

Then since $n = j + t + s + 2r$,

$$\begin{aligned} F(u, v, x, y) &= \sum_{0 \leq r} 2^r \frac{\Gamma(r + \frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{u^r}{(1 - x - y - v)^{1+2r}} \\ &= \sum_{0 \leq t, j, r, s} (-1)^{t+j+s} 2^r \binom{-1-2r}{t, j, s, -1-n} \frac{\Gamma(r + \frac{1}{2})}{\Gamma(\frac{1}{2})} u^r v^s x^t y^j. \end{aligned}$$

The second equality follows from generalized multinomial series: Newton's binomial series $(1 - \tilde{w})^\alpha = \sum_{0 \leq k} \binom{\alpha}{k} (-\tilde{w})^k$ holds for $\alpha \in \mathbb{C}$ and $|\tilde{w}| < 1$ (see [29]), and we set $\tilde{w} = x + y + v$ to express $(1 - x - y - v)^{-1-2r}$ via the multinomial coefficients $\binom{-1-2r}{t, j, s, -1-n}$, see Eq. (4.6) and [21, (26.4.2)]. Euler's reflection formula for the Gamma function implies that

$$(-1)^{t+j+s} \binom{-1-2r}{t, j, s, -1-n} = \binom{n}{t, j, s, 2r},$$

and hence in the ring of ordinary differential operators

$$(\partial + c_1 z + c_2)^n = \sum_{0 \leq p \leq n} \sum_{\substack{0 \leq t, r, s \\ t+s+2r=p}} 2^r \binom{n}{t, s, 2r, n-p} \frac{\Gamma(r + \frac{1}{2})}{\Gamma(\frac{1}{2})} c_1^r c_2^s (c_1 z)^t \partial^{n-p}.$$

To obtain a description of $(\partial + c_1 z + c_2)^n$ when $n < 0$, we observe that the coefficients in a similar formula must be polynomials in n , which follows by recursion after expanding $(\partial + c_1 z + c_2)^n (\partial + c_1 z + c_2)^{-n} = 1$. We may obtain this formula for $n < 0$ by extending the last sum given for $n \geq 0$ to a sum over all $0 \leq p$ noting that the multinomial coefficient is zero for $p > n$, and then, for each fixed p , extending the coefficients on the right hand side as polynomials in $n \geq 0$ to all n in $\mathbb{Z}$. □

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