AUTOMORPHISMS OF QUANTUM POLYNOMIAL RINGS AND DRINFELD HECKE ALGEBRAS

A.V. SHEPLER AND C. UHL

Abstract. We consider quantum (skew) polynomial rings and observe that their graded automorphisms coincide with those of quantum exterior algebras. This allows us to define a quantum determinant giving a homomorphism of groups acting on quantum polynomial rings. We use quantum subdeterminants to classify the resulting Drinfeld Hecke algebras for the symmetric group, other infinite families of Coxeter and complex reflection groups, and mystic reflection groups (which satisfy a version of the Shephard-Todd-Chevalley Theorem). This direct combinatorial approach replaces the technology of Hochschild cohomology used by Naidu and Witherspoon over fields of characteristic zero and allows us to extend some of their results to fields of arbitrary characteristic and also locate new deformations of skew group algebras.

1. Introduction

One challenge to investigating noncommutative rings remains some mystery surrounding their automorphism groups. We consider here *quantum polynomial rings*, also sometimes known as *quantum symmetric algebras* or *skew polynomial rings*. For a finite-dimensional vector space $V \cong \mathbb{F}^n$ over a field $\mathbb{F}$, the noncommutative algebra $S_Q(V)$ is generated by a basis $v_1, \dots, v_n$ of V with multiplication $v_j v_i = q_{ij} v_i v_j$ for some quantum scalars $Q = \{q_{ij}\} \subset \mathbb{F}$ with $q_{ii} = 1$, $q_{ij} = q_{ji}^{-1}$. One may view $S_Q(V)$ as the coordinate ring of the n-dimensional quantum affine space. We take $S_Q(V)$ as a graded algebra with $\deg v_i = 1$ for all i.

In the nonquantum setting, every graded automorphism of the commutative polynomial ring $S(V) \cong \mathbb{F}[v_1, \dots, v_n]$ defines a general linear transformation of V and vice versa. This fails in the noncommutative setting: Every graded automorphism of $S_Q(V)$ defines an element of $\mathrm{GL}(V)$, but every not every element of $\mathrm{GL}(V)$ extends to a graded automorphism. The graded automorphisms of quantum polynomial rings have been classified in low dimension (see [3] and [19]). Kirkman, Kuzmanovich, and Zhang [18] investigated finite groups of these automorphisms satisfying a version of the Shephard-Todd-Chevalley Theorem. More recently, Bao, He, and Zhang [5] showed a version of the Auslander Theorem for these groups. Related investigations include [32], [10], [9], [8], [2], [4].

For a finite group G of graded automorphisms of a quantum polynomial ring $S_Q(V)$, deformations of the natural semidirect product algebra $S_Q(V) \rtimes G$ (skew group algebra) include quantum Drinfeld Hecke algebras. These analogs of graded affine Hecke algebras and symplectic reflection algebras can be studied using Hochschild cohomology, but previous results have depended on an extra hypothesis that the given group G act not only on $S_Q(V)$ but also on the associated quantum exterior algebra $\bigwedge_Q(V)$ (see [24], [27], [25], [31]). In

---

Date: August 26, 2021.

Mathematica Subject Classification (2020). 16S36, 16W20, 16S35, 16S80, 16E40.

Key words. Automorphism groups, quantum polynomial ring, skew polynomial ring, quantum determinant, deformations, graded Hecke algebras, Drinfeld Hecke algebras, Hochschild cohomology.

The first author was partially supported by Simons grant 429539 and is the corresponding author.

addition, many computations in Hochschild cohomology have relied on the characteristic $\mathrm{char}(\mathbb{F})$ of the underlying field not dividing $|G|$.

Thus one asks how the group $\mathrm{Aut}_{\mathrm{gr}}(S_Q(V))$ of graded automorphisms of the quantum polynomial ring compares with that of the associated quantum exterior algebra,

$$\textstyle\bigwedge_Q(V) = \mathbb{F}\text{-span}\{v_{i_1} \wedge_Q \cdots \wedge_Q v_{i_m} : 1 \leq i_1, \cdots, i_m \leq n\},$$

with quantum exterior product $v_j \wedge_Q v_i = -q_{ij}\ v_i \wedge_Q v_j$. The classification of groups acting on quantum polynomial rings in low dimension (see [19, Theorem 11.1]) implies that $\mathrm{Aut}_{\mathrm{gr}}(S_Q(V)) = \mathrm{Aut}_{\mathrm{gr}}(\bigwedge_Q(V))$ for $\dim_{\mathbb{F}} V \leq 3$. Computer calculations using [15] and [23] verify the same when $\dim_{\mathbb{F}} V = 4$. We show a more general fact: For any set of quantum scalars Q and any finite-dimensional $\mathbb{F}$-vector space V,

$$\mathrm{Aut}_{\mathrm{gr}}\big(\textstyle\bigwedge_Q(V)\big) = \mathrm{Aut}_{\mathrm{gr}}\big(S_Q(V)\big). \tag{1.1}$$

We make no assumptions on the characteristic of $\mathbb{F}$ except that $\mathrm{char}(\mathbb{F}) \neq 2$. This result implies that previous tools in characteristic 0 of [25] and [31] using Koszul resolutions to explore some Hochschild cohomology of skew group algebras apply to all finite groups of graded automorphisms acting on $S_Q(V)$; extra hypotheses that groups act on both the symmetric and exterior quantum algebras are not needed.

Observation (1.1) also allows us to to define a *quantum determinant* that behaves in some ways like the usual determinant for linear groups. For graded transformations acting on the quantum exterior algebra by graded automorphisms, we verify that this quantum determinant is simply the scalar by which the quantum volume form changes. As a direct corollary, we see that this notion of quantum determinant defines a homomorphism of matrix groups acting on quantum polynomial rings. (Note that this formulation of quantum determinant is defined for any matrix with entries in $\mathbb{F}$; it is not the notion usually employed for quantum matrices alá Manin [22].)

As an application of these ideas, we explore deformations of $S_Q(V) \rtimes G$ for G a finite group of graded automorphisms that are modeled on Lusztig's graded affine Hecke algebra and symplectic reflection algebras. We classify quantum Drinfeld Hecke algebras (or "quantum graded Hecke algebras") for the infinite family of monomial reflection groups (including infinite families of Coxeter groups and complex reflection groups) and mystic reflection groups using techniques of [33]. We recover some results of Naidu and Witherspoon [25] over $\mathbb{C}$ for $\dim_{\mathbb{F}} V \geq 4$ who used Hochschild cohomology. The advantage of our approach is 4-fold. First, we bypass analysis of various cochain complexes in Hochschild cohomology. Second, we show results hold even in the modular setting when $\mathrm{char}(\mathbb{F})$ divides $|G|$. (Note that those previous calculations in Hochschild cohomology relied on $\mathrm{char}(\mathbb{F}) = 0$; the group algebra $\mathbb{F}G$ may not be semi-simple in the modular setting.) Third, we classify algebras in the delicate setting when $\dim_{\mathbb{F}} V = 3$ (certain parameters are forced to vanish in higher dimension). Fourth, we find new families of algebras when $\dim_{\mathbb{F}} V = 4$ for the complex reflection groups $G(r, r, 4)$.

Notation. We fix a vector space $V \cong \mathbb{F}^n$ over a field $\mathbb{F}$ of characteristic not 2 throughout. All algebras are associative $\mathbb{F}$-algebras. We identify the identity $1_{\mathbb{F}}$ of the field with the group identity 1_G in any group ring $\mathbb{F}G$. We use left superscripts to indicate the action of a group G on a set S, writing $s \mapsto {}^g s$ for g in G, s in S, to distinguish from the multiplication in algebras containing $\mathbb{F}G$. We also fix a set of quantum parameters $Q := \{q_{ij}\}_{1 \leq i,j \leq n}$ with $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ throughout.

Outline. In Section 2, we highlight some conditions for a finite linear group G to act on a quantum polynomial ring $S_Q(V)$ and for G to act on the associated quantum exterior algebra $\bigwedge_Q(V)$. In Section 3, we show that a linear transformation acts as a graded automorphism of $S_Q(V)$ if and only if it acts as a graded automorphism of $\bigwedge_Q(V)$. We introduce the quantum sign and quantum determinant of a matrix in Section 4 and show how to use inversions to simplify. We also show that this notion of quantum determinant is a homomorphism of groups of graded automorphisms of quantum polynomial rings. We consider quantum Drinfeld Hecke algebras in Section 5. In Sections 6 and 7, we classify these deformations for symmetric groups and the infinite family of complex reflection groups $G(r, p, n)$ (the Shephard-Todd family of monomial groups) which include the Weyl groups of type B_n/C_n and D_n. We show how to use cycle type to give quick combinatorial proofs for classification results of Naidu and Witherspoon [25] and extend results to fields of characteristic not 2. We take up the mystic reflection groups of Kirkman, Kuzmanovich, and Zhang [18] and Bazlov and Berenstein [7] in Section 8. We end in Section 9 with a quick discussion of direct sums of groups.

2. Automorphisms of quantum polynomial rings and determinants

We recall conditions describing the graded automorphisms of a quantum (or skew) polynomial ring. We fix throughout an $\mathbb{F}$-basis $v_1, \dots, v_n$ of $V \cong \mathbb{F}^n$ and assume every matrix in $\mathrm{GL}_n(\mathbb{F})$ acting on V is written with respect to this basis. We also have fixed throughout a *quantum system of parameters* (or a set of *quantum scalars*)

$$Q := \{q_{ij}\}_{1 \le i,j \le n} \subset \mathbb{F},$$

i.e., a set of nonzero scalars with $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for all i, j.

Quantum polynomial rings. The *quantum polynomial algebra* (or *skew polynomial ring*) $S_Q(V)$ is the noncommutative $\mathbb{F}$-algebra generated by $v_1, \dots, v_n$ with relations $v_j v_i = q_{ij} v_i v_j$ for all $1 \le i, j \le n$:

$$S_Q(V) = \mathbb{F}\langle v_1, \dots, v_n \rangle / (v_j v_i - q_{ij} v_i v_j : 1 \le i, j \le n).$$

Thus $S_Q(V) \cong T_{\mathbb{F}}(V)/(v_j \otimes v_i - q_{ij} v_i \otimes v_j : 1 \le i, j \le n)$ for $T_{\mathbb{F}}(V)$ the tensor algebra of V over $\mathbb{F}$. (We use the index convention of [18] and [19]). Note that the algebra $S_Q(V)$ has the *PBW property* with respect to this presentation: $S_Q(V)$ has $\mathbb{F}$-vector space basis $\{v_1^{m_1} v_2^{m_2} \cdots v_n^{m_n} : m_i \in \mathbb{Z}_{\ge 0}\}$.

Groups acting as graded automorphisms. We view $S_Q(V)$ as a graded algebra with $\deg v = 1$ for all $v \in V$. The set of graded automorphisms of $S_Q(V)$ is

$$\mathrm{Aut}_{\mathrm{gr}}\big(S_Q(V)\big) = \{h \in \mathrm{GL}(V) : {}^h v_j \, {}^h v_i = q_{ij} \, {}^h v_i \, {}^h v_j \quad \text{for } 1 \le i, j \le n\}.$$

Diagonal matrix groups on V always extend to an action by automorphisms on $S_Q(V)$, but many other group actions do not extend. When $q_{ij} = -1$ for all $i \ne j$ any subgroup of monomial matrices in $\mathrm{GL}_n(\mathbb{F})$ acts as graded automorphisms on $S_Q(V)$. Recall that a matrix is *monomial* if each row and each column has exactly one nonzero entry. Groups of monomial matrices are sometimes called *permutation groups*; they often take the form $H \rtimes \mathfrak{S}_n$ for some diagonal group H and the symmetric group $\mathfrak{S}_n$ acting by permutation of basis vectors $v_1, \dots, v_n$ of V. In fact, we identify $\mathfrak{S}_n$ with its permutation representation as $n \times n$ matrices: π in $\mathfrak{S}_n$ acts via $v_i \mapsto v_{\pi(i)}$.

The group $\mathrm{Aut_{gr}}(S_Q(V))$ has been determined for $n = 1, 2, 3$ (see [3] and [19]) and in some other cases (see [1], [2], and [4]). For example, for $n = 2$,

$$\mathrm{Aut_{gr}}\big(S_Q(\mathbb{F}^2)\big) = \begin{cases} \mathrm{GL}_2(\mathbb{F}) & \text{for } q_{12} = 1, \\ \text{Diagonal Matrices} \cong (\mathbb{F}^*)^2 & \text{for } q_{12} \neq \pm 1, \\ \text{Monomial Matrices} \subset \mathrm{GL}_2(\mathbb{F}) & \text{for } q_{12} = -1. \end{cases} \tag{2.1}$$

The next lemma can be checked directly. Recall that $Q = \{q_{ij}\}$ is fixed throughout.

Lemma 2.2. *The automorphism group* $\mathrm{Aut_{gr}}(S_Q(V))$ *can unveil some quantum scalars:*

- *If some* $g \in \mathrm{Aut_{gr}}(S_Q(V))$ *has nonzero entries in the same row in columns* i, j, *then* $q_{ij} = 1$.
- *If* $\mathrm{Aut_{gr}}(S_Q(V))$ *contains* $\mathfrak{S}_n$, *then either* $q_{ij} = -1$ *for all* $i \neq j$ *or else* $q_{ij} = 1$ *for all* i, j.
- *If* $\mathrm{Aut_{gr}}(S_Q(V))$ *contains* $\mathfrak{S}_n$ *and a nonmonomial matrix, then* $q_{ij} = 1$ *for all* i, j.

We give an example of a monomial and a non-monomial group acting.

Example 2.3. *The group* $G = \left\langle h = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \omega \end{smallmatrix}\right)\right\rangle \subset GL_3(V)$ *for* $V = \mathbb{C}^3$ *and* $\omega = e^{\frac{2\pi i}{3}} \in \mathbb{C}$ *(see Example 5.7) acts as graded automorphisms, for* $Q = \{q_{ij}\}$ *with* $q_{13} = \omega = q_{23}$, $q_{12} = -1$, *on*

$$S_Q(V) = \mathbb{C}\langle v_1, v_2, v_3 : v_2v_1 = -v_1v_2,\ v_3v_1 = \omega\, v_1v_3,\ v_3v_2 = \omega\, v_2v_3\rangle\,.$$

Example 2.4. *The group* $G = \left\langle \begin{pmatrix} -\sqrt{1-\eta^3} & \eta^2 & 0 \\ \eta & \sqrt{1-\eta^3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \subset GL_3(V)$ *for* $V = \mathbb{C}^3$ *and* $\eta = e^{\frac{2\pi i}{5}} \in \mathbb{C}$ *acts as graded automorphisms on* $S_Q(V)$ *for* $Q = \{q_{ij}\}$ *with* $q_{12} = 1$, $q_{13} = -1 = q_{23}$. *See Example 10.3.*

Quantum minor determinant. We define the *quantum minor determinant* of a matrix $h = \{h_i^k\}_{1 \le k,i \le n}$ in $\mathrm{GL}_n(\mathbb{F})$ with $h(v_k) = \sum_i h_i^k v_i$ (i.e., subscript denotes row) by

$$\det{}_{ijkl,Q}(h) = h_k^i h_l^j - q_{ij} h_l^i h_k^j\,.$$

We drop the subscript Q, writing $\det_{ijkl}$ for $\det_{ijkl,Q}$, when no confusion should arise. Straightforward computation verifies the next lemma; the one after is from [19].

