STEINBERG'S THEOREM
FOR CRYSTALLOGRAPHIC COMPLEX REFLECTION GROUPS

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ABSTRACT. Popov classified crystallographic complex reflection groups by determining lattices they stabilize. These analogs of affine Weyl groups have infinite order and are generated by reflections about affine hyperplanes; most arise as the semi-direct product of a finite complex reflection group and a full rank lattice. Steinberg's fixed point theorem asserts that the regular orbits under the action of a reflection group are exactly the orbits lying off of reflecting hyperplanes. This theorem holds for finite reflection groups (real or complex) and also affine Weyl groups but fails for some crystallographic complex reflection groups. We determine when Steinberg's theorem holds for the infinite family of crystallographic complex reflection groups. We include crystallographic groups built on finite Coxeter groups.

1. INTRODUCTION

We investigate Steinberg's fixed point theorem for affine complex reflection groups, analogs of affine Weyl groups called *crystallographic complex reflection groups*. These discrete groups were classified by Popov [9] in 1982 and are generated by reflections about affine hyperplanes, i.e., mirrors that do not necessarily include the origin. They combine the isometries of a finite complex reflection group $G \leq \mathrm{GL}(V)$ with translation along a G-invariant lattice Λ in $V = \mathbb{C}^n$, but they can not always be described as a semi-direct product $G \ltimes \Lambda$. The adjective *crystallographic* indicates that the orbit space V/W for the action of W on V is compact.

A point under the action of a group is called *regular* if its stabilizer in the group is trivial. Steinberg [15, 14] showed that the regular points under the action of a finite reflection group (real or complex) or an affine Weyl group are precisely those lying off of the reflecting hyperplanes. A famous (or infamous) series of exercises in Bourbaki [2, Ch. V, §5, Ex. 8] also outlines a proof of this fact using Auslander [1]. More recently, Lehrer [6] gave a proof based on elementary invariant theory. Steinberg [14] remarked that this result is sometimes known as "Chevalley's Theorem" in the case of Coxeter groups although it was known to Cartan and Weyl. We show that Steinberg's theorem holds for most crystallographic complex reflection groups but not all. In fact, it fails for some groups whose underlying linear parts are finite Coxeter groups.

Every affine complex reflection group is the direct sum of irreducible ones (or trivial groups), and the conclusion of Steinberg's theorem is preserved under direct sum. Thus one asks: For which irreducible groups does Steinberg's theorem hold? Popov's classification [9] of irreducible crystallographic reflection groups comprises one infinite family of groups $[G(r,p,n)]_k = G(r,p,n) \ltimes \Lambda$ depending on 4 parameters and some exceptional groups. The 4-parameter family combines the 3-parameter family of finite complex reflection groups $G(r,p,n)$ (which includes the infinite families of Coxeter groups) with various invariant lattices Λ in $\mathbb{C}^n$. We classify those groups in this family for which non-regular orbits all lie on reflecting hyperplanes, thus determining when Steinberg's theorem holds. We treat the case of crystallographic groups built on Coxeter groups separately.

Theorem 1.1. *Let Λ be a $G(r,p,n)$-invariant lattice of full rank $2n$ in $\mathbb{C}^n$ for $r,p,n \geq 1$. Assume $G(r,p,n)$ is not a Coxeter group. The set of nonregular points for $W = G(r,p,n) \ltimes \Lambda$ acting on $\mathbb{C}^n$ is the union of reflecting affine hyperplanes for W if and only if $r \neq p$ and $W \neq [G(3,1,n)]_2$ and $W \neq [G(6,3,2)]_2$.*

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We also determine those crystallographic reflection groups $G \ltimes \Lambda$ for which Steinberg's theorem fails when G lies in an infinite family of Weyl groups, see Theorem 7.1: The theorem fails except when G is $W(A_{n-1})$ or $W(B_n)$, and even then it does not always hold. Our arguments rely on analysis of orbits under various reflection groups of infinite order acting on subsets of $\mathbb{C}$. The next two examples are those mentioned in the theorem. We use the standard basis $e_1, \dots, e_n$ of $V = \mathbb{C}^n$.