Lemma 2.5. *For any matrix* $h \in GL_n(\mathbb{F})$,

$$\det{}_{ijkl}(h) - q_{lk} \det{}_{ijlk}(h) = \det{}_{lkji}(h^t) - q_{ij} \det{}_{lkij}(h^t)\,.$$

Lemma 2.6. *A matrix* $h \in GL_n(\mathbb{F})$ *acts on* $S_Q(V)$ *if and only if*

$$\det{}_{ijkl}(h) = -q_{lk} \det{}_{ijlk}(h) \quad \textit{for all}\ \ h \in G \quad \textit{and } 1 \le i,j,k,l \le n\,.$$

Quantum exterior algebra. The *quantum exterior algebra* determined by Q is

$$\bigwedge_Q(V) = \mathbb{F}\text{-span}\{v_{i_1} \wedge_Q \cdots \wedge_Q v_{i_m} : 1 \le i_1, \ldots, i_m \le n\}$$

with multiplication determined by $v_j \wedge_Q v_i = -q_{ij}\ v_i \wedge_Q v_j$ for all i, j. Formally,

$$\bigwedge_Q(V) \cong T_{\mathbb{F}}(V)/(v_j \otimes v_i + q_{ij}\, v_i \otimes v_j : 1 \le i, j \le n)\,,$$

and we consider $\bigwedge_Q(V)$ as a graded algebra with $\deg v_i = 1$ for each i. Note that $v_i \wedge_Q v_i = 0$ as $\mathrm{char}(\mathbb{F}) \neq 2$. The set of graded automorphisms of $\bigwedge_Q(V)$ is

$$\mathrm{Aut_{gr}}\big(\textstyle\bigwedge_Q(V)\big) = \{h \in \mathrm{GL}(V) :\ {}^h v_j \wedge_Q {}^h v_i = -q_{ij}\ {}^h v_i \wedge_Q {}^h v_j \ \text{ for } 1 \le i, j \le n\}.$$

A *quantum 2-form* is an element of $\bigwedge_Q^2 V^* \cong (\bigwedge_{Q^{-1}}^2 V)^*$ for $Q^{-1} = \{q_{ij}^{-1}\}$, i.e., a function $\theta : V \otimes V \to \mathbb{F}$ which is *anti-quantum-linear*:

$$\theta(v_j \otimes v_i) = -q_{ji}\,\theta(v_i \otimes v_j) \quad \text{for all } i, j\,. \tag{2.7}$$

Remark 2.8. One might ask if opposite quantum scalars are helpful in comparing automorphisms of quantum polynomial versus exterior rings. Generally they are not as often

$$\mathrm{Aut}_{\mathrm{gr}}\big(S_{Q'}(V)\big) \not\subset \mathrm{Aut}_{\mathrm{gr}}\big(\textstyle\bigwedge_Q(V)\big) \quad \text{or} \quad \mathrm{Aut}_{\mathrm{gr}}\big(\textstyle\bigwedge_Q(V)\big) \not\subset \mathrm{Aut}_{\mathrm{gr}}\big(S_{Q'}(V)\big)$$

for $Q' = \{q'_{ij}\}$ with $q'_{ij} = -q_{ij}$ for $i \neq j$ and $q'_{ii} = 1$. For example, take $n = 2$: If $q_{12} = -1$, every subgroup of $\mathrm{GL}(V)$ acts on $S_{Q'}(V)$, but only monomial groups act on $\bigwedge_Q(V)$ as graded automorphisms; if $q_{12} = 1$, then any group of linear transformations acts on both $S_Q(V)$ and $\bigwedge_Q(V)$ as graded automorphisms, but only monomial groups act on $S_{Q'}(V)$.

3. Actions on the quantum polynomial ring versus exterior algebra

Connections between quantum Drinfeld Hecke algebras and Hochschild cohomology have thus far required a hypothesis that the finite subgroup G of $\mathrm{GL}(V)$ act on *both* the quantum polynomial ring $S_Q(V)$ and the quantum exterior algebra $\bigwedge_Q(V)$ as graded automorphisms. (This assumption is sometimes implicit.) We develop some conditions for a group to act on both $S_Q(V)$ and $\bigwedge_Q(V)$ as graded automorphisms in this section. By the classification [19, Theorem 11.1] and these conditions, we observe that any element of $\mathrm{GL}_3(\mathbb{F})$ acting as a graded automorphism on $S_Q(V)$ also acts as a graded automorphism on $\bigwedge_Q(V)$, and vice versa. We show in this section that this observation holds in arbitrary dimension.

We rephrase and coalesce some conditions from [19] (as a subscript Q' was omitted in Corollaries 3.3 and 9.1 and Corollary 9.2(i) contained a typo).

Lemma 3.1. *A matrix in $GL_n(\mathbb{F})$ acts as an automorphism on $S_Q(V)$ if and only if its transpose acts as an automorphism on $\bigwedge_Q(V)$.*

Proof. By Lemma 2.6 (with indices exchanged), we need only show that h in $\mathrm{GL}(V)$ acts on $\bigwedge_Q(V)$ exactly when

$$\det{}_{klji,Q}(h^t) = -q_{ij}\ \det{}_{klij,Q}(h^t) \quad \text{for all}\ \ 1 \leq i, j, k, l \leq n\,. \tag{3.2}$$

For fixed $i \neq j$, we expand ${}^h v_j \wedge_Q {}^h v_i + q_{ij}\, {}^h v_i \wedge_Q {}^h v_j$ as

$$\sum_{k,l} h^j_k h^i_l\, v_k \wedge_Q v_l + q_{ij} \sum_{k,l} h^i_k\, h^j_l\, v_k \wedge_Q v_l \ = \ \sum_{k,l} (h^j_k h^i_l + q_{ij} h^i_k h^j_l)\, v_k \wedge_Q v_l.$$

Since $\sum_{k>l} (h^j_k h^i_l + q_{ij} h^i_k h^j_l)\, v_k \wedge_Q v_l = \sum_{k<l} -q_{kl}\ (h^j_l h^i_k + q_{ij} h^i_l h^j_k)\, v_k \wedge_Q v_l$, this is just

$$\sum_{k<l} \big(\det{}_{klji,Q}(h^t) + q_{ij}\det{}_{klij,Q}(h^t)\big)\, v_k \wedge_Q v_l + \sum_k (h^j_k h^i_k + q_{ij}\ h^i_k h^j_k)\, v_k \wedge_Q v_k\ .$$

As the second sum lies in the ideal of relations defining $\bigwedge_Q(V)$, the element h acts on $\bigwedge_Q(V)$ if and only if the first sum vanishes, giving Eq. (3.2) for $k < l$. The result follows, as Eq. (3.2) holds for $k = l$ as well. □

The next lemma gives a necessary and sufficient condition for a transformation in $\mathrm{GL}_n(\mathbb{F})$ to act as a graded automorphism on *both* the quantum polynomial ring $S_Q(V)$ and the exterior algebra $\bigwedge_Q(V)$: For any pair of nonzero entries in the matrix, the quantum scalar tracking the rows must coincide with the quantum scalar tracking the columns (see part (c)). Note that we require this stronger version of [25, Lemma 4.3] for Theorem 3.6 and the next section.

Lemma 3.3. *The following are equivalent for any $h \in GL_n(\mathbb{F})$:*

(a) h acts as a graded automorphisms on both $S_Q(V)$ and on $\bigwedge_Q(V)$;

(b) for all $1 \le i,j,k,l \le n$, $\det_{ijkl}(h) \;=\; \det_{lkji}(h^t)$;

(c) for all $1 \le i,j,k,l \le n$, either $q_{ij} = q_{lk}$ or $h^i_l h^j_k = 0$.

Proof. We use Lemmas 2.5, 2.6, and 3.1. Condition (a) implies that for all i,j,k,l

$$\det_{ijkl,Q}(h) = -q_{lk}\det_{ijlk,Q}(h) \qquad \text{and} \quad \det_{ijkl,Q}(h^t) = -q_{lk}\det_{ijlk}(h^t)\,.$$

We rewrite the second equation after exchanging i and l and exchanging j and k:

$$(h^t)^l_j(h^t)^k_i - q_{lk}(h^t)^l_i(h^t)^k_j \;=\; -q_{ij}\big((h^t)^l_i(h^t)^k_j - q_{lk}(h^t)^l_j(h^t)^k_i\big)\,.$$

Condition (a) thus implies that for all i,j,k,l

$$h^i_k h^j_l - q_{ij}h^i_l h^j_k + q_{lk}h^i_l h^j_k - q_{lk}q_{ij}h^i_k h^j_l \;=\; 0 \;=\; h^j_l h^i_k - q_{lk}h^i_l h^j_k + q_{ij}h^i_l h^j_k - q_{ij}q_{lk}h^j_l h^i_k\,,$$

and Condition (c) follows from adding the expression on the left to that on the right (as $\mathrm{char}(\mathbb{F}) \neq 2$). Notice that Conditions (c) and (b) are equivalent since the vanishing of $(1 - q_{lk}q_{ij})h^i_k h^j_l$ is equivalent (again, as $\mathrm{char}(\mathbb{F}) \neq 2$) to that of

$$\begin{aligned} h^i_k h^j_l + (-q_{ij}h^i_l h^j_k + q_{ij}h^i_l h^j_k) - q_{lk}q_{ij}h^i_k h^j_l \;&=\; \det_{ijkl,Q}(h) + q_{ij}\det_{lkij}(h^t) \\ =\; \det_{ijkl,Q}(h) + q_{ij}(-q_{ji})\det_{lkji}(h^t) \;&=\; \det_{ijkl,Q}(h) - \det_{lkji}(h^t)\,. \end{aligned}$$

Finally, Condition (c) implies that h acts on $S_Q(V)$ as it compels the vanishing of

$$\begin{aligned} \det_{ijkl}(h) + q_{lk}\det_{ijlk}(h) &= h^i_k h^j_l - q_{ij}h^i_l h^j_k + q_{lk}h^i_l h^j_k - q_{lk}q_{ij}h^i_k h^j_l \\ &= h^i_k h^j_l\,(1 - q_{ij}q_{lk}) + (q_{lk} - q_{ij})\,h^i_l h^j_k \end{aligned}$$

and also that h acts on $\bigwedge_Q(V)$ as it compels the vanishing of

$$\begin{aligned} \det_{ijkl}(h^t) + q_{lk}\det_{ijlk}(h^t) &= (h^t)^i_k(h^t)^j_l - q_{ij}(h^t)^i_l(h^t)^j_k + q_{lk}(h^t)^i_l(h^t)^j_k - q_{lk}q_{ij}(h^t)^i_k(h^t)^j_l \\ &= h^k_i h^l_j - q_{ij}h^l_i h^k_j + q_{lk}h^l_i h^k_j - q_{lk}q_{ij}h^k_i h^l_j \\ &= (1 - q_{lk}q_{ij})\,h^k_i h^l_j + (q_{lk} - q_{ij})\,h^l_i h^k_j\,. \end{aligned}$$

$\square$

Remark 3.4. For any group G of *monomial* matrices, $G \subset \mathrm{Aut}_{\mathrm{gr}}(S_Q(V))$ implies that $G \subset \mathrm{Aut}_{\mathrm{gr}}(\bigwedge_Q(V))$ by Lemma 3.3. Indeed, if $h = \{h^i_j\}$ in $\mathrm{GL}_n(\mathbb{F})$ is monomial with $h^i_l h^j_k \neq 0$, then $q_{ij} = q_{lk}$ since

$$q_{ij}\ h^i_l h^j_k\ v_l v_k = q_{ij}\ {}^h(v_i v_j) = {}^h(v_j v_i) = h^i_l h^j_k\ v_k v_l = q_{lk}\ h^i_l h^j_k\ v_l v_k\,.$$

We generalize this fact to arbitrary groups in Corollary 3.7.

Lemmas 2.6 and 3.1 with $k = \ell$ imply the next observation (as $\mathrm{char}(\mathbb{F}) \neq 2$).

Lemma 3.5. *If $h \in \mathrm{Aut}_{\mathrm{gr}}(\bigwedge_Q(V))$ is nonmonomial, then $q_{ij} = 1$ for any pair of rows i, j of h with nonzero entries in the same column.*

We have been unable to find an easy argument for showing the next theorem. The proof relies on a series of careful reductions.

Theorem 3.6. *Any element of $GL(V)$ that acts on $\bigwedge_Q(V)$ as a graded automorphism also acts on $S_Q(V)$ as a graded automorphism:*

$$\mathrm{Aut}_{\mathrm{gr}}\big(\textstyle\bigwedge_Q(V)\big) \subset \mathrm{Aut}_{\mathrm{gr}}\big(S_Q(V)\big).$$

Proof. Say h in $\mathrm{GL}_n(\mathbb{F})$ is a graded automorphism of $\bigwedge_Q(V)$. For any pair of nonzero entries in the matrix h, we use Lemma 3.3 and verify that the quantum scalar tracking the rows coincides with quantum scalar tracking the columns: We fix a pair of columns ℓ, k and pair of rows i, j of h such that $h_i^\ell h_j^k \neq 0$ and show that $q_{ij} = q_{\ell k}$ by chasing nonmonomial submatrices in h and their corresponding quantum scalars.

First notice that if $k = \ell$, then h contains a column with nonzero entries in rows i and j implying that $q_{ij} = 1 = q_{kk} = q_{\ell k}$ by Lemma 3.5 for $i \neq j$. (If $i = j$, then $q_{ij} = 1 = q_{\ell k}$.) Thus we may assume $k \neq \ell$.

Now let M be the submatrix of h with columns ℓ and k and rows i and j (not necessarily distinct). We argue that we may assume the entries of M are all nonzero and that $q_{ij} = 1$. If $i = j$, then h_i^ℓ and $h_i^k = h_j^k$ are both nonzero (as $h_i^\ell h_j^k \neq 0$) and $q_{ij} = q_{ii} = 1$. If $i \neq j$ and an entry of M is zero, then by Lemmas 2.6 and 3.1

$$\begin{aligned} h_i^\ell h_j^k &= h_i^\ell h_j^k - q_{ij} h_i^k h_j^\ell = \det_{ij\ell k}(h^t) = -q_{k\ell} \det_{ijk\ell}(h^t) \\ &= -q_{k\ell}\,(h_i^k h_j^\ell - q_{ij} h_i^\ell h_j^k) = q_{k\ell} q_{ij}\ h_i^\ell h_j^k \end{aligned}$$

implying that $q_{ij} = q_{\ell k}$. So for $i \neq j$, we may assume the entries of M are all nonzero, and Lemma 3.5 implies that $q_{ij} = 1$ in this case as well.

The submatrix M may not be invertible, but we may replace M by an invertible 2×2 submatrix M' of h by replacing the row j by some row j' of h since h is invertible. (Note that if $j = i$, then $j' \neq i$.) Then $q_{ij'} = 1$ by Lemma 3.5 as the two entries in row j' of M' can not both vanish. As $q_{ij'} = 1$, Lemmas 2.6 and 3.1 (with j' instead of j) then implies that

$$\det M' = \det_{ij'\ell k}(h^t) = (-q_{k\ell}) \det_{ij'k\ell}(h^t) = (-q_{k\ell})\,(-\det M') = q_{k\ell}\ \det M',$$

and $q_{k\ell} = 1 = q_{ji}$ since $\det M' \neq 0$, concluding the proof. □

Theorem 3.6 together with Lemma 3.1 implies

Corollary 3.7. *An element of $GL(V)$ acts on $S_Q(V)$ as a graded automorphism if and only if it acts on $\bigwedge_Q(V)$ as a graded automorphism:*

$$\mathrm{Aut}_{\mathrm{gr}}\big(\textstyle\bigwedge_Q(V)\big) = \mathrm{Aut}_{\mathrm{gr}}\big(S_Q(V)\big).$$

Corollary 3.7 and Lemmas 3.1 and 3.5 imply

Corollary 3.8. *Suppose a nonmomial matrix in $GL_n(\mathbb{F})$ acts on $\bigwedge_Q(V)$ or $S_Q(V)$ as a graded automorphism. Then $q_{ij} = 1$ for any pair of columns i, j with nonzero entries in the same row and for any pair of rows i, j with nonzero entries in the same column.*

Remark 3.9. Theorem 4.2 of [31] assumes that the finite group G acts on *both* $S_Q(V)$ and on $\bigwedge_Q(V)$ as graded automorphisms (this assumption is implicit in Section 4). Corollary 3.7 implies that Theorem 4.2 of [31] holds for *all* groups acting on $S_Q(V)$.