Example 1.2. The group $[G(6,3,2)]_2$ acting on $\mathbb{C}^2$ is $G(6,3,2) \ltimes \Lambda$ for $\mathbb{Z}$-lattice in $\mathbb{C}^2$ of rank 4

$$\Lambda = \mathbb{Z}[2\xi](\xi e_1 - e_2) + \mathbb{Z}[2\xi](1-\xi)(e_1 - e_2),$$

where $\xi = e^{2\pi i/6}$ and where $G(6,3,2)$ is the linear group generated by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \xi^3 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & \xi \\ \xi^{-1} & 0 \end{smallmatrix}\right)$.

Example 1.3. The group $[G(3,1,n)]_2$ acting on $\mathbb{C}^n$ is $W = G(3,1,n) \ltimes \Lambda$ for $\mathbb{Z}$-lattice of rank $2n$

$$\Lambda \;=\; \mathbb{Z}[\omega]e_1 \oplus \Sigma_{k=2}^{n} \;\mathbb{Z}[\omega]\,\left(\tfrac{1}{1-\omega}\right)(e_{k-1} - e_k)$$

where $\omega = e^{2\pi i/3}$. Here, $G(3,1,n)$ is the finite complex reflection group generated by the $n \times n$ permutation matrices together with the diagonal matrix $\mathrm{diag}(\omega, 1, \dots, 1)$.

Outline. We give basic notions and recall Steinberg's fixed point theorem for reflection groups in Section 2. We review Popov's classification in Section 3. The case of crystallographic groups built upon the symmetric group $\mathfrak{S}_n$ appears in Section 4. We determine the genuine groups $G(r,p,n) \ltimes \Lambda$ satisfying Steinberg's theorem in Section 5 and those for which the theorem fails in Section 6. In Section 7, we consider crystallographic groups built upon Coxeter groups.

2. Crystallographic reflection groups

We fix a positive definite inner product on $V = \mathbb{C}^n$ and standard basis $e_1, \dots, e_n$ of V with dual basis $x_1, \dots, x_n$ of V^*. All lattices are $\mathbb{Z}$-lattices and a lattice in $\mathbb{C}^n$ has *full rank* if it has rank $2n$.

Affine transformations. We identify the set of affine transformations on V with $\mathrm{M}_n(\mathbb{C}) \ltimes V$ so that an *affine transformation* g of V is the composition of a linear transformation and a translation:

$$g(v) = \mathrm{Lin}(g)(v) + \mathrm{Tran}(g) \quad \text{for } v \text{ in } V,$$

for some fixed matrix $\mathrm{Lin}(v) \in M_n(\mathbb{C})$, the *linear part* of g, and a fixed vector $\mathrm{Tran}(g) = g(0) \in V$, the *translational part* of g. Note that $\mathrm{Lin}(g)$ is the map $v \mapsto g(v) - g(0)$. The set of invertible affine transformations $A(V)$ is identified with $GL(V) \ltimes V$ via $g \mapsto (\mathrm{Lin}(g), g(0))$ and we have a map $\mathrm{Lin} : A(V) \to \mathrm{GL}(V)$.

Affine reflections. An *affine reflection* (or just *reflection*) on V is a non-identity affine isometry s fixing an affine hyperplane H_s in $\mathbb{C}^n$ pointwise, called the *reflecting hyperplane* of s. A reflection s is *central* if $s(0) = 0$ or, equivalently, if $H_s = \ker(s - 1_V)$ is a linear subspace of V and s is a linear transformation. When s is a central reflection of finite order, there is a vector $\alpha_s \perp H_s$ with $s(\alpha_s) = \xi \alpha_s$ for some primitive m-th root-of-unity ξ in $\mathbb{C}$, the non-identity eigenvalue of s, where m is the order of s.