Recall that the Hochschild cohomology of an algebra A is its cohomology as a bimodule over itself, $\mathrm{HH}^\bullet(\mathrm{A}) = \mathrm{HH}^\bullet_{\mathrm{A}\otimes\mathrm{A}^{\mathrm{op}}}(\mathrm{A},\mathrm{A})$. Corollary 3.7 and [25, Theorem 4.4] imply

Corollary 3.10. *Suppose char*$(\mathbb{F}) = 0$ *and that* $G \subset GL_n(\mathbb{F})$ *is a finite group acting by automorphisms on* $S_Q(V)$*. Then each constant Hochschild 2-cocycle on* $S_Q(V) \rtimes G$ *gives rise to a quantum Drinfeld Hecke algebra.*

4. A Quantum Determinant

We define a quantum determinant in this section and show it defines a homomorphism of groups acting by graded automorphisms on quantum polynomial rings. This notion of quantum determinant differs from that for quantum matrices (see [16], [11], [22]).

Quantum sign and determinant. We use the action of the symmetric group $\mathfrak{S}_n$ on the basis $v_1, \dots, v_n$ of V by permutation of indices to define the *quantum sign*, even when $\mathfrak{S}_n \not\subset \mathrm{Aut}_{\mathrm{gr}}(\bigwedge_Q(V)$ under this action. Recall the *inversion set* of a permutation, $\mathrm{Inv}(\sigma) = \{(i,j) : i < j \text{ but } \sigma(i) > \sigma(j)\}$.

Definition 4.1. Define the *quantum sign* or Q*-sign* of a permutation σ in $\mathfrak{S}_n$ as

$$\mathrm{sgn}_Q(\sigma) \;=\; \mathrm{sgn}(\sigma) \prod_{(i,j)\in \mathrm{Inv}(\sigma)} q_{\sigma(j)\sigma(i)} \;=\; \mathrm{sgn}(\sigma) \prod_{(i,j)\in \mathrm{Inv}(\sigma^{-1})} q_{ij}\ .$$

Define the *quantum determinant* for any $h \in \mathrm{GL}_n(\mathbb{F})$ as the scalar

$$\det{}_{\mathrm{Q}}(h) \;=\; \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sgn}_{\mathrm{Q}}(\sigma)\; h^1_{\sigma(1)} h^2_{\sigma(2)} \cdots h^n_{\sigma(n)} \quad \in \mathbb{F}\,.$$

For example, $\mathrm{sgn}_{\mathrm{Q}}\big((1\ 2\ 3)(4\ 5)\big) = -q_{12}\, q_{13}\, q_{45}$. For $n = 3$ and $h = \{h^i_j\}$ in $\mathrm{GL}_3(\mathbb{F})$, the quantum determinant $\det_Q(h)$ is

$$h^1_1h^2_2h^3_3 + q_{13}q_{12}\, h^1_2h^2_3h^3_1 + q_{13}q_{23}\, h^1_3h^2_1h^3_2 - q_{23}\, h^1_1h^2_3h^3_2 - q_{12}\, h^1_2h^2_1h^3_3 - q_{12}q_{23}q_{13}\, h^1_3h^2_2h^3_1\,.$$

Recall that $\mathrm{sgn}(\sigma) = (-1)^{|\mathrm{Inv}(\sigma)|}$ and that σ can be factored into the product over all (i,j) in $\mathrm{Inv}(\sigma)$ of transpositions $(i\ j)$.

Quantum determinant as a homomorphism. One may check directly that the quantum determinant $\det_Q$ gives the scalar by which an automorphism of $\bigwedge_Q(V)$ acts on the quantum volume form:

Lemma 4.2. *For any permutation* σ *in* $\mathfrak{S}_n$*,*

$$v_{\sigma(1)} \wedge_Q \cdots \wedge_Q v_{\sigma(n)} \;=\; \mathrm{sgn}_{\mathrm{Q}}(\sigma)\; v_1 \wedge_Q \cdots \wedge_Q v_n \quad \textit{and} \quad \mathrm{sgn}_{\mathrm{Q}}(\sigma) = \det{}_Q(\sigma)\,.$$

Furthermore, for all h *in* $\mathrm{Aut}_{\mathrm{gr}}(\bigwedge_Q V)$*,*

$$h(v_1 \wedge_Q \cdots \wedge_Q v_n) = \det{}_Q(h)\; v_1 \wedge_Q \cdots \wedge_Q v_n\,.$$

Corollary 4.3. *The quantum determinant* $\det_Q$ *is a group homomorphism on* $\mathrm{Aut}_{\mathrm{gr}}(S_Q(V))$*:*

$$\det{}_{\mathrm{Q}}(gh) = \det{}_{\mathrm{Q}}(g)\det{}_{\mathrm{Q}}(h) \;\; \textit{for all } g,h \textit{ in } \mathrm{Aut}_{\mathrm{gr}}(S_Q(V))\,.$$

Proof. By Corollary 3.7 and Lemma 4.2,

$$\begin{aligned}\det_Q(gh)(v_1 \wedge_Q \cdots \wedge_Q v_n) &= {}^{gh}(v_1 \wedge_Q \cdots \wedge_Q v_n) \;=\; {}^{g}\big({}^{h}(v_1 \wedge_Q \cdots \wedge_Q v_n)\big)\\ &= {}^{g}\big(\det_Q(h)(v_1 \wedge_Q \cdots \wedge_Q v_n)\big) = \det_Q(h)\;\; {}^{g}(v_1 \wedge_Q \cdots \wedge_Q v_n)\\ &= \det_Q(h)\det_Q(g)\; v_1 \wedge_Q \cdots \wedge_Q v_n\,. \end{aligned}$$ □

Note that the quantum determinant $\det_Q$ is not a group homomorphism on other groups. For example, when $G = \mathfrak{S}_3$ and $q_{13} \neq q_{23}$, $G \not\subset \mathrm{Aut}_{\mathrm{gr}}(S_Q(V))$, and

$$\det_Q((1\ 2)(2\ 3)) = q_{12}\, q_{13} \neq q_{12}\, q_{23} = \det_Q((1\ 2))\det_Q((2\ 3))\,.$$

Remark 4.4. Graded automorphisms of $S_Q(V)$ have nonzero quantum determinants: If $h \in \mathrm{Aut}_{\mathrm{gr}}(S_Q(V))$, then Corollary 4.3 implies that

$$1 = \det_Q(1_V) = \det_Q(hh^{-1}) = \det_Q(h)\ \det_Q(h^{-1})\,.$$

The converse is false of course. Indeed, the matrix $h = \left(\begin{smallmatrix} 1 & 1\\ -1 & 1\end{smallmatrix}\right)$ doesn't act on $S_Q(V)$ as a graded automorphism when $q_{12} \neq 1$ although $\det_Q(h) \neq 0$.

Remark 4.5. One asks how $\mathrm{Aut}_{\mathrm{gr}}(S_Q(V))$ overlaps with the quantum-special linear group,

$$\mathrm{SL}_{n,Q}(\mathbb{F}) = \{g \in \mathrm{GL}_n(\mathbb{F}) : \det_Q(g) = 1\}.$$

For $n = 2$ with $q_{12} = q$,

$$\mathrm{SL}_{n,Q}(\mathbb{F}) \cap \mathrm{Aut}_{\mathrm{gr}}(S_Q(V)) = \begin{cases} \mathrm{SL}_2(\mathbb{F}) & \text{for } q = 1,\\ \{\mathrm{Diag}(a,b) : ab = 1\} \cup \{\mathrm{AntiDiag}(a,b) : ab = 1\} & \text{for } q = -1,\\ \{\mathrm{Diag}(a,b) : ab = 1\} & \text{for } q \neq \pm 1\,.\end{cases}$$

Remark 4.6. When the classical Shephard-Todd complex reflection group $G(r,r,n)$ acts as graded automorphisms on a nontrivial $S_Q(V)$, then necessarily every $q_{ij} = -1$ for $i \neq j$ by Lemma 2.2 and all group elements have quantum determinant 1 (one may use Corollary 4.3):

$$G(r,r,n) \subset \mathrm{SL}_{n,Q}(\mathbb{F})\,.$$

Note that any g in the mystic reflection group $M(n,1,\beta)$ (see 9.1) has $\det_Q(g) = \pm 1$.

A simplification of the quantum determinant. We give a simplification of the quantum determinant for matrices that act as graded automorphisms on a quantum polynomial ring. This simplification implies a version of the familiar down-up rule for determinants of 3×3 matrices.

For an odd cycle π in the symmetric group $\mathfrak{S}_n$ of order $|\pi|$, we define a set of quantum parameters that records certain elements of the cycle paired together with their "halfway partners":

$$Q_\pi = \{q_{ab} : (a,b) \in \mathrm{Inv}(\pi) \text{ and } (a\ b) \text{ appears in the disjoint cycle decomposition of } \pi^{|\pi|/2}\}.$$

E.g., if $\pi = (1\ 11\ 9\ 2\ 5\ 7\ 4\ 8)$, then $|\pi| = 8$, $\pi^4 = (1\ 5)(11\ 7)(9\ 4)(2\ 8)$, and $Q_\pi = \{q_{15}, q_{49}, q_{28}\}$ as $(7,11) \notin \mathrm{Inv}(\pi)$. (Note that $|\pi|$ is always even since π is an odd cycle.)

In the next proposition, we take a product over the odd cycles π of a permutation σ, i.e., all the odd cycles π appearing in a decomposition of σ into the product of disjoint cycles. For example, if $\sigma = (1\ 11\ 9\ 2\ 5\ 7\ 4\ 8)(3\ 6)(10\ 12\ 13)$, then in the statement, we may choose $c_\sigma = q_{15}q_{36}$ or $c_\sigma = q_{49}q_{36}$ or $c_\sigma = q_{28}q_{36}$.

Proposition 4.7. *The quantum determinant simplifies as*

$$\det_Q(h) = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\ c_\sigma\ h^1_{\sigma(1)}\cdots h^n_{\sigma(n)} \quad \textit{for all} \quad h\in \operatorname{Aut}_{\mathrm{gr}}(S_Q(V)),$$

where

$$c_\sigma = \prod_{\textit{odd cycles } \pi \textit{ of } \sigma} q_\pi \quad \in \mathbb{F}$$

for any choice of element q_π in Q_π.

Proof. Fix a permutation $\sigma \neq 1$ in $\mathfrak{S}_n$. Lemma 3.3(c) implies that

$$q_{ij}\, h^1_{\sigma(1)}\cdots h^n_{\sigma(n)} = q_{\sigma(i)\,\sigma(j)}\ h^1_{\sigma(1)}\cdots h^n_{\sigma(n)} \qquad \text{for } i\neq j. \tag{4.8}$$

We use this key observation to cancel factors of q_{ij} appearing in the quantum determinant. Indeed, in the coefficient of the σ-summand of $\det_Q(h)$ (see Definition 4.1),

$$\operatorname{sgn}_Q(\sigma) \ = \ \operatorname{sgn}(\sigma) \prod_{(i,j)\in \operatorname{Inv}(\sigma)} q_{\sigma(j)\sigma(i)}\,,$$

a factor q_{ij} cancels with $q_{\sigma(j)\,\sigma(i)} = q^{-1}_{\sigma(i)\,\sigma(j)}$ provided both (i,j) and $(\sigma(j),\sigma(i))$ lie in $\operatorname{Inv}(\sigma)$. In order to pair and cancel factors appropriately, we consider the orbits O of σ acting diagonally on the set of ordered pairs $P=\{(i,j) : i\neq j\}$ and the swap bijection $\tau : P\to P$, $(i,j)\mapsto (j,i)$, noting that $\operatorname{Inv}(\sigma)$ is the disjoint union over orbits O of the sets $O\cap \operatorname{Inv}(\sigma)$.

Fix an orbit $O\subset P$ with $O\cap\operatorname{Inv}(\sigma)\neq\varnothing$. Say (i,j) lies in $O\cap \operatorname{Inv}(\sigma)$ and consider any (a,b) in $\tau(O)\cap\operatorname{Inv}(\sigma)$. Then (b,a) lies in O and hence $(b,a)=(\sigma^k(i),\sigma^k(j))$ for some k. Thus

$$q_{ab}\ h^1_{\sigma(1)}\cdots h^n_{\sigma(n)} = q^{-1}_{ba}\ h^1_{\sigma(1)}\cdots h^n_{\sigma(n)} = q^{-1}_{\sigma^k(i)\,\sigma^k(j)}\ h^1_{\sigma(1)}\cdots h^n_{\sigma(n)} = q^{-1}_{ij}\ h^1_{\sigma(1)}\cdots h^n_{\sigma(n)}$$

by Eq. (4.8), and hence

$$q_{ij}q_{ab}\ h^1_{\sigma(1)}\cdots h^n_{\sigma(n)} = h^1_{\sigma(1)}\cdots h^n_{\sigma(n)}\,. \tag{4.9}$$

Hence we investigate how the elements of $O\cap\operatorname{Inv}(\sigma)$ may be paired with the elements of $\tau(O)\cap\operatorname{Inv}(\sigma)$ in order to simplify the formula for $\det_Q(h)$. Note that the set $\tau(O)$ is again an orbit, and hence either $O=\tau(O)$ or $O\cap\tau(O)=\varnothing$.

First suppose $O\cap\tau(O)=\varnothing$. It is not difficult to see that the sets $O\cap\operatorname{Inv}(\sigma)$ and $\tau(O)\cap\operatorname{Inv}(\sigma)$ are in bijection, so each element of $O\cap\operatorname{Inv}(\sigma)$ may be paired with a unique element of $\tau(O)\cap\operatorname{Inv}(\sigma)$ in the factorization of $\operatorname{sgn}_Q(\sigma)$. This implies that O and $\tau(O)$ taken together contribute no quantum scalars to the σ-summand $\operatorname{sgn}_Q(\sigma)h^1_{\sigma(1)}\cdots h^n_{\sigma(n)}$ of $\det_Q(h)$ after simplifying by Eq. (4.9). Indeed, one may define a bijection

$$O\cap\operatorname{Inv}(\sigma)\to\tau(O)\cap\operatorname{Inv}(\sigma),$$

for example, by $(i,j)\mapsto(\sigma^m(j),\sigma^m(i))$ where $0<m<|\sigma|$ is the minimal integer such that $(\sigma^m(j),\sigma^m(i))$ lies in $\operatorname{Inv}(\sigma)$.

Now suppose $O=\tau(O)$. Then there is a unique cycle π in a decomposition of σ into the product of disjoint cycles that does not fix any entry of any element in O. We claim that π has even length ℓ and that

$$O=\{(i,j) : i,j \text{ are not fixed by } \pi \text{ and } j=\pi^{\ell/2}(i)\}\,.$$

Consider some (i,j) in $O=\tau(O)$. Then i and j both appear in the cycle π, i.e., are not fixed by π, and $(i,j)=(\sigma^k(j),\sigma^k(i))$ for some $k>0$, say minimal. Then π must have even length

$2k$ (since $\pi = (i\ a_1\ \ \cdots\ a_{k-1}\ j\ a_{k+1}\ \cdots\ a_{2k-1})$ for some a_m). Conversely, if (i,j) lies in the given set, then (i,j) lies in O and O has the description claimed.

We argue that the set $O \cap \mathrm{Inv}(\sigma) = O \cap \mathrm{Inv}(\pi)$ has odd size. By Eq. (4.9), this implies (as $O = \tau(O)$) that all but one of the elements of $O \cap \mathrm{Inv}(\sigma)$ may be paired so as to avoid contributing any quantum scalars to $\mathrm{sgn}_Q(\sigma)$ in the formula for $\det_Q(h)$. Furthermore, by Eq. (4.8), it does not matter which lone element of this set contributes a quantum scalar to $\mathrm{sgn}_Q(\sigma)$ in the formula, and we obtain the advertised description of the quantum determinant.

To see that $O \cap \mathrm{Inv}(\pi)$ has odd size, first note that $|\mathrm{Inv}(\pi)|$ is odd because π is an odd permutation. The set $\mathrm{Inv}(\pi)$ is the disjoint union of the sets $O' \cap \mathrm{Inv}(\pi)$ over all the orbits O' of the group $\langle \pi \rangle$ acting on the set P. Replacing σ by π throughout the above arguments, we see that if O' is an orbit with $O' \cap \tau(O') = \varnothing$, then there is a bijection between $(O' \cap \mathrm{Inv}(\pi))$ and $(\tau(O') \cap \mathrm{Inv}(\pi))$ and hence the two orbits O' and $\tau(O')$ together contribute an even number of elements to $\mathrm{Inv}(\pi)$. In addition, the arguments above for π in place of σ show there is exactly one orbit O' under the action of π with $O' = \tau(O')$ (since π itself is a *single* cycle of even length) and $O' = O$. Hence the parity of $|\mathrm{Inv}(\pi)|$ is that of $|O \cap \mathrm{Inv}(\pi)|$ and thus $|O \cap \mathrm{Inv}(\pi)|$ must also be odd.

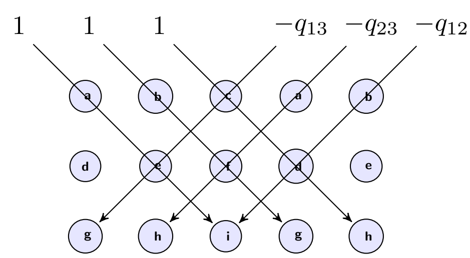
□

Proposition 4.7 implies a simplification of the *down-up diagonal-antidiagonal* pattern for computing determinants of 3×3 matrices. Recall that a matrix lies in $\mathrm{Aut}_{\mathrm{gr}}(S_Q(V))$ if and only if it lies in $\mathrm{Aut}_{\mathrm{gr}}(\bigwedge_Q(V))$ (Corollary 3.7).

Corollary 4.10. *For $n = 3$, if $h = \{h^i_j\} \in GL_3(\mathbb{F})$ lies in $\mathrm{Aut}_{\mathrm{gr}}(S_Q(V))$, then*

$$\det_Q(h) = h^1_1 h^2_2 h^3_3 + h^1_2 h^2_3 h^3_1 + h^1_3 h^2_1 h^3_2 - q_{23}\ h^1_1 h^2_3 h^3_2 - q_{12}\ h^1_2 h^2_1 h^3_3 - q_{13}\ h^1_3 h^2_2 h^3_1\,.$$

Figure 1. Quantum determinant of a graded automorphism in dimension 3



5. Quantum Drinfeld Hecke algebras

We now turn to quantum Drinfeld Hecke algebras and fix a finite group $G \subset \mathrm{GL}_n(\mathbb{F})$ acting on $V \cong \mathbb{F}^n$. Recall that if G acts on an $\mathbb{F}$-algebra A by automorphisms (for example, the quantum symmetric algebra $A = S_Q(V)$ or the tensor algebra $A = T_{\mathbb{F}}(V)$), the natural semidirect product algebra $A \rtimes G$ is the $\mathbb{F}$-vector space $A \otimes_{\mathbb{F}} \mathbb{F}G$ with multiplication

$$(a \otimes g)(b \otimes h) = a\ {}^g b \otimes gh \qquad \text{for } a, b \in A \text{ and } g, h \in G.$$

This algebra is alternatively often called the *skew group algebra* or *smash product algebra* (written $A\#G$). We identify $A \rtimes G$ with the $\mathbb{F}$-algebra generated by A and $\mathbb{F}G$ with relations $g\ a = {}^g a\ g$ for all $a \in A$ and $g \in G$ by suppressing tensor signs, $a \otimes g = ag$.

Parameter functions. We view $T_{\mathbb{F}}(V) \rtimes G$ as a graded algebra after assigning group elements in G degree 0 and vectors in V degree 1. We consider a quotient by relations that lower the degree of q-commutators $v_j v_i - q_{ij} v_i v_j$ recorded by a parameter function $\kappa : V \otimes V \to \mathbb{F}G$. We abbreviate $\kappa(v, w) = \kappa(v \otimes w)$ for ease with notation throughout.

Quantum Drinfeld orbifold algebras. Given the quantum system of parameters Q and a linear parameter function $\kappa : V \otimes V \to \mathbb{F}G$, we define the $\mathbb{F}$-algebra

$$\mathcal{H}_{Q,\kappa} := (T(V) \rtimes G)/\big(v_j v_i - q_{ij} v_i v_j - \kappa(v_i, v_j) : 1 \leq i, j \leq n\big).$$

We say $\mathcal{H}_{Q,\kappa}$ is a *quantum Drinfeld Hecke algebra* if it satisfies the *PBW property*, i.e., if

$$\{v_1^{m_1} v_2^{m_2} \cdots v_n^{m_n} g : m_i \in \mathbb{Z}_{\geq 0},\ g \in G\}$$

is a basis for $\mathcal{H}_{Q,\kappa}$ as an $\mathbb{F}$-vector space. This is equivalent to $\mathcal{H}_{Q,\kappa}$ defining a *graded deformation* of $S_Q(V) \rtimes G$. For related work, see Jing and Zhang [17], Shakalli [27], Levandovskyy and Shepler [19], Shroff and Witherspoon [31], and Naidu and Witherspoon [25].

Remark 5.1. The PBW algebras $\mathcal{H}_{Q,\kappa}$ include the *braided Cherednik algebras* of Bazlov and Berenstein [7]. In the special case that $q_{ij} = 1$ for all i, j, they also include Lusztig's graded Hecke algebras [20, 21], the symplectic reflection algebras explored by Etingof and Ginzburg [14], the Drinfeld Hecke algebras of [13], and the noncommutative deformations of Kleinian singularities studied by Crawley-Boevey and Holland [12].

Support of parameter. For any parameter $\kappa : V \otimes V \to \mathbb{F}G$, we fix linear functions $\kappa_g : V \otimes V \to \mathbb{F}$ for g in G decomposing κ as

$$\kappa(v_i, v_j) = \sum_{g \in G} \kappa_g(v_i, v_j)\, g \quad \text{for} \quad 1 \leq i, j \leq n. \tag{5.2}$$

We say κ is *supported* on a subset of group elements $S \subset G$ if $\kappa_g \equiv 0$ for all $g \notin S$.

Group action on parameters. A group G acts on any parameter function $\kappa : V \otimes V \to \mathbb{F}G$ in the standard way, where G acts on $\mathbb{F}G$ by conjugation:

$$(\,{}^g\kappa)(u, v) = {}^g\big(\kappa({}^{g^{-1}}u,\, {}^{g^{-1}}v)\big) \quad \text{for } g \text{ in } G\,.$$

PBW conditions. We recall necessary and sufficient conditions for $\mathcal{H}_{Q,\kappa}$ to satisfy the PBW property. The following strengthens a theorem of Levandovskyy and Shepler [19]. A version appears in [30] and [31] with the extra (implicit) hypothesis that G acts on *both* $S_Q(V)$ and on $\bigwedge_Q(V)$; we give a quick proof showing how Corollary 3.7 is used. Recall that κ is a quantum 2-form when $\kappa(v_j, v_i) = -q_{ji}\ \kappa(v_i, v_j)$ for all i, j (see Eq. (2.7)).

Theorem 5.3. *Let G be a finite subgroup of $GL_n(\mathbb{F})$. The algebra $\mathcal{H}_{Q,\kappa}$ satisfies the PBW property if and only if*

(1) G acts by graded automorphisms on $S_Q(V)$,
(2) $\kappa : V \otimes V \to \mathbb{F}G$ is a quantum 2-form,
(3) the quantum Jacobi identity holds for all $1 \leq i < j < k \leq n$ and g in G,

$$0 = \sum_{\sigma \in \mathrm{Alt}_3} \kappa_g(v_{\sigma(i)}, v_{\sigma(j)})\ \big(q_{\sigma(j)\sigma(k)}\ {}^g v_{\sigma(k)} - q_{\sigma(k)\sigma(i)}\, v_{\sigma(k)}\big)\,,$$

(4) κ is G-invariant.

Proof. By [19, Theorem 7.6], we need only check that Condition (4) is equivalent to

$$\kappa_{h^{-1}gh}(v_i, v_j) = \sum_{1\leq k<l\leq n} \det_{ijkl}(h)\ \kappa_g(v_k, v_l) \text{ for all } g, h \in G \text{ and } 1 \leq i < j \leq n, \tag{5.4}$$

assuming Conditions (1), (2), (3) already hold. As κ is bilinear, straightforward calculation (as in the proof of [19, Lemma 3.2]) using Eq. (2.7) confirms that κ is invariant exactly when

$$\kappa_{h^{-1}gh}(\ v_i, v_j) = \kappa_g(\ {}^h v_i,\ {}^h v_j) = \sum_{k<l} \det_{lkji}(h^t)\ \kappa_g(v_k, v_l) \text{ for all } g, h \in G \text{ and } i \neq j.$$

But this is just Eq. (5.4) since $\det_{lkji}(h^t) = \det_{ijkl}(h)$ by Lemma 3.3 and Corollary 3.7. □

Parameter space. A parameter κ is *admissible* if it defines a quantum Drinfeld Hecke algebra $\mathcal{H}_{Q,\kappa}$, i.e., defines a PBW algebra (see [13]). Note that any linear combination of admissible parameters is again admissible (see Theorem 5.3). We call the $\mathbb{F}$-vector space

$$P = P_G = \{\kappa \in \mathrm{Hom}_{\mathbb{F}}(V \otimes V, \mathbb{F}G) : \kappa \text{ is admissible}\}$$

of all admissible parameters the *parameter space of quantum Drinfeld Hecke algebras*. We denote its dimension by $\dim_{\mathbb{F}} P = \dim_{\mathbb{F}}(P_G)$ for a specific finite group G (with fixed Q).

By Theorem 5.3 (see Eq. (5.4)), we can write any parameter $\kappa \in \mathrm{Hom}_{\mathbb{F}}(V \otimes V, \mathbb{F}G)$ as the sum over the conjugacy classes C of G of parameter functions κ_C, each supported only on C:

$$\kappa = \sum_{\text{conj. classes } C} \kappa_C \quad \text{with} \quad \kappa_C(v, w) = \sum_{g\in C} \kappa_g(v, w)\ g \quad \text{for } v, w \in V.$$

By Theorem 5.3, κ is admissible exactly when each κ_C is admissible. Thus to find the dimension of P_G, we need only find the dimension of admissible parameters κ supported on a fixed conjugacy class C and then add over all conjugacy classes C of G:

$$\dim_{\mathbb{F}} P = \sum_{\text{conj. classes } C} \dim_{\mathbb{F}}\{\kappa \in \mathrm{Hom}_{\mathbb{F}}(V \otimes V, \mathbb{F}G) : \kappa_g \equiv 0 \text{ for } g \notin C,\ \kappa \text{ admissible}\}. \tag{5.5}$$

Basis matters. Bilinearity of the parameter κ plays no role here; we only ever evaluate κ on the given basis, as another choice of basis for V may define a non-isomorphic algebra. Consider a linear action of the Klein 4-group on $\mathbb{C}^3$ with two parameters worth of nontrivial quantum Drinfeld Hecke algebras using one basis of $\mathbb{C}^3$ but none using another: Set

$$G = \left\langle \begin{pmatrix} -1 & & \\ & & -1 \\ & -1 & \end{pmatrix}, \begin{pmatrix} -1 & & \\ & & 1 \\ & 1 & \end{pmatrix} \right\rangle \quad \text{and} \quad G' = \left\langle \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \right\rangle.$$

Here, G and G' give equivalent representations but $\dim_{\mathbb{F}}(P_G) = 2$ when $q_{23} = -1$ and $q_{12}q_{13} = \pm 1$, whereas $\dim_{\mathbb{F}}(P_{G'}) = 0$ for all choices of Q.

Examples. We end this section with a few examples.

Example 5.6. Consider the symmetric group $G = \mathfrak{S}_2$ acting on $V = \mathbb{F}^2$ by permuting basis elements x, y. Then G acts on $S_Q(V) = \mathbb{F}_Q[x, y]/(xy + yx)$ for $q_{12} = -1$. For any a, b in $\mathbb{F}$, the $\mathbb{F}$-algebra $\mathcal{H}_{Q,\kappa}$ generated by symbols g, x, y (for g the transposition) with relations

$$g^2 = 1,\ gx = yg,\ gy = xg,\ xy = -yx + a + bg$$

exhibits the PBW property and is a quantum Drinfeld Hecke algebra. Notice that the parameter function $\kappa : V \otimes V \to \mathbb{F}G$ is defined by $\kappa(x, y) = a + bg$.

Example 5.7. Recall the monomial group G and quantum scalars Q from Example 2.3. Every quantum Drinfeld Hecke algebra $\mathcal{H}_{Q,\kappa}$ is generated by v_1, v_2, v_3 and h with relations

$$h^6 = 1,\ hv_1 = v_2 h, \quad hv_2 = v_1 h, \quad hv_3 = \omega\, v_3 h,$$
$$v_2 v_1 = -v_1 v_2 + m_1 h + m_2 h^4, \quad v_3 v_1 = \omega\, v_1 v_3, \quad \text{and} \quad v_3 v_2 = \omega\, v_2 v_3,$$

for some parameters $m_1, m_2 \in \mathbb{C}$. Hence $\dim_{\mathbb{C}}(P_G) = 2$.

Example 5.8. Consider the monomial group $G \subset \mathrm{GL}(V)$ generated by $g = \left(\begin{smallmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ 4 & 0 & 0 \end{smallmatrix}\right)$ for $V = \mathbb{C}^3$. When $q_{ij} = -1$ for $i \neq j$, $\dim_{\mathbb{F}}(P_G) = 3$: Each quantum Drinfeld Hecke algebra $\mathcal{H}_{Q,\kappa}$ is generated by v_1, v_2, v_3 and $\mathbb{C}G$ with relations $gv_i = {}^g v_i g$ for all i and these relations given by parameters $m_1, m_2, m_3 \in \mathbb{C}$:

$$v_3 v_2 = -v_2 v_3 + m_1 + m_2 g + m_3 g^2,$$
$$v_2 v_1 = -v_1 v_2 + 4m_1 + 4m_2 g + 4m_3 g^2,$$
$$v_3 v_1 = -v_1 v_3 + 2m_1 + 2m_2 g + 2m_3 g^2.$$

6. Some combinatorial lemmas

Before any classification results, we first collect some preliminary observations giving combinatorial ways to investigate quantum Drinfeld Hecke algebras. We will use these results to classify algebras for the symmetric group acting by permutation matrices, the infinite family of complex reflection groups $G(r, p, n)$, and the mystic reflection groups in later sections. Every monomial matrix g can be written as the product $d\sigma$ of a diagonal matrix d and a permutation matrix σ in the symmetric group $\mathfrak{S}_n$. If σ defines a k-cycle in $\mathfrak{S}_n$, we say that g has *k-cycle type σ*. When σ is the product of two disjoint transpositions, we say g is the *product of two disjoint 2-cycle type elements*. The next lemma explains why we are primarily interested in 2-cycle and 3-cycle types. We use some ideas from [33].

Lemma 6.1. *Say $n \geq 3$ and $G \subset GL_n(\mathbb{F})$ is a monomial matrix group. If $\mathcal{H}_{Q,\kappa}$ is a quantum Drinfeld Hecke algebra, then for any g in G, $\kappa_g \not\equiv 0$ implies*

- *g is diagonal, or*
- *g has 2-cycle or 3-cycle type, or*
- *g is the product of two disjoint 2-cycle type elements.*

Proof. Say g is not diagonal and write $g = d\sigma$ as above with d diagonal and $\sigma \neq 1$ a permutation. For $i \neq j$, we may judiciously choose $k \notin \{i, j\}$ with $\sigma(k)$ not in $\{i, j, k\}$ in Theorem 5.3(3) to force $\kappa_g(v_i, v_j) \equiv 0$ except when σ is a 2-cycle, 3-cycle, or product of two disjoint 2-cycles. □

Notice that when $q_{ij} = -1$ for all $i \neq j$, $\det_{ijkl}(g) = \det_{ijlk}(g) = \det_{jikl}(g) = \det_{jilk}(g)$ for any matrix g. In fact, one may verify the next two lemmas directly.