The lemma below shows that every affine reflection is obtained by composing a central reflection with translation by a vector perpendicular to the central reflecting hyperplane (compare with Popov [9, Subsection 1.2]). We give a proof of this fact and related observations we need later for completeness.

Lemma 2.1. *Suppose g is an affine transformation on V and $\mathrm{Lin}(g)$ has finite order. Then*

(1) The transformation g has finite order if and only if g fixes some point of V.

(2) The transformation g is a reflection if and only if

$$g(v) = s(v) + b \quad \textit{for all } v \in V$$

for some central reflection $s \in \mathrm{GL}(V)$ of finite order and $b \in H_s^{\perp}$. Here, $H_g = H_s + h$ for any $h \in H_g$.

(3) The transformation g is a reflection if and only if g fixes a point of V and $\mathrm{Lin}(g)$ is a reflection.

Proof. For (1), if g has finite order, then it fixes the average of the elements in the orbit of the zero vector in V under the cyclic group $\langle g\rangle$. Conversely, suppose g fixes a point u in V and $\mathrm{Lin}(g)$ has finite order. Then $u = g(u) = \mathrm{Lin}(g)u + \mathrm{Tran}(g)$ and $g(v) = \mathrm{Lin}(g)v + \mathrm{Tran}(g) = \mathrm{Lin}(g)(v-u)+u$ for all $v \in V$. Then g has finite order as $\langle g\rangle$ is conjugate to the finite group $\langle \mathrm{Lin}(g)\rangle$.

For (2), let $\mathrm{Lin}(g) = s$ and $\mathrm{Tran}(g) = b$. Suppose g is a reflection. Then $H_g = H_0 + c$ for some central hyperplane H_0 in V and some c in V, and for any $h_0 \in H_0$,

$$h_0 + c = g(h_0 + c) = s(h_0) + s(c) + b = s(h_0) + g(c) = s(h_0) + c.$$

Hence, $s \neq 1$ is a reflection in $\mathrm{GL}(V)$ with $H_s = H_0$. Furthermore, $b \in \mathrm{Im}(1-s) = H_0^{\perp}$. Conversely, assume $s \in \mathrm{GL}(V)$ is a reflection of finite order and $b \in H_s^{\perp}$. Then $s(b) = \xi b$ for some primitive m-th root-of-unity where m is the order of s. Then

$$g^m(v) = s^m(v) + \Sigma_{k=0}^{m-1}\, s^k(b) = v + \Sigma_{k=0}^{m-1}\, \xi^k b = v \quad \text{for all } v \in V$$

and g also has finite order. By part (1), g must fix a point u in V. Then $g(h+u) = s(h+u)+b = s(h)+s(u)+b = h+u$ for any h in H_s and g is a reflection about the hyperplane $H_s + u$. Part (3) follows from part (2). □

Reflection groups. An *affine reflection group* (or just *reflection group*) is a subgroup of $A(V)$ generated by affine reflections acting discretely on $V = \mathbb{C}^n$. The *reflecting hyperplanes* for a group W are the hyperplanes fixed by reflections in W. For any reflection group W, we set (following Popov [9])

$$\mathrm{Lin}(W) = \{\mathrm{Lin}(g) : g \in W\} \quad \text{and} \quad \mathrm{Tran}(W) = \{g \in W : g = \mathrm{Tran}(g)\}.$$

Every affine reflection group W is the product of irreducible affine reflection groups (or trivial groups) and W is irreducible exactly when $\mathrm{Lin}(W)$ is irreducible, see [9]. If $G \le \mathrm{GL}_n(\mathbb{C})$ is an irreducible finite reflection group and Λ is a G-invariant lattice in V, then $W = G \ltimes \Lambda$ is an affine reflection group with $\mathrm{Lin}(W) = G$ and $\mathrm{Tran}(W) = \Lambda$.