Lemma 6.2. *Let $G \subset GL_n(\mathbb{F})$ be a monomial group and $q_{ij} = -1$ for all $i \neq j$. Then for all g, h in G, if $\det_{ijkl}(g \cdot h) \neq 0$, there exists a unique pair $1 \leq a < b \leq n$ with*

$$\det_{ijkl}(gh) = \det_{ijab}(h) \cdot \det_{abkl}(g).$$

And for any pair $a < b$, the product $\det_{ijab}(h) \cdot \det_{abkl}(g)$ either is zero or is $\det_{ijkl}(gh)$.

Lemma 6.3. *Let $G \subset GL_n(\mathbb{F})$ be a monomial matrix group and $q_{ij} = -1$ for all $i \neq j$. To check Eq.* (5.4)*, it suffices to consider g in a set of conjugacy class representatives.*

Proof. Assume Eq. (5.4) holds for a fixed g. Say $g' = z^{-1}gz$ for some z in G and fix h in G and $i \neq j$. As G is monomial, there is a unique pair $a < b$ with $\det_{ijab}(zh) \neq 0$ and

$$\kappa_{h^{-1}g'h}(v_i, v_j) = \kappa_{(zh)^{-1}g(zh)}(v_i, v_j) = \det_{ijab}(zh)\ \kappa_g(v_a, v_b)\,.$$

There is also a unique pair $c < d$ so that $0 \neq \det_{ijab}(zh) = \det_{ijcd}(h)\ \det_{cdab}(z)$ (by Lemma 6.2). Then $a < b$ is the unique pair for $c < d$ with $\det_{cdab}(z) \neq 0$, and hence the last display gives

$$\det_{ijcd}(h)\ \det_{cdab}(z)\ \kappa_g(v_a, v_b) = \det_{ijcd}(h)\ \kappa_{z^{-1}gz}(v_c, v_d) = \det_{ijcd}(h)\ \kappa_{g'}(v_c, v_d).$$

Also $c < d$ is the unique pair for $i \neq j$ with $\det_{ijcd}(h) \neq 0$, so Eq. (5.4) holds with g' in place of g, i.e., $\kappa_{h^{-1}g'h}(v_i, v_j) = \sum_{k<l} \det_{ijkl}(h)\ \kappa_{g'}(v_k, v_l)$. □

We will use the next technical lemma for the infinite family of Shephard-Todd groups $G(r, p, n)$ in Section 8 and the mystic reflection groups in Section 9. We denote the centralizer of each g in G by $C_G(g)$.

Lemma 6.4. *Let $G \subset GL_n(\mathbb{F})$, $n \geq 3$, be a finite monomial group with $q_{ij} = -1$ for $i \neq j$ containing the 3-cycle $g = (1\ 2\ 3)$. Suppose that the centralizer $C_G(g)$ is a subgroup of $\langle g, g \cdot (-I)\rangle$ upon restriction of each group to $V' = \mathbb{F}v_1 + \mathbb{F}v_2 + \mathbb{F}v_3$. Then there is a nontrivial quantum Drinfeld Hecke algebra $\mathcal{H}_{Q,\kappa}$ with $\kappa : V \otimes V \to \mathbb{F}G$ supported on the conjugacy class of g. In fact, for any quantum Drinfeld Hecke algebra and any $i \neq j$, $\kappa_g(v_i, v_j) = \kappa_g(v_1, v_2)$ for i, j distinct in $\{1, 2, 3\}$ and $\kappa_g(v_i, v_j) = 0$ otherwise.*

Proof. For $0 \neq m \in \mathbb{F}$, define a quantum 2-form κ supported on the conjugacy class of g by setting $\kappa_g(v_i, v_j) = 0$ for i or $j \notin \{1, 2, 3\}$, $\kappa_g(v_1, v_2) = \kappa_g(v_2, v_3) = \kappa_g(v_3, v_1) = m$, and

$$\kappa_{h^{-1}gh}(v_i, v_j) = \sum_{k<l} \det_{ijkl}(h)\ \kappa_g(v_k, v_l) \quad \text{for} \quad 1 \leq i < j \leq n,\ h \in G\,.$$

We argue that $\kappa_{h^{-1}gh}$ is well-defined. Say $h^{-1}gh = z^{-1}gz$ and $i < j$. On one hand, there is a unique pair $a < b$ (since G is monomial) with $\det_{ijab}(h) \neq 0$, and

$$\kappa_{h^{-1}gh}(v_i, v_j) := \sum_{k<l} \det_{ijkl}(h)\ \kappa_g(v_k, v_l) = \det_{ijab}(h)\ \kappa_g(v_a, v_b).$$

On the other hand, there is a unique pair $c < d$ with $\det_{ijcd}(z) \neq 0$, and

$$\kappa_{z^{-1}gz}(v_i, v_j) := \sum_{k<l} \det_{ijkl}(z)\ \kappa_g(v_k, v_l) = \det_{ijcd}(z)\ \kappa_g(v_c, v_d).$$

We show

$$\det_{ijab}(h)\ \kappa_g(v_a, v_b) = \det_{ijcd}(z)\ \kappa_g(v_c, v_d). \tag{6.5}$$

Since G is monomial, Lemma 6.2 implies that

$$\det_{abcd}(zh^{-1}) = \det_{abij}(h^{-1})\ \det_{ijcd}(z) = (\det_{ijab}(h))^{-1}\ \det_{ijcd}(z) \neq 0 \tag{6.6}$$

and $\{a, b\} \subset \{1, 2, 3\}$ exactly when $\{c, d\} \subset \{1, 2, 3\}$ since $zh^{-1} \in C_G(g)$. If $\{a, b\} \not\subset \{1, 2, 3\}$, then $\kappa_g(v_a, v_b) = \kappa_g(v_c, v_d) = 0$ by construction of κ. So we assume $a, b, c, d \in \{1, 2, 3\}$ and $\kappa_g(v_a, v_b) = \kappa_g(v_c, v_d)$. But $zh^{-1}|_{V'} \in \langle \pm g\rangle_{V'}$, hence $1 = \det_{abcd}(zh^{-1})$ and $\det_{ijab}(h) = \det_{ijcd}(z)$ by Eq. (6.6) implying Eq. (6.5). One can then verify that κ is admissible using Theorem 5.3.

Now suppose that $\mathcal{H}_{Q,\kappa}$ is a quantum Drinfeld Hecke algebra. For $i \neq j$ with i or j not in $\{1,2,3\}$, we may find an index k so that Theorem 5.3(3) forces $\kappa_g(v_i, v_j) = 0$. For $i \neq j$ with $i, j \in \{1,2,3\}$, Theorem 5.3(4) implies that $\kappa_{(1\ 2\ 3)}(v_i, v_j) = \kappa_g(v_1, v_2)$. $\square$

The next lemma is used for the Coxeter groups $\mathfrak{S}_n = G(1,1,n)$, $\mathcal{W}(B_n) = G(2,1,n)$, and $\mathcal{W}(D_n) = G(2,2,n)$ in Sections 7 and 8.

Lemma 6.7. *Suppose $G \subset GL_n(\mathbb{F})$, $n \geq 3$, is a finite monomial group with $q_{ij} = -1$ for $i \neq j$. Say G contains the transposition $(1\ 2)$ with $\det_{1212}(c) = 1$ for all c in $C_G((1\ 2))$. Then for any parameter m in $\mathbb{F}$, there is a quantum Drinfeld Hecke algebra $\mathcal{H}_{Q,\kappa}$ with $\kappa : V \otimes V \to \mathbb{F}G$ supported on transpositions with $\kappa_{(1\ 2)}(v_1, v_2) = m$ and $\kappa_{(1\ 2)}(v_1, v_3) = 0$.*

Proof. Define a quantum 2-form κ supported on the conjugacy class of $(1\ 2)$ by setting $\kappa_{h^{-1}(1\ 2)h}(v_i, v_j) = \det_{ij12}(h)\ \kappa_{(1\ 2)}(v_1, v_2)$ for $i < j$. We argue that this does not depend on choice of h. Indeed, if $h^{-1}(1\ 2)h = z^{-1}(1\ 2)z$ for h, z in G, then $zh^{-1} \in C_G((1\ 2))$ and $\det_{1212}(zh^{-1}) = 1$. Since this is nonzero, Lemma 6.2 gives a unique pair $i < j$ with

$$1 = \det_{1212}(zh^{-1}) = \det_{12ij}(h^{-1})\ \det_{ij12}(z) = (\det_{ij12}(h))^{-1}\ \det_{ij12}(z)$$

and $\det_{ij12}(h) = \det_{ij12}(z)$ so κ is well-defined. One may then check the conditions of Theorem 5.3 directly. Note that Theorem 5.3(4) holds by Eq. (5.4) using Lemma 6.3 since $\det_{ij12}(h)$ is nonzero for only one fixed pair $i < j$ and so $\det_{ijkl}(h)$ is nonzero with $k < l$ only for $k = 1$, $l = 2$:

$$\kappa_{h^{-1}(1\ 2)h}(v_i, v_j) = \det_{ij12}(h)\, \kappa_{(1\ 2)}(v_1, v_2) = \sum_{k<l} \det_{ijkl}(h)\, \kappa_{(1\ 2)}(v_k, v_l)\,. \quad \square$$

Remark 6.8. For the Coxeter groups $\mathfrak{S}_n = G(1,1,n)$, $\mathcal{W}(B_n) = G(2,1,n)$, and $\mathcal{W}(D_n) = G(2,2,n)$ for $n \geq 2$, the proof of Lemma 6.7 gives an admissible parameter κ for any $m \in \mathbb{F}$ defined by

$$\kappa_g(v_i, v_j) = \begin{cases} m & \text{for } g = (i\ j) \\ -m & \text{for } g = t_i t_j (i\ j) \\ 0 & \text{otherwise}\,, \end{cases}$$

where t_i is the identity matrix except -1 in the i-th slot. Note here that all conjugates of $(1\ 2)$ take the form $(i\ j)$ or $t_i t_j (i\ j)$ for some $i \neq j$.

In the next two lemmas, we again consider the the symmetric group $\mathfrak{S}_n$ acting by permuting basis elements of V, i.e., we identify $\mathfrak{S}_n$ with the group of permutation matrices in $\mathrm{GL}_n(\mathbb{F})$.

Lemma 6.9. *Say $G = \mathfrak{S}_n$ for $n \geq 3$ and $q_{ij} = -1$ for $i \neq j$. Then for any parameter m in $\mathbb{F}$, there is a quantum Drinfeld Hecke algebra $\mathcal{H}_{Q,\kappa}$ for $\kappa : V \otimes V \to \mathbb{F}G$ supported on transpositions with $\kappa_{(1\ 2)}(v_1, v_2) = 0$ and $\kappa_{(1\ 2)}(v_1, v_3) = m$.*

Proof. Notice that $\det_{ijkl}(g) \in \{0, 1\}$ for all $g \in G$. Say $m \neq 0$ and define a quantum 2-form κ supported on the conjugacy class of $(1\ 2)$ by

$$\kappa_{(a\ b)}(v_i, v_j) = \begin{cases} m & \text{for } i \text{ or } j \text{ in } \{a, b\} \text{ but not both}, \\ 0 & \text{otherwise}\,. \end{cases}$$

We argue that κ satisfies Theorem 5.3(4) by verifying Eq. (5.4) using Lemma 6.3:

$$\kappa_{h^{-1}(1\ 2)h}(v_i, v_j) = \sum_{k<l} \det_{ijkl}(h)\ \kappa_{(1\ 2)}(v_k, v_l) \qquad \text{for all } h \in G\,, 1 \le i < j\ \le n.$$

For fixed $i < j$ and h in G, set $a = h^{-1}(1)$ and $b = h^{-1}(2)$ so $h^{-1}(1\ 2)h = (a\ b)$. There is a unique pair $i' < j'$ with $0 \neq \det_{iji'j'}(h) = \det_{i'j'ij}(h^{-1})$, so $(h(i)\ h(j)) = (i'\ j')$, and we need only verify that

$$\kappa_{(a\ b)}(v_i, v_j) = \det_{iji'j'}(h)\ \kappa_{(1\ 2)}(v_{i'}, v_{j'})\ = \kappa_{(1\ 2)}(v_{i'}, v_{j'})\,.$$

Each side is either m or zero. The scalar $\kappa_{(a\ b)}(v_i, v_j)$ is nonzero exactly when the set $\{i, j\} \cap \{a, b\}$ has size 1, i.e., exactly when $\{i', j'\} \cap \{1, 2\} = \{h(i), h(j)\} \cap \{h(a), h(b)\}$ has size 1. But this is exactly the condition that $\kappa_{(1\ 2)}(v_{i'}, v_{j'})$ is nonzero and Theorem 5.3(4) holds. The other conditions of Theorem 5.3 may be checked directly. Note that for the quantum Jacobi identity, we verify that

$$({}^h v_k - v_k)\kappa_h(v_i, v_j) + ({}^h v_j - v_j)\kappa_h(v_i, v_k) + ({}^h v_i - v_i)\kappa_h(v_j, v_k) = 0$$

by taking $h = (a\ b)$ and considering various overlaps of $\{i, j, k\}$ with $\{a, b\}$. □

Lemma 6.10. *Say $G = \mathfrak{S}_n$ for $n > 3$ and $q_{ij} = -1$ for $i \neq j$. There is a nontrivial quantum Drinfeld Hecke algebra $\mathcal{H}_{Q,\kappa}$ for $\kappa : V \otimes V \to \mathbb{F}G$ supported on products of two disjoint transpositions.*

Proof. Suppose $0 \neq m \in \mathbb{F}$ and define a quantum 2-form κ supported on the conjugacy class of $(1\ 2)(3\ 4)$ by setting, for disjoint 2-cycles $(a\ b)$ and $(c\ d)$ and $i < j$,

$$\kappa_{(a\ b)(c\ d)}(v_i, v_j) = \begin{cases} m & \text{for } (a\ b) \neq (i\ j) \neq (c\ d) \text{ and } i, j \in \{a, b, c, d\}, \\ 0 & \text{otherwise.} \end{cases}$$

We argue that κ satisfies Theorem 5.3(4) by verifying Eq. (5.4) using Lemma 6.3:

$$\kappa_{h^{-1}(1\ 2)(3\ 4)h}(v_i, v_j) = \sum_{k<l} \det_{ijkl}(h)\ \kappa_{(1\ 2)(3\ 4)}(v_k, v_l) \qquad \text{for all } 1 \le i < j \le n,\ h \in G\,.$$

Fix $i < j$ and h in G and set $(a\ b)(c\ d) = h^{-1}(1\ 2)(3\ 4)h$. There is a unique pair $i' < j'$ with $0 \neq \det_{iji'j'}(h) = \det_{i'j'ij}(h^{-1})$, so $(h(i)\ h(j)) = (i'\ j')$ and we need only check that

$$\kappa_{(a\ b)(c\ d)}(v_i, v_j) = \det_{iji'j'}(h)\ \kappa_{(1\ 2)(3\ 4)}(v_{i'}, v_{j'})\ = \kappa_{(1\ 2)(3\ 4)}(v_{i'}, v_{j'})\,.$$

We verify as in the proof of Lemma 6.9, noting that $(a\ b) \neq (i\ j) \neq (c\ d)$ with $i, j \in \{a, b, c, d\}$ exactly when $(1\ 2) \neq (i'\ j') \neq (3\ 4)$ with $i', j' \in \{1, 2, 3, 4\}$ (just apply h to each index). □

7. Symmetric group acting by permutation of basis vectors

We consider quantum Drinfeld Hecke algebras for the action of the symmetric group $\mathfrak{S}_n$ by permutations in this section. We assume at least one quantum parameter q_{ij} is not 1, else we are in the non-quantum setting and may use the classification of algebras from [29]. This forces $q_{ij} = -1$ for all $i \neq j$ by Lemma 2.2. Thus we assume throughout this section that

$$Q = \{q_{ij} = -1, q_{ii} = 1 : 1 \le i \neq j \le n\}.$$

The dimension of the parameter space of quantum Drinfeld Hecke algebras depends on whether $n > 3$, so we gives the cases $n = 3$ and $n = 4$ explicitly before generalizing to arbitrary n. Here, as before, $\mathfrak{S}_n$ acts on $V \cong \mathbb{F}^n$ by permutation of basis vectors $v_1, \dots, v_n$, i.e., ${}^{\sigma}v_i = v_{\sigma(i)}$. We recover results of Naidu and Witherspoon [25] who worked over the complex numbers $\mathbb{C}$ and used Hochschild cohomology; our combinatorial approach (following ideas of [33]) allows us to extend results to arbitrary fields $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$.

3-dimensional space. A careful analysis using Theorem 5.3 gives a 4-parameter family when $n = 3$: The quantum Drinfeld Hecke algebras are the $\mathbb{F}$-algebras generated by v_1, v_2, v_3 and $\mathbb{F}\mathfrak{S}_3$ with relations $\sigma v_k = v_{\sigma(k)}\sigma$ for all k and all $\sigma \in \mathfrak{S}_3$ and, for some fixed scalars $m_1, \dots, m_4$ in $\mathbb{F}$,

$$\begin{aligned}
v_2 v_1 &= -v_1 v_2 + m_1 + m_2(1\ 2) + m_3\big((1\ 3) + (2\ 3)\big) + m_4\big((1\ 2\ 3) + (1\ 3\ 2)\big),\\
v_3 v_2 &= -v_2 v_3 + m_1 + m_2(2\ 3) + m_3\big((2\ 1) + (3\ 1)\big) + m_4\big((2\ 3\ 1) + (2\ 1\ 3)\big),\\
v_1 v_3 &= -v_3 v_1 + m_1 + m_2(3\ 1) + m_3\big((3\ 2) + (1\ 2)\big) + m_4\big((3\ 1\ 2) + (3\ 2\ 1)\big).
\end{aligned}$$

4-dimensional space. Theorem 7.1 below gives a 5-parameter family when $n = 4$: The quantum Drinfeld Hecke algebras are precisely the $\mathbb{F}$-algebras generated by v_1, v_2, v_3, v_4 and $\mathbb{F}\mathfrak{S}_4$ with relations $\sigma v_k = v_{\sigma(k)}\sigma$ for all k and all $\sigma \in \mathfrak{S}_4$ and, for some fixed scalars $m_1, \dots, m_5$ in $\mathbb{F}$,

$$\begin{aligned}
v_2 v_1 = -v_1 v_2 &+ m_1 + m_2(1\ 2) + m_3\big((1\ 3) + (1\ 4) + (2\ 3) + (2\ 4)\big)\\
&+ m_4\big((1\ 2\ 3) + (2\ 1\ 3) + (1\ 2\ 4) + (2\ 1\ 4)\big) + m_5\big((1\ 3)(2\ 4) + (1\ 4)(2\ 3)\big),\\
v_3 v_1 = -v_1 v_3 &+ m_1 + m_2(1\ 3) + m_3\big((1\ 2) + (1\ 4) + (2\ 3) + (3\ 4)\big)\\
&+ m_4\big((1\ 3\ 2) + (3\ 1\ 2) + (1\ 3\ 4) + (3\ 1\ 4)\big) + m_5\big((1\ 2)(3\ 4) + (1\ 4)(2\ 3)\big),\\
v_4 v_1 = -v_1 v_4 &+ m_1 + m_2(1\ 4) + m_3\big((1\ 2) + (1\ 3) + (2\ 4) + (3\ 4)\big)\\
&+ m_4\big((1\ 4\ 2) + (4\ 1\ 2) + (1\ 4\ 3) + (4\ 1\ 3)\big) + m_5\big((1\ 2)(3\ 4) + (1\ 3)(2\ 4)\big),\\
v_3 v_2 = -v_2 v_3 &+ m_1 + m_2(2\ 3) + m_3\big((1\ 2) + (1\ 3) + (2\ 4) + (3\ 4)\big)\\
&+ m_4\big((2\ 3\ 1) + (3\ 2\ 1) + (2\ 3\ 4) + (3\ 2\ 4)\big) + m_5\big((1\ 2)(3\ 4) + (1\ 3)(2\ 4)\big),\\
v_4 v_2 = -v_2 v_4 &+ m_1 + m_2(2\ 4) + m_3\big((1\ 2) + (1\ 4) + (2\ 3) + (3\ 4)\big)\\
&+ m_4\big((2\ 4\ 1) + (4\ 2\ 1) + (2\ 4\ 3) + (4\ 2\ 3)\big) + m_5\big((1\ 2)(3\ 4) + (1\ 4)(2\ 3)\big),\\
v_4 v_3 = -v_3 v_4 &+ m_1 + m_2(3\ 4) + m_3\big((1\ 4) + (2\ 3) + (2\ 4) + (1\ 3)\big)\\
&+ m_4\big((3\ 4\ 1) + (4\ 3\ 1) + (3\ 4\ 2) + (4\ 3\ 2)\big) + m_5\big((1\ 3)(2\ 4) + (1\ 4)(2\ 3)\big).
\end{aligned}$$

Arbitrary dimension. The quantum Drinfeld Hecke algebras constitute a 5-parameter family for the symmetric group $\mathfrak{S}_n$ with $n \geq 4$:

Theorem 7.1. *Let $G = \mathfrak{S}_n$ act on $V \cong \mathbb{F}^n$ by permutation of basis vectors for $n \geq 4$. The quantum Drinfeld Hecke algebras are precisely the $\mathbb{F}$-algebras generated by $v_1, \dots, v_n$ and $\mathbb{F}\mathfrak{S}_n$ with relations $\sigma v_k = v_{\sigma(k)}\sigma$ for all k and*

$$\begin{aligned}
v_{\sigma(2)} v_{\sigma(1)} = &- v_{\sigma(1)} v_{\sigma(2)} + m_1 + m_2\ \big(\sigma(1)\ \sigma(2)\big) + m_3 \sum_{i \neq \sigma(1),\, \sigma(2)} \Big(\big(\sigma(1)\ i\big) + \big(\sigma(2)\ i\big)\Big)\\
&+ m_4 \sum_{i \neq \sigma(1), \sigma(2)} \Big(\big(\sigma(1)\ \sigma(2)\ i\big) + \big(\sigma(2)\ \sigma(1)\ i\big)\Big) + m_5 \sum_{i,j \notin \{\sigma(1),\, \sigma(2)\};\ i \neq j} \big(\sigma(1)\ i\big)\big(\sigma(2)\ j\big),
\end{aligned}$$

for all $\sigma \in G$, for some fixed scalars $m_1, \dots, m_5$ in $\mathbb{F}$.

Note that the right side indeed only depends only on $\sigma(1)$ and $\sigma(2)$ (as an unordered pair).

Proof. Suppose κ is admissible. Theorem 5.3(4) implies that κ is invariant, i.e., for any σ in $\mathfrak{S}_n$, $\kappa(v_{\sigma(1)}, v_{\sigma(2)}) = \sum_{g\in G} \kappa_g(v_1, v_2)\, \sigma g \sigma^{-1}$, and κ is determined by $\kappa(v_1, v_2)$. As cycle type determines the conjugacy classes in $\mathfrak{S}_n$, Lemma 6.1 implies that if $\kappa_g \not\equiv 0$, then g is conjugate to I (the identity), (1 2), (1 2 3), or (1 3)(2 4).

By Theorem 5.3(3) and (4), $0 = \kappa_{(k\ l)}(v_1, v_2) = \kappa_{(1\ 2)(i\ j)}(v_1, v_2) = \kappa_{(i\ j)(k\ l)}(v_1, v_2)$ when $k, l \notin \{1, 2\}$ and $(i\ j) \neq (1\ 2)$. In addition (by Eq. (5.4)), $\kappa_{(1\ l)(2\ k)}(v_1, v_2) = \kappa_{(1\ 3)(2\ 4)}(v_1, v_2)$ for all $l \neq k$ with $l, k \notin \{1, 2\}$. In fact, one may show (using Lemma 6.4 and Eq. (5.5)) that κ is determined by

$$
\begin{aligned}
&m_1 = \kappa_I(v_1, v_2), \qquad m_2 = \kappa_{(1\ 2)}(v_1, v_2),\\
&m_3 = \kappa_{(1\ 3)}(v_1, v_2), \quad m_4 = \kappa_{(1\ 2\ 3)}(v_1, v_2), \quad \text{and} \quad m_5 = \kappa_{(1\ 3)(2\ 4)}(v_1, v_2)\,.
\end{aligned}
$$

Conversely, using Eq. (5.5), the identity I in G contributes one parameter worth of quantum Drinfeld Hecke algebras, the conjugacy class of (1 2) contributes two parameters worth by Lemmas 6.7 and 6.9, and the conjugacy classes of (1 2 3) and (1 2)(3 4) each contribute another parameter of freedom by Lemmas 6.4 and 6.10. The proofs of these lemmas give the algebras in the statement of the theorem explicitly. □

8. Infinite families of reflection groups $G(r, p, n)$

We consider the infinite family $G(r, p, n)$ of reflection groups (see Shephard and Todd [28] when $\mathbb{F} = \mathbb{C}$), which includes

- the symmetric group acting as permutations, $\mathfrak{S}_n = G(1, 1, n)$,
- the Weyl groups $\mathcal{W}(B_n) = G(2, 1, n)$ acting on $\mathbb{R}^n$ or $\mathbb{C}^n$,
- the Weyl groups $\mathcal{W}(D_n) = G(2, 2, n)$ acting on $\mathbb{R}^n$ or $\mathbb{C}^n$, and
- the symmetry group $G(r, 1, n)$ of the regular r-cube polytope in $\mathbb{C}^n$.

We describe the quantum Drinfeld Hecke algebras, recovering results of Naidu and Witherspoon [25] in the case $\mathbb{F} = \mathbb{C}$ and $n \geq 4$. The combinatorial approach chosen here over a cohomological approach has certain advantages: This combinatorial avenue

- allows us to extend results to fields $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$,
- helps us classify algebras in the delicate case when $n = 3$,
- reveals an extra parameter of algebras for $G(r, r, 4)$ when r is odd, and
- extends to other groups, like mystic reflection groups (examined in the next section).

We fix $r, p, n \in \mathbb{Z}$ with p dividing r, and assume $\mathbb{F}$ contains a primitive r-th root-of-unity ω in this section. The finite group $G(r, p, n) \subset \mathrm{GL}(\mathbb{F})$ consists of the $n \times n$ monomial matrices whose nonzero entries are r-th roots-of-unity in $\mathbb{F}$ and whose product of nonzero entries is 1 when raised to the power r/p. The group $G(r, p, n)$ has order $n!\, r^n/p$ and is the semidirect product $D(r, p, n) \rtimes \mathfrak{S}_n$ for $D(r, p, n)$ the subgroup of diagonal matrices in $G(r, p, n)$. Note that $G(r, p, n)$ contains the symmetric group $\mathfrak{S}_n = G(1, 1, n)$ as a subgroup.

By Lemma 2.2, the group $G = G(r, p, n)$ acts on the quantum polynomial ring $S_Q(V)$ by automorphisms if and only if either $q_{ij} = 1$ for all i, j or else $q_{ij} = -1$ for all $i \neq j$. In the trivial case, when all $q_{ij} = 1$, [26] gives a classification of quantum Drinfeld Hecke algebras. Thus we assume throughout this section that the set of quantum parameters is

$$Q = \{q_{ij} = -1, q_{ii} = 1 : 1 \leq i \neq j \leq n\}.$$

Define a diagonal matrix Λ_i which is ωI except with ω^{1-n} as i-th entry:

$$\Lambda_i = \mathrm{diag}(\omega, \ldots, \omega, \omega^{1-n}, \omega, \ldots, \omega)\,. \tag{8.1}$$

Lemma 8.2. *Say $G = G(r, r, 4)$ for $r \geq 1$ with r odd. There is a nontrivial quantum Drinfeld Hecke algebra $\mathcal{H}_{Q,\kappa}$ for $\kappa : V \otimes V \to \mathbb{F}G$ supported on the conjugacy class of* $(1\ 2)(3\ 4)$.

Proof. Let $g = (1\ 2)(3\ 4)$ and $0 \neq m \in \mathbb{F}$. Define a quantum 2-form κ supported on the conjugacy class of g by setting $\kappa_g(v_i, v_j) = 0$ for $\{i, j\} = \{1, 2\}$ or $\{3, 4\}$, $\kappa_g(v_i, v_j) = m$ otherwise, and extending via

$$\kappa_{h^{-1}gh}(v_i, v_j) = \sum_{k<l} \det{}_{ijkl}(h)\ \kappa_g(v_k, v_l) \quad \text{for} \quad 1 \leq i, j \leq 4,\ h \in G\,. \tag{8.3}$$

We argue that $\kappa_{h^{-1}gh}$ is well-defined. Say $h^{-1}gh = z^{-1}gz$. As in the proof of Lemma 6.4, we show that $\kappa_{h^{-1}gh}(v_i, v_j) = \kappa_{z^{-1}gz}(v_i, v_j)$ using the fact that

$$\kappa_{h^{-1}gh}(v_i, v_j) = \det{}_{ijab}(h)\ \kappa_g(v_a, v_b) \quad \text{whereas} \quad \kappa_{z^{-1}gz}(v_i, v_j) = \det{}_{ijcd}(z)\ \kappa_g(v_c, v_d) \tag{8.4}$$

for some pairs $a < b$ and $c < d$ with $\det_{ijab}(h) \neq 0 \neq \det_{ijcd}(z)$ and

$$\det{}_{abcd}(zh^{-1}) = \det{}_{abij}(h^{-1})\ \det{}_{ijcd}(z) = (\det{}_{ijab}(h))^{-1}\ \det{}_{ijcd}(z) \neq 0. \tag{8.5}$$

If a matrix lies in the centralizer $C_G(g)$, then the entries in its first two columns coincide and also the entries in its last two columns coincide and the matrix is the product of a diagonal matrix with (1), (1 2), (3 4), (1 2)(3 4), (1 3)(2 4), (1 4)(2 3), (1 4 2 3), or (1 3 2 4). In addition, the entry in the first column is inverse to that in the last column because $r = p$ is odd, and thus there are exactly r elements in $C_G(g)$ corresponding to each permutation listed. Then as zh^{-1} lies in $C_G(g)$ with $\det_{abcd}(zh^{-1}) \neq 0$, we conclude after careful examination of the centralizer that either $\{(1\ 2), (3\ 4)\}$ contains both $(a\ b)$ and $(c\ d)$ and hence $\kappa_g(v_a, v_b) = \kappa_g(v_c, v_d) = 0$ or else $\{(1\ 2), (3\ 4)\}$ contains neither $(a\ b)$ nor $(c\ d)$ and hence $\kappa_g(v_a, v_b) = \kappa_g(v_c, v_d) \neq 0$ with $\det_{abcd}(zh^{-1}) = 1$. Eqs. (8.4) and (8.5) then imply that $\kappa_{h^{-1}gh}(v_i, v_j) = \kappa_{z^{-1}gz}(v_i, v_j)$.

One may then use Theorem 5.3 to verify that κ is admissible with some straightforward computations. For Theorem 5.3(3), note that any conjugate $h^{-1}gh$ of g is the product of a diagonal matrix with (1 2)(3 4) or (1 3)(2 4) or (1 4)(2 3). For example, if $h^{-1}gh$ enacts $v_1 \mapsto \lambda v_2$, $v_2 \mapsto \lambda^{-1} v_1$, $v_3 \mapsto \eta v_4$, and $v_4 \mapsto \eta^{-1} v_3$ for λ, η in $\mathbb{F}$, we may take h to be the diagonal matrix $(\lambda^{1/2}, \lambda^{-1/2}, \eta^{1/2}, \eta^{-1/2})$ and check Theorem 5.3(3) directly using Eq. (8.3). Here, we use the fact that the squaring map on the multiplicative group of r-th roots-of-unity is onto since r is odd. □

Naidu and Witherspoon [25, Theorem 6.9] proved the following when $\mathbb{F} = \mathbb{C}$ and $n \geq 4$ by computing Hochschild cohomology. The homological techniques used do not extend directly to arbitrary fields. We give a direct combinatorial proof that holds for all fields $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$, including the case when $\mathrm{char}(\mathbb{F})$ divides $|G|$, and all $n \geq 3$.