Crystallographic groups. An affine reflection group W acting on $V = \mathbb{C}^n$ is *crystallographic* if its space of orbits V/W is compact; otherwise it is *noncrystallographic*. Popov [9] showed that if W is an irreducible affine reflection group of infinite order, then $\mathrm{Tran}(W)$ is a lattice of rank $n = \dim V$ or $2n$, and the crystallographic groups are those whose lattices have full rank $2n$.

Genuine groups. A *Coxeter group* is a group of general linear transformations generated by reflections on $\mathbb{R}^n$. We assume all Coxeter groups act discretely. Every Coxeter group defines a reflection group on $\mathbb{C}^n$ by extension of scalars. The infinite Coxeter groups acting on $\mathbb{R}^n$ are the *affine Weyl groups*; they define affine reflection groups acting on $\mathbb{C}^n$ which are not crystallographic since the underlying lattices have rank n instead of $2n$. But the Weyl groups acting on $\mathbb{R}^n$ define groups acting on $\mathbb{C}^n$ which stabilize various lattices of full rank $2n$. Following Malle [7], we call the resulting affine complex reflection groups W *non-genuine*: they are crystallographic but have linear part $\mathrm{Lin}(W)$ a Coxeter group. We say a crystallographic reflection group W is *genuine* when $\mathrm{Lin}(W)$ is not merely obtained as the complexification of a finite Coxeter group. See Section 7.

Steinberg's fixed point theorem. We recall Steinberg's fixed point theorem:

Theorem 2.2 (Steinberg [15], [14])**.** *Let W be a Coxeter group (of finite or infinite order) or a finite complex reflection group acting on V. Then a vector v in V is fixed by some nonidentity group element of W if and only if v lies on a reflecting hyperplane for W.*

Definition 2.3. We say an affine reflection group W has the *Steinberg property* if the set of nonregular points is the union of reflecting hyperplanes for W, i.e., if the conclusion of Theorem 2.2 holds.

Recall that affine transformations which are conjugate under some a in $\mathrm{GL}(V)$ have fixed point spaces in the same a-orbit. Thus to show that the fixed point space V^g of some element g in a reflection group W lies on a reflecting hyperplane for W, we may replace g by any conjugate of g in W. We may also replace g by any power of g, as $V^g \subset V^{g^j}$ for all j.

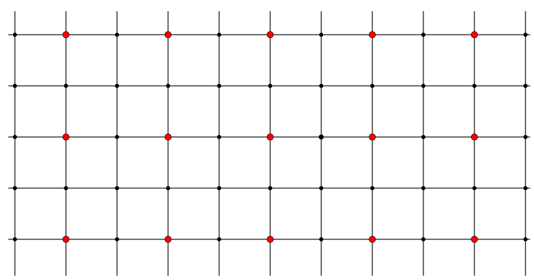
3. CLASSIFICATION

3.1. **Rank 1 crystallographic groups.** Every rank $n = 1$ reflection group W trivially satisfies the Steinberg property since affine hyperplanes are just points in $V = \mathbb{C}^1$ and the reflections of W are exactly those group elements in W fixing a point of V. Our arguments later use orbits of various rank 1 reflection groups acting on certain sets, so we give a few more details on this case here. Every rank $n = 1$ crystallographic reflection group is