Proposition 8.6. *Let $G = G(r, p, n)$ for $n \geq 3$. Then the dimension of the parameter space of quantum Drinfeld Hecke algebras is*

$$\dim_{\mathbb{F}}(P_G) = \begin{cases} 5 & \text{if } r = 1, n \geq 4, \text{ i.e., } G = G(1,1,n) = \mathfrak{S}_{n\geq 4}, \\ 4 & \text{if } r = 1, n = 3, \text{ i.e., } G = G(1,1,3) = \mathfrak{S}_3, \\ 2 & \text{if } r = 2, \text{ i.e., } G = \mathcal{W}(B_n) = G(2,1,n) \text{ or } G = \mathcal{W}(D_n) = G(2,2,n), \\ 1 & \text{if } r > 2, \ 3 \nmid r, \text{ and } G = G(r, r/2, 3) \text{ with } r \text{ even}, \\ 1 & \text{if } r > 2, \ 3 \nmid r, \text{ and } G = G(r,r,3), \\ 1 & \text{if } r > 2, \ r \text{ odd, and } G = G(r,r,4), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. In the case $r = p = 1$, $G = G(1,1,n)$ is the symmetric group $\mathfrak{S}_n$ and we appeal to Section 7. So we assume $r > 1$.

We use Theorem 5.3 with Eq. (5.5). Assume that κ is admissible and that g in $G(r,p,n)$ has one of the cycle types given in Lemma 6.1.

First consider the case when $r = 2$ so that G is the Weyl group $G(2,2,n)$ or $G(2,1,n)$. Theorem 5.3(3) and (4) imply that $\kappa_g \equiv 0$ unless g lies in the conjugacy classes of (1 2) or (1 2 3) (one must check the quantum minor determinants of elements in the centralizer of g). In fact, $\kappa_{(1\ 2)}(v_i, v_j) = 0$ unless $(i,j) = (1,2)$. (For example, $h = (-1 \oplus -1 \oplus I)$ commutes with (1 2) forcing $\kappa_{(1\ 2)}(v_1, v_3) = 0$). The conjugacy class of (1 2) contributes one parameter of freedom to the family of quantum Drinfeld Hecke algebras by Lemma 6.7 and the conjugacy class of (1 2 3) contributes another parameter by Lemma 6.4, hence $\dim_{\mathbb{F}}(P_G) = 2$. Hence we assume $r > 2$.

The group $G(r,p,n)$ contains the set of diagonal matrices $\{\Lambda_1, \ldots, \Lambda_n\}$ from Eq. (8.1); Eq. (5.4) forces $\kappa_g \equiv 0$ for any g a diagonal matrix and for any g with 2-cycle type since certain Λ_i lie in the centralizer of g. For example, if g has cycle-type (1 2), then Λ_k lies in the centralizer of g for $k \notin \{1,2\}$ with $\kappa_g(v_i, v_j) = \det_{ijij}(\Lambda_k)\, \kappa_g(v_i, v_j)$ by Theorem 5.3(4); this forces $\kappa_g(v_i, v_j) = 0$ for $i, j \neq k$ (since $\det_{ijij}(\Lambda_k) = \omega^2 \neq 1$). If $n \geq 5$ this forces $\kappa_g(v_i, v_j) = 0$ for all i, j as k can vary over $3 \leq k \leq n$. When $n = 3, 4$, one can verify that $\det_{ijij}(\Lambda_3) \neq 1$ for all $i \neq j$ forcing $\kappa_g(v_i, v_j) = 0$. Hence we may assume g has 3-cycle type or is the product of two disjoint 2-cycle type elements.

Now assume $n = 3$. If $3 \mid r$, then the center of G contains the scalar matrix $h = \omega^{r/3} I$, which forces $\kappa \equiv 0$ by Theorem 5.3(4) (see Eq. (5.4)) since $\det_{ijij}(h) = \omega^{r2/3} \neq 1$ for all $i \neq j$ and $\dim_{\mathbb{F}}(P_G) = 0$. If $r/p > 2$, then the scalar matrix $h = \omega^p I$ lies in the center and likewise forces $\kappa \equiv 0$ since $\det_{ijij}(h) = \omega^{2p} \neq 1$ for all $i \neq j$ as $r \nmid 2p$ and $\dim_{\mathbb{F}}(P_G) = 0$.

For $n = 3$, that leaves the cases $r/p \leq 2$ and $3 \nmid r$. Theorem 5.3(3) forces $\kappa_g \equiv 0$ for g of 3-cycle type unless the product of nonzero entries in g is 1. Such elements all lie in the conjugacy class of (1 2 3) which generates its own centralizer in G, and hence $\dim_{\mathbb{F}}(P_G) = 1$ in this case by Lemma 6.4.

Now assume $n \geq 4$. Suppose g has 3-cycle type $(a\ b\ c)$. Then g commutes with any Λ_d for $d \in \{1, \ldots, n\}/\{a, b, c\}$ and $\kappa_g \equiv 0$ by Theorem 5.3(4) (see Eq. (5.4)) since $\det_{abab}(\Lambda_d) = \det_{bcbc}(\Lambda_d) = \det_{acac}(\Lambda_d) = \omega^2 \neq 1$ (as $r > 2$). This forces $\kappa_g(v_a, v_b) = \kappa_g(v_b, v_c) = \kappa_g(v_c, v_a) = 0$. Theorem 5.3(3) forces $\kappa_g(v_i, v_j) = 0$ for i or j not in $\{a, b, c\}$ (just choose $k \in \{a, b, c\}$).

Thus we may assume $n \geq 4$ and g is the product of two disjoint 2-cycle type elements, say g is the product of a diagonal matrix and (1 2)(3 4). If r is even, then G contains the diagonal matrix $h = \mathrm{diag}(-1, -1, 1, \ldots, 1)$ which commutes with g and arguments similar to above show that $\kappa_g \equiv 0$. If r is odd but $n > 4$, then G contains the diagonal matrix

$h = \text{diag}(\omega, \omega, \omega, \omega, \omega^{-4}, 1, \ldots, 1)$ which commutes with g and we may show $\kappa_g \equiv 0$. This leaves the case that $n = 4$ and r is odd: again, Theorem 5.3(3) forces $\kappa_g \equiv 0$ unless g is conjugate to (1 2)(3 4) and by Lemma 8.2, $\dim_{\mathbb{F}}(P_G) = 1$. □

Example 8.7. Let $G = G(2,1,3)$ over $\mathbb{F} = \mathbb{R}$, the Weyl group $\mathcal{W}(B_3)$. Every quantum Drinfeld Hecke algebra $\mathcal{H}_{Q,\kappa}$ is generated by v_1, v_2, v_3 and $\mathbb{R}G$ with relations $gv_k = {}^g v_k g$ for all k and $g \in G$ and

$$\begin{aligned}
v_2v_1 &= -v_1v_2 + m(s_1 - t_1t_2s_1) \\
&\quad + m'(s_1s_2 - t_2t_3s_1s_2 - t_1t_2s_1s_2 + t_1t_3s_1s_2 + s_2s_1 - t_1t_3s_2s_1 + t_2t_3s_2s_1 - t_1t_2s_2s_1), \\
v_3v_2 &= -v_2v_3 + m(s_2 - t_2t_3s_2) \\
&\quad + m'(s_1s_2 - t_2t_3s_1s_2 + t_1t_2s_1s_2 - t_1t_3s_1s_2 + s_2s_1 + t_1t_3s_2s_1 - t_2t_3s_2s_1 - t_1t_2s_2s_1), \\
v_3v_1 &= -v_1v_3 + m(s_1s_2s_1 - t_1t_3s_1s_2s_1) \\
&\quad + m'(s_1s_2 + t_2t_3s_1s_2 - t_1t_2s_1s_2 - t_1t_3s_1s_2 + s_2s_1 - t_1t_3s_2s_1 - t_2t_3s_2s_1 + t_1t_2s_2s_1)
\end{aligned}$$

for some $m, m' \in \mathbb{R}$. Thus $\dim_{\mathbb{R}}(P_G) = 2$. Here, s_i is the transposition $(i\ i+1)$ and t_i is the identity matrix except with -1 in the i-th entry.

Example 8.8. The group $G(2,2,3)$ over $\mathbb{F} = \mathbb{R}$ is the Weyl group $\mathcal{W}(D_3)$. One compares the conjugacy classes in $G(2,2,3)$ to those in $G(2,1,3)$ to see that every quantum Drinfeld Hecke algebra for $G(2,1,3)$ is a quantum Drinfeld Hecke algebra for $G(2,2,3)$ and vice versa. So the dimension of the parameter space is also 2 for $G(2,2,3)$.

Here are two examples over a field of characteristic 5, the first in the nonmodular setting and the second in the modular setting.

Example 8.9. In the group $G = G(4,2,3)$ over $\mathbb{F} = \mathbb{F}_{25}$, the product of nonzero entries in each matrix is ± 1 (here, 2 is a primitive 4-th root-of-unity). The dimension of the parameter space P_G of quantum Drinfeld Hecke algebras is 1. Indeed, every PBW algebra is supported on the 32-element conjugacy class of (1 2 3): we just fix $\kappa_{(1\ 2\ 3)}(v_1, v_2) \in \mathbb{F}_{25}$, use Eq. (5.4) to determine κ_g for g conjugate to (1 2 3), and set $\kappa_g \equiv 0$ for all other g in G.

Example 8.10. In the group $G = G(2,2,5)$ over $\mathbb{F} = \mathbb{F}_{25}$, the product of nonzero entries in each matrix is 1. Here, $\dim_{\mathbb{F}}(P_G) = 2$. Indeed, every PBW algebra is supported on the conjugacy class of (1 2 3) or the conjugacy class of (1 2): we fix $\kappa_{(1\ 2\ 3)}(v_1, v_2) \in \mathbb{F}_{25}$ and $\kappa_{(1\ 2)}(v_1, v_2) \in \mathbb{F}_{25}$, use Eq. (5.4) to determine κ_g for g conjugate to (1 2 3) or (1 2), and set $\kappa_g \equiv 0$ for all other g in G.

9. Mystic Reflection Groups

Another natural family to consider is the infinite family of mystic reflection groups described by Kirkman, Kuzmanovich, and Zhang [18] and also Bazlov and Berenstein [6]. Following their constructions, we assume throughout this section that $\mathbb{F} = \mathbb{C}$, $\alpha, \beta \in \mathbb{Z}_{>0}$ with $\alpha \mid \beta$ and $2 \mid \beta$, and

$$Q = \{q_{ij} = -1, q_{ii} = 1 : 1 \le i \ne j \le n\}.$$

Definition 9.1 ([18])**.** For $1 \le i, j \le n$, $i \ne j$, $\lambda \ne 1$, define $\theta_{i,\lambda}$, $\tau_{i,j,\lambda}$ in $\text{Aut}_{\text{gr}}(S_Q(V))$ by

$$\theta_{i,\lambda}(v_l) = \begin{cases} v_l & l \ne i \\ \lambda v_l & l = i \end{cases} \qquad \text{and} \qquad \tau_{i,j,\lambda}(v_l) = \begin{cases} v_l & l \ne i,j \\ \lambda v_j & l = i \\ -\lambda^{-1} v_i & l = j \end{cases}.$$

The $\theta_{i,\lambda}$ are called *standard reflections* and the $\tau_{i,j,\lambda}$ *standard mystic reflections*. Then the *mystic reflection group* $M(n,\alpha,\beta)$ is the following subgroup of $\mathrm{Aut}_{\mathrm{gr}}(S_Q(V))$:

$$M(n,\alpha,\beta) = \left\langle \{\theta_{i,\lambda} \mid \lambda^\alpha = 1, 1 \le i \le n\} \cup \{\tau_{i,j,\lambda} \mid \lambda^\beta = 1, 1 \le i \ne j \le n\} \right\rangle .$$

This provides infinite families of groups with nontrivial quantum Drinfeld Hecke algebras:

Theorem 9.2. *The dimension of the parameter space P_G of quantum Drinfeld Hecke algebras for $G = M(n,\alpha,\beta)$ for $n \ge 3$ is*

$$\mathit{dim}_{\mathbb{F}}(P_G) = \begin{cases} 2 & \text{if } \ G = M(n,2,2), \\ 1 & \text{if } \ G = M(n,1,2), \\ 1 & \text{if } \ \ n = 3 \text{ with } \alpha \le 2,\ 3 \nmid \beta, \text{ and } G \ne M(3,2,2), \\ 0 & \text{otherwise}. \end{cases}$$

Proof. For $G = M(n,2,2) = G(2,1,n)$, we appeal to Proposition 8.6. For the other cases, we use Theorem 5.3 with Eq. (5.5). Assume that κ is admissible and that g in $M(n,\alpha,\beta)$ has one of the cycle types given in Lemma 6.1.

First suppose that $G = M(n,1,2) \subset G(2,1,n) \cap \mathrm{SL}_n(\mathbb{C})$. Theorem 5.3(3) and (4) imply that $\kappa_g \equiv 0$ unless g lies in the conjugacy class of $(1\ 2\ 3)$. (We check the quantum minor determinants of elements in the centralizer of g in addition to judiciously choosing indices in Theorem 5.3(3): For example, for $g = \tau_{i,j,-1}$ with $i \ne j$ and $k,l \notin \{i,j\}$, the elements g and $h = g^2$ commute with g with $\det_{ijij}(g) \ne 1 \ne \det_{ikik}(h)$; this forces $\kappa_g(v_i,v_j) = \kappa_g(v_i,v_k) = 0$, whereas Theorem 5.3(3) forces $\kappa_g(v_k,v_l) = 0$.) One may check that the hypotheses of Lemma 6.4 hold (see [33]) and thus the conjugacy class of $(1\ 2\ 3)$ contributes one parameter of freedom to P_G.

Thus we assume $\beta > 2$. We also assume α is 1 or 2, else the center of G contains cI for $c \ne \pm 1$, forcing $\kappa \equiv 0$ and $\dim_{\mathbb{F}}(P_G) = 0$.

Eq. (5.4) forces $\kappa_g \equiv 0$ when g is a diagonal matrix or has 2-cycle type since certain diagonal matrices Λ_i (see Eq. (8.1)) lie in the centralizer of g with $\det_{jkjk}(\Lambda_i) = e^{\frac{4\pi i}{\beta}} \ne 1$ for distinct $j,k \ne i$ as $2 \mid \beta > 2$. For example, the group contains

$$\tau_{2,1,\omega} \cdot \tau_{2,1,-1} \cdot \tau_{3,2,\omega^2} \cdot \tau_{3,2,-1} \cdot \dots \cdot \tau_{n,n-1,\omega^{n-1}} \cdot \tau_{n,n-1,-1} = \Lambda_n \qquad \text{for} \qquad \omega = e^{\frac{2\pi i}{\beta}}.$$

So we assume g has 3-cycle type or else is the product of two disjoint 2-cycle type elements.

Consider the case $n = 3$. If $3 \mid \beta$, then the center of G contains the scalar matrix $\tau_{2,1,\gamma} \cdot \tau_{2,1,-1} \cdot \tau_{3,2,\gamma^2} \cdot \tau_{3,2,-1} = \lambda^{\beta/3} I \ne \pm I$ for $\lambda = e^{2\pi i/\beta}$ which forces $\kappa \equiv 0$. So we assume $3 \nmid \beta$. In fact, $\kappa_g \equiv 0$ unless g is conjugate to $(1\ 2\ 3)$ and there is one parameter worth of algebras by Lemma 6.4 (see [33]). Note that $G \ne M(3,2,2)$ here as $\beta > 2$.

Lastly, we consider the case $n \ge 4$. Suppose g has 3-cycle type $(a\ b\ c)$. Then g commutes with Λ_d for $d \in \{1,\dots,n\}/\{a,b,c\}$ and $\omega = e^{2\pi i/\beta}$, which forces $\kappa_g(v_i,v_j) = 0$ for $i,j \in \{a,b,c\}$ by Theorem 5.3(4) (see Eq. (5.4)) since $\det_{abab}(\Lambda_d) = \det_{bcbc}(\Lambda_d) = \det_{acac}(\Lambda_d) = \omega^2 \ne 1$ (as $\beta > 2$). Theorem 5.3(3) forces $\kappa_g(v_i,v_j) = 0$ for i or j not in $\{a,b,c\}$ (just choose $k \in \{a,b,c\}$). Hence $\kappa_g \equiv 0$ in this case. Suppose instead that g is the product of two disjoint 2-cycle type elements, say g is the product of a diagonal matrix and $(1\ 2)(3\ 4)$. Then $\kappa_g \equiv 0$ as well since Theorem 5.3(3) forces $\kappa_g(v_1,v_2) = \kappa_g(v_3,v_4) = \kappa_g(v_i,v_j) = 0$ for i or $j \notin \{1,2,3,4\}$ and Theorem 5.3(4) forces $\kappa_g(v_i,v_j) = 0$ for $i,j \in \{1,2,3,4\}$ but $(1\ 2) \ne (i\ j) \ne (3\ 4)$ since the centralizer of g in G contains the diagonal matrix $\mathrm{diag}(-1,-1,1,\dots,1)$. $\square$

Example 9.3. The group $M(n,1,2)$ for $n \geq 3$ is generated by mystic reflections $\tau_{i,j,-1}$ and contains the 3-cycle $(1\ 2\ 3)$. The quantum Drinfeld Hecke algebras constitute a 1-parameter family with each algebra $\mathcal{H}_{Q,\kappa}$ supported on the conjugacy class of $(1\ 2\ 3)$: Fix $\kappa_{(1\ 2\ 3)}(v_1, v_2) = m \in \mathbb{C}$, use Eq. (5.4) to determine κ_g on the conjugacy class of $(1\ 2\ 3)$, and set $\kappa_g \equiv 0$ otherwise.

10. Direct Sums

We end by observing that there is no way *a priori* to predict how the dimension of the parameter space of quantum Drinfeld Hecke algebras will change when taking direct sums of acting groups. We demonstrate by simply adding on a 1-dimensional group action.

In the proposition below, we take a fixed basis v_1, v_2, v_3 of $V = V_2 \oplus V_1 \cong \mathbb{F}^3$ for $V_2 = \mathbb{F}^2$ and $V_1 = \mathbb{F}^1$ with v_1, v_2 spanning V_2 and v_3 spanning V_1. We write $q_{31}q_{32} \in G_1$ to mean multiplication by $q_{31}q_{32}$ lies in G_1, i.e., G_1 contains the 1×1 matrix $[q_{31}q_{32}]$.

Proposition 10.1. *Let G_k be a group of graded automorphisms acting on $S_{Q_k}(V_k)$ for $V_k = \mathbb{F}^k$ and $Q_k = \{q_{ij}^{(k)}\}$ for $k = 1, 2$. Suppose $G = G_2 \oplus G_1$ acts by graded automorphisms on $S_Q(V)$ for $V = V_2 \oplus V_1$ and $Q = \{q_{ij}\}$ with $q_{12} = q_{12}^{(2)}$. Then*

(a) if $|G_1| > 1$ and $q_{31}q_{32} \in G_1$, then $\dim_{\mathbb{F}}(P_G) = \dim_{\mathbb{F}}(P_{G_2})$,

(b) if $|G_1| = 1$ and $q_{31}q_{32} \in G_1$, then $\dim_{\mathbb{F}}(P_G) \geq \dim_{\mathbb{F}}(P_{G_2})$,

(c) if $|G_1| > 1$ and $q_{31}q_{32} \notin G_1$, then $\dim_{\mathbb{F}}(P_G) = 0$,

(d) if $|G_1| = 1$ and $q_{31}q_{32} \notin G_1$, then $\dim_{\mathbb{F}}(P_G)$ is not bound above or below by $\dim_{\mathbb{F}}(P_{G_2})$.

Proof. First note that in parts (a) and (c), the center $Z(G)$ contains a nonidentity matrix $z = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & * \end{smallmatrix}\right)$ with $\det_{1313,Q}(z) \neq 1 \neq \det_{2323,Q}(z)$ and thus $\kappa_h(v_1, v_3) \equiv 0 \equiv \kappa_h(v_2, v_3)$ for all $h \in G$ by Theorem 5.3(4) for any admissible parameter κ for G. In fact, for part (c), Theorem 5.3(3) forces $\kappa(v_1, v_2) = 0$ as well.

Now suppose we are in case (a) or (b). For each $g \in G_2$, define $h(g) = g \oplus [q_{31}q_{32}] \in G$. If κ' is an admissible parameter for G_2, then we may define an admissible parameter κ for G with $\kappa_{h(g)}(v_1, v_2) = \kappa'_g(v_1, v_2)$ for g in G_2 and κ zero otherwise. (One can check that κ satisfies Theorem 5.3(3) and (4)). Thus $\dim_{\mathbb{F}}(P_G) \geq \dim_{\mathbb{F}}(P_{G_2})$. In case (a), if κ is an admissible parameter for G, we may define an admissible parameter κ' for G_2 with $\kappa'_g(v_1, v_2) = \kappa_{h(g)}(v_1, v_2)$ and hence $\dim_{\mathbb{F}}(P_G) \leq \dim_{\mathbb{F}}(P_{G_2})$. Note that in part (b), we may have a strict inequality (see Example 10.3) or equality (Example 10.4). The claim in part (d) is verified with Example 10.2.

□

Example 10.2. To justify Proposition 10.1(d), we fix $G_1 = \{1\} \subset \mathrm{GL}_1(\mathbb{F})$ with $q_{31}q_{32} \neq 1$ and give three groups $G_i \subset \mathrm{GL}_2(\mathbb{F})$ with which to compare $\dim_{\mathbb{F}}(P_{G_i \oplus G_1})$ to $\dim_{\mathbb{F}}(P_{G_i})$. Set

$$G_2 = \left\{ \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\},\ G_3 = \left\{ \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\},\ G_4 = \left\{ \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\}.$$

Then

$$\begin{aligned}
&\dim_{\mathbb{F}}(P_{G_2}) = 2 > 0 = \dim_{\mathbb{F}}(P_{G_2 \oplus G_1}) && \text{if } q_{13} = -1, q_{23} = 1, \text{ and } q_{12} = 1, \\
&\dim_{\mathbb{F}}(P_{G_3}) = 0 < 1 = \dim_{\mathbb{F}}(P_{G_3 \oplus G_1}) && \text{if } q_{13} = -1, q_{23} = 1, \text{ and } q_{12} = -1, \quad \text{and} \\
&\dim_{\mathbb{F}}(P_{G_4}) = 0 = \dim_{\mathbb{F}}(P_{G_4 \oplus G_1}) && \text{if } q_{13} = -1, q_{23} = 1, \text{ and } q_{12} = 1.
\end{aligned}$$

Note that the inequality in Proposition 10.1(b) is often strict, as we see next.

Example 10.3. Consider $G = G_2 \oplus G_1 \subset \mathrm{GL}_3(\mathbb{C})$ for $G_1 = 1$ the trivial group and

$$G_2 = \left\{ \begin{pmatrix} -\sqrt{1-\eta^3} & \eta^2 \\ \eta & \sqrt{1-\eta^3} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \qquad \text{for } \eta = e^{\frac{2\pi i}{5}},\ q_{12} = 1,\ q_{23} = -1 = q_{13}.$$

Then $\dim_{\mathbb{C}}(P_{G_2}) = 0$ by Theorem 5.3 (4) as G_2 is abelian containing g with $\det_Q(g) = -1$. However, $\dim_{\mathbb{C}}(P_G) = 1$ as the set of quantum Drinfeld Hecke algebras for G comprises the algebras $\mathcal{H}_{Q,\kappa,G}$ generated by v_1, v_2, v_3 and $\mathbb{C}G$ with relations $gv_i = {}^g v_i g$ for all $g \in G$ and

$$v_2 v_1 = v_1 v_2, \quad v_3 v_2 = -v_2 v_3 + mg, \quad v_3 v_1 = -v_1 v_3 + m\,\eta^3\,(1 - \sqrt{1-\eta^3})\,g\,,$$

with parameter $m \in \mathbb{C}$. Thus $\dim_{\mathbb{C}}(P_G) > \dim_{\mathbb{C}}(P_{G_2})$.

We end with a classical complex reflection group, namely, the 2-dimensional tetrahedral group G_4 of order 24 as classified by Shephard and Todd [28]. We consider the direct sum of G_4 with a trivial group to demonstrate the equality in Proposition 10.1(b).

Example 10.4. Set $q_{12} = -1 = q_{13}$, $q_{23} = 1$, and $\omega = e^{2\pi i/3}$ in $\mathbb{C}$. Consider $G = \{1\} \oplus G_4$ (using a reflection representation of G_4 perhaps equivalent to your favorite) generated by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix} \quad \text{and} \quad B = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & i & \sqrt{2}i\omega^2 \\ 0 & \sqrt{2}i\omega^2 & -i\omega \end{pmatrix} \quad \text{with } g = B^2A^2B^2,\ g_2 = B^2A,\ g_3 = AB^2\,.$$

Then for any m in $\mathbb{C}$, the $\mathbb{C}$-algebra $\mathcal{H}_{Q,\kappa}$ generated by v_1, v_2, v_3 and $\mathbb{C}G$ with relations $gv_i = {}^g v_i g$ for all $g \in G$ and

$$v_3 v_2 = v_2 v_3 + m(g + g^{-1} + \omega^2 g_2 + \omega^2 g_2^{-1} + \omega g_3 + \omega g_3^{-1}), \quad v_3 v_1 = -v_1 v_3, \quad v_2 v_1 = -v_1 v_2$$

is a quantum Drinfeld Hecke algebra. By Theorem 5.3, these are all the quantum Drinfeld Hecke algebras. Thus $\dim_{\mathbb{C}}(P_G) = 1$. Note $\dim_{\mathbb{C}}(P_{G_4}) = 1$ as well.

References

[1] J. Alev and F. Dumas, “Rigidité des plongements des quotients primitifs minimaux de Uq(sl(2)) dans l’algèbre quantique de Weyl-Hayashi”, Nagoya Math. J., 143 (1996), 119–146.

[2] V.A. Artamonov, “Quantum Polynomial Algebras”, J. Math. Sci. (N.Y.), 87 (1997), 3, 3441–3462.

[3] J. Alev and M. Chamarie, “Dérivations et automorphismes de quelques algèbres quantiques”, Comm. Algebra, 20 (1992), 6, 1787–1802.

[4] V. Artamonov and R. Wisbauer, “Homological Properties of Quantum Polynomials”, Algebr. Represent. Theory, 4 (2001), 219–247.

[5] Y.-H. Bao, J.-W. He, and J. J. Zhang, “Noncommutative Auslander theorem”, Trans. Amer. Math. Soc., 370 (2018), 8613–8638.

[6] Y. Bazlov and A. Berenstein, “Mystic Reflection Groups”, Symmetry, Integrability and Geometry: Methods and Applications, SIGMA, 10 (2014), 040, 11 pages.

[7] Y. Bazlov and A. Berenstein, “Noncommutative Dunkl operators and braided Cherednik algebras”, Selecta Math., 14 (2009), 3–4, 325–372.

[8] S. Çeken, J. H. Palmieri, Y.-H. Wang, and J. J. Zhang, “The discriminant controls automorphism groups of noncommutative algebras”, Adv. Math., 269 (2015), 10, 551–584.

[9] K. Chan, A.A. Young, and J.J. Zhang, “Discriminant formulas and applications”, Algebra Number Theory, 10 (2016), 3, pp. 557–596.

[10] K. Chan, A.A. Young, and J.J. Zhang, “Discriminants and automorphism groups of Veronese subrings of skew polynomial rings”, Math. Z., 288 (2018), 1395–1420.

[11] A. Chervova, G. Falquib, and V. Rubtsovc, “Algebraic properties of Manin matrices 1”, Adv. Appl. Math., 43 (2009), 3, 239–315.

[12] W. Crawley-Boevey and M. P. Holland, “Noncommutative deformations of Kleinian singularities”, Duke Math. J., 92 (1998), 3, 605–635.

[13] V. G. Drinfeld, “Degenerate affine Hecke algebras and Yangians”, Funct. Anal. Appl., 20 (1986), 58–60.

[14] P. Etingof and V. Ginzburg, "Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism", Invent. Math., 147 (2002), 2, 243–348.

[15] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.11.0*; 2020, `https://www.gap-system.org`.

[16] I. M. Gel'fand, V. S. Retakh, "Determinants of matrices over noncommutative rings", Funct. Anal. Appl., 25 (1991), 91–102.

[17] N. Jing and J. Zhang, "Gorensteinness of Invariant Subrings of Quantum Algebras", J. of Algebra, 221 (1999), 669–691.

[18] E. Kirkman, J. Kuzmanovich, and J. J. Zhang, "Shephard-Todd-Chevalley Theorem for skew polynomial rings", Algebr. Represent. Theory, 13 (2010), 127–158.

[19] V. Levandovskyy and A. Shepler, "Quantum Drinfeld Hecke algebras", Canad. J. Math., 66 (2014), 4, 874–901.

[20] G. Lusztig, "Cuspidal local systems and graded Hecke algebras. I", Inst. Haute Etudes Sci. Publ. Math., 67 (1988), 145–202.

[21] G. Lusztig, "Affine Hecke algebras and their graded version", J. Amer. Math. Soc., 2 (1989), no. 3, 599–635.

[22] Yu. I. Manin, "Some remarks on Koszul algebras and quantum groups", Ann. Inst. Fourier, 37 (1987), 4, 191–205.

[23] Wolfram Research, Inc., Mathematica, Version 11.1.1.0, Champaign, IL (2020).

[24] D. Naidu, P. Shroff, and S. Witherspoon, "Hochschild cohomology of group extensions of quantum symmetric algebras", Proc. Amer. Math. Soc., 139 (2011), 1553–1567.

[25] D. Naidu and S. Witherspoon, "Hochschild cohomology and quantum Drinfeld Hecke algebras," Sel. Math. New Ser., 22 (2016), 3, 1537–1561.

[26] A. Ram and A. Shepler, "Classification of graded Hecke algebras for complex reflection groups", Comment. Math. Hel., 78 (2003), 308—334.

[27] J. Shakalli, "Deformations of quantum symmetric algebras extended by groups", J. of Algebra, 370 (2012), 15, 79–99.

[28] G. C. Shephard and J. A. Todd, "Finite unitary reflection groups", Canad. J. Math., 6 (1954), 274–304.

[29] A. Shepler and S. Witherspoon, "Hochschild cohomology and graded Hecke algebras", Trans. Amer. Math. Soc., 360 (2008), 8, 3975–4005.

[30] P. Shroff, "Quantum Drinfeld orbifold algebras", Commun. Algebra, 43 (2015), 1563–1570.

[31] P. Shroff and S. Witherspoon, "PBW deformations of quantum symmetric algebras and their group extensions", J. Algebra Appl., 15 (2016), 3, 15 pp.

[32] M. Suárez-Alvarez and Q. Vivas. "Automorphisms and isomorphisms of quantum generalized Weyl algebras", J. of Algebra, 424 (2015), 540–552.

[33] C. Uhl, "Quantum Drinfeld Hecke Algebras", University of North Texas, Ph.D., 2016.

A.V. Shepler, Department of Mathematics, University of North Texas, Denton, Texas 76203, USA

Email address: `ashepler@unt.edu`

C. Uhl, Department of Mathematics, St. Bonaventure University, St Bonaventure, NY 14778

Email address: `cuhl@sbu.edu`