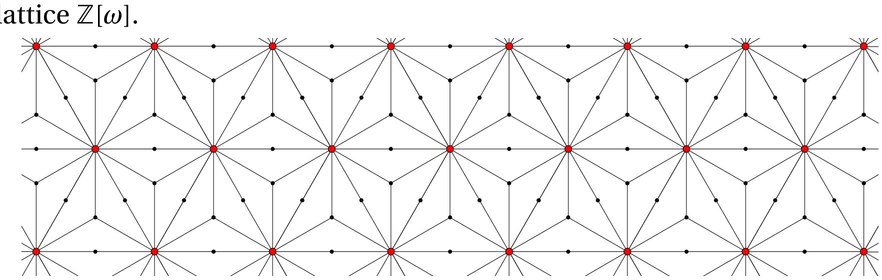
$$W = G(r, 1, 1) \ltimes \Lambda$$

for some cyclic group $G(r, 1, 1) = \langle \xi \rangle \subset \mathrm{GL}(V)$ acting on $V = \mathbb{C}$ for $\xi = e^{\frac{2\pi i}{r}}$ with $r \geq 2$ and some lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\zeta$ stable under multiplication by ξ with $\zeta \in \mathbb{C}$. When W is nongenuine, $r = 2$ and Λ is equivalent to $\mathbb{Z} + \mathbb{Z}\alpha$ for some $\alpha \in \mathbb{C}$ in the modular strip (see Section 7). When W is genuine, only 2 possible lattices Λ arise, with ζ a third or fourth root-of-unity in $\mathbb{C}$, and $r = 3$, 4 or 6. In this case, $\mathbb{Z} + \mathbb{Z}\zeta = \mathbb{Z}[\zeta]$ and $W \cong \mathbb{Z}/r\mathbb{Z} \ltimes \mathbb{Z}[\zeta]$. In fact, every genuine crystallographic reflection group acting on $\mathbb{C}$ is equivalent to a subgroup of one of the 3 examples below (see Popov [9]).

Example 3.1. The reflection group $W = G(4, 1, 1) \ltimes \mathbb{Z}[i]$ acting on $V = \mathbb{C}$ is crystallographic with $\frac{1}{2}\mathbb{Z}[i]$ the set of reflecting hyperplanes. Larger dots indicate points in the lattice $\mathbb{Z}[i]$.



Example 3.2. The reflection group $W = G(6, 1, 1) \ltimes \mathbb{Z}[\omega]$ for $\omega = e^{2\pi i/3}$ acting on $V = \mathbb{C}$ is crystallographic with $\frac{1}{1-\omega}(\mathbb{Z} + \mathbb{Z}\omega) \cup \frac{1}{2}(\mathbb{Z} + \mathbb{Z}\omega)$ the set of reflecting hyperplanes. Again, larger dots indicate points in the lattice $\mathbb{Z}[\omega]$.



Example 3.3. The group $W = G(3, 1, 1) \ltimes \mathbb{Z}[e^{2\pi i/3}]$ is a crystallographic reflecting group acting on $V = \mathbb{C}$.

The 3-parameter family of finite complex reflection groups. Shephard and Todd [13] classified the irreducible finite complex reflection groups. They give a 3-parameter family $G(r, p, n)$ and 34 exceptional groups denoted by G_i for $4 \leq i \leq 37$. The group $G(r, 1, n)$ consists of $n \times n$ monomial matrices (i.e., matrices with a single nonzero entry in each row and column) whose nonzero entries are complex r-th roots-of-unity, for r a positive integer. Note that as an abstract group,

$$G(r, 1, n) \cong \mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n,$$

where $\mathfrak{S}_n$ is the symmetric group. Each group $G(r,1,n)$ is generated by reflections on $V=\mathbb{C}^n$ of order 2 and order r (see [8]). In fact, $G(r,1,n)$ is the symmetry group of the *cross-polytope* in $\mathbb{C}^n$, a regular complex polytope studied by Shephard [11] and Coxeter [4]. Note that $G(r,1,n)$ acts by isometries with respect to the standard inner product.

For any integer $p\ge 1$ dividing r, the group $G(r,p,n)$ is the subgroup of $G(r,1,n)$ consisting of those matrices whose product of nonzero entries is 1 when raised to the power r/p. The groups $G(r,p,n)$ are also generated by reflections and include the infinite families of Coxeter groups acting on $\mathbb{R}^n$:

- $G(2,1,n)$ is the Weyl group $\mathrm{W}(B_n)$,
- $G(2,2,n)$ is the Weyl group $\mathrm{W}(D_n)$,
- $G(r,r,2)$ is the dihedral group $\mathrm{W}(I_r)$ of order $2r$ after change-of-basis,
- $G(1,1,n)$ is the symmetric group $\mathfrak{S}_n$ acting by permutation matrices with irreducible reflection representation $W(A_{n-1})$.

Note that $G(3,3,2)$ is equivalent to the complexification of the Weyl group $W(A_2)$. Also, $G(4,4,2)$ and $G(2,1,2)$ are equivalent and $G(2,2,2)=G(2,1,1)\times G(2,1,1)$.

Example 3.4. The group $G=G(4,1,2)$ is generated by matrices $\left(\begin{smallmatrix} i & 0\\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\right)$. The reflections in G are the diagonal matrices with only one diagonal entry not equal to 1 and the antidiagonal matrices whose nonzero entries are inverse. The reflecting hyperplanes for G are $H_j=\ker(x_j)$ for $j=1,2$ and $H_{1,2}(\zeta)=\ker(x_1-\zeta x_2)$ for ζ a 4-th root-of-unity.

Popov's Classification of crystallographic groups. Popov [9] classified the crystallographic reflection groups W using Shephard and Todd's [13] notation for their linear parts, showing that if W is irreducible, then $\mathrm{Lin}(W)$ fixes $\mathrm{Tran}(W)$ set-wise and $\mathrm{Lin}(W)$ is one of

$$W(A_{n-1}),\ G(2,p,n),\ G(3,p,n),\ G(4,p,n),\ \text{or } G(6,p,n),$$

or one of the 16 exceptional groups

$$G_4,\ G_5,\ G_8,\ G_{12},\ G_{24},\ G_{25},\ G_{26},\ G_{28},\ G_{29},\ G_{31},\ G_{32},\ G_{33},\ G_{34},\ G_{35},\ G_{36},\ G_{37}.$$

Each nongenuine crystallographic complex reflection group stabilizes a moduli space of full rank lattices in $\mathbb{C}^n$, see Section 7. If W is an irreducible crystallographic complex reflection group and $W\not\cong \mathrm{Lin}(W)\ltimes \mathrm{Tran}(W)$, then $\mathrm{Lin}(W)$ is $G(4,2,n)$, $G(6,2,2)$, G_{12}, or G_{31}. (Goryunov [5] gives the group $[G(6,2,2)]^*$ left out of the classification.) We do not consider these exceptional cases here.

Popov [9] determined that each irreducible genuine finite complex reflection group W stabilizes at most three full rank lattices, up to equivalence. His notation

$$W=[G_i]_k$$

indicates that W has linear part $\mathrm{Lin}(W)=G_i$ in the notation of Shephard and Todd with the index $k=1$, 2, or 3 indicating one of possibly three different lattices $\mathrm{Tran}(W)=\Lambda$ stabilized by G_i. When there is only one such lattice, we write $[G_i]_1$, although Popov merely writes $[G_i]$.

4-parameter family of crystallographic groups. The 3-parameter family of finite groups $G(r,p,n)$ gives rise to a 4-parameter family of crystallographic reflection groups $[G(r,p,n)]_k=G(r,p,n)\ltimes\Lambda$ where Λ is a $G(r,p,n)$-invariant lattice of rank $2n$ in $\mathbb{C}^n$. The affine reflections in $G(r,p,n)\ltimes\Lambda$ have the form $s(v)=\sigma(v)+b$ for $v\in V$ where σ is a reflection in $G(r,p,n)$ fixing some central hyperplane H_σ of the form $\ker(x_j-\xi^p x_k)$ or $\ker(x_j)$ for $1\le j<k\le n$ and where $b\in\Lambda\cap(H_\sigma)^\perp$ (see Lemma 2.1).

Table 1 gives the genuine crystallographic groups in the infinite family $[G(r,p,n)]_k=G(r,p,n)\ltimes\Lambda$ and specifies those for which Steinberg's theorem holds. Here, $k=1,2,3$ indicates choice of lattice Λ. We omit non-genuine groups W, as they appear in Section 7, and thus we assume $r>2$ and omit the groups $[G(r,r,2)]$. Again, ξ is a primitive r-th root-of-unity in $\mathbb{C}$; note that $\mathbb{Z}[\xi]=\mathbb{Z}+\mathbb{Z}\xi$ for $r=3,4,6$.

TABLE 1. Genuine crystallographic complex reflection groups $W = [G(r,p,n)]_k = G(r,p,n) \ltimes \Lambda$ acting on $\mathbb{C}^n$

Group W	**dim**	$G(r,p,n)$**-invariant lattice** Λ	**Steinberg's thm**
$[G(r,1,1)]_1$	$n=1$	$\mathbb{Z}[\xi]e_1$	✓
$[G(3,1,n)]_1$	$n\geq 2$	$\mathbb{Z}[\xi]e_1 + \Sigma_{j=2}^n \mathbb{Z}[\xi](e_{j-1}-e_j)$	✓
$[G(3,1,n)]_2$	$n\geq 2$	$\mathbb{Z}[\xi]e_1 + \Sigma_{j=2}^n \mathbb{Z}[\xi]\left(\frac{1}{1-\xi}\right)(e_{j-1}-e_j)$	✗
$[G(3,3,n)]_1$	$n\geq 3$	$\mathbb{Z}[\xi](\xi e_1 - e_2) + \Sigma_{k=j}^n \mathbb{Z}[\xi](e_{j-1}-e_j)$	✗
$[G(4,1,n)]_1$	$n\geq 2$	$\mathbb{Z}[\xi]e_1 + \Sigma_{k=j}^n \mathbb{Z}[\xi](e_{j-1}-e_j)$	✓
$[G(4,1,n)]_2$	$n\geq 2$	$\mathbb{Z}[\xi]e_1 + \Sigma_{j=2}^n \mathbb{Z}[\xi]\left(\frac{1}{1-\xi}\right)(e_{j-1}-e_j)$	✓
$[G(4,2,n)]_1$	$n\geq 2$	$\mathbb{Z}[\xi](\xi e_1 - e_2) + \Sigma_{j=2}^n \mathbb{Z}[\xi](e_{j-1}-e_j)$	✓
$[G(4,2,n)]_2$	$n\geq 2$	$\mathbb{Z}[\xi](\xi e_1 - e_2) + \Sigma_{j=2}^n \mathbb{Z}[\xi](e_{j-1}-e_j) + \mathbb{Z}[\xi]e_n$	✓
$[G(4,2,2)]_3$	$n=2$	$\mathbb{Z}[\xi](\xi e_1 - e_2) + \mathbb{Z}[\xi](1+\xi)(e_1-e_2)$	✓
$[G(4,4,n)]_1$	$n\geq 3$	$\mathbb{Z}[\xi](\xi e_1 - e_2) + \Sigma_{j=2}^n \mathbb{Z}[\xi](e_{j-1}-e_j)$	✗
$[G(6,1,n)]_1$	$n\geq 2$	$\mathbb{Z}[\xi]e_1 + \Sigma_{j=2}^n \mathbb{Z}[\xi](e_{j-1}-e_j)$	✓
$[G(6,2,n)]_1$	$n\geq 2$	$\mathbb{Z}[\xi](\xi e_1 - e_2) + \Sigma_{j=2}^n \mathbb{Z}[\xi](e_{j-1}-e_j)$	✓
$[G(6,3,n)]_1$	$n\geq 2$	$\mathbb{Z}[\xi](\xi e_1 - e_2) + \Sigma_{j=2}^n \mathbb{Z}[\xi](e_{j-1}-e_j)$	✓
$[G(6,2,2)]_2$	$n=2$	$\mathbb{Z}[\xi](\xi e_1 - e_2) + \mathbb{Z}[\xi](1+\xi)(e_1-e_2)$	✓
$[G(6,3,2)]_2$	$n=2$	$\mathbb{Z}[2\xi](\xi e_1 - e_2) + \mathbb{Z}[2\xi](1-\xi)(e_1-e_2)$	✗
$[G(6,6,n)]_1$	$n\geq 3$	$\mathbb{Z}[\xi](\xi e_1 - e_2) + \Sigma_{j=2}^n \mathbb{Z}[\xi](e_{j-1}-e_j)$	✗

4. THE SYMMETRIC GROUP

The group $G(1,1,n)$ is the symmetric group $\mathfrak{S}_n$ acting on $V = \mathbb{C}^n$ in its natural reflection representation by permutation of basis vectors $e_1,\dots,e_n$. It is reducible and not the linear part of any crystallographic affine reflection group. We identify the irreducible Weyl group $W(A_{n-1}) \cong \mathfrak{S}_n$ with the restriction of $G(1,1,n)$ to the subspace $V' = \mathbb{C}\text{-span}\{e_2-e_1,\dots,e_n-e_{n-1}\} \cong \mathbb{C}^{n-1}$ of V. Every irreducible crystallographic complex reflection group with linear part $W(A_{n-1})$ lies in the 1-parameter family

$$[W(A_{n-1})]_1^\alpha = W(A_{n-1}) \ltimes \Lambda_\alpha \cong \mathfrak{S}_n \ltimes \Lambda_\alpha \quad \text{for } \Lambda_\alpha = \Sigma_{j=2}^n(\mathbb{Z}+\mathbb{Z}\alpha)(e_{j-1}-e_j), \quad \text{with parameter } \alpha\in\mathbb{C}.$$

In fact, every $W[(A_{n-1})]_1^\alpha$ is equivalent to a group with α in the modular strip (see Section 7).

We use the next proposition to streamline arguments later for genuine and nongenuine groups.

Lemma 4.1. *Let $W = G(r,p,n) \ltimes \Lambda$ for some $G(r,p,n)$-invariant lattice Λ of full rank in $\mathbb{C}^n$, for $r,p\geq 1$, $n\geq 2$. Suppose that g in W has linear part a nontrivial cycle in the symmetric group $G(1,1,n)$. Then any vector fixed by g lies on a reflecting hyperplane for W.*

Proof. The claim follows for $n=1$, see Subsection 3.1, so we assume $n>1$. After conjugation, we may assume $\mathrm{Lin}(g) = (m \;\; m+1 \;\; m+2 \;\cdots\; \ell) \neq 1$ in $\mathfrak{S}_n$ for some $1 \le m \le n$, using $\mathfrak{S}_n$-notation for elements of $G(1,1,n)$. Let s be the affine reflection about the hyperplane $H_s = \ker(x_m - x_{m+1} + \beta)$ in V defined by

$$s(v) = (m \;\;\; m+1)\, v + \beta(e_{m+1} - e_m) \quad \text{for } \; v \in V\,,$$

with $\beta = x_{m+1}(\mathrm{Tran}(g)) \in \mathbb{C}$. Then $\mathrm{Tran}(g) \in \Lambda$ implies $\beta(e_{m+1} - e_m) \in \Lambda$ as well, upon inspection of the possible lattices, see Tables 1 and 3, and s lies in W. If g fixes $u \in V$, then $x_{m+1}(u) = x_m(u) + \beta$ and u lies on the reflecting hyperplane H_s of W. □

In the next proposition, we see that Steinberg's theorem holds for all reflection groups of the form $W = W(A_{n-1}) \ltimes \Lambda$ for any $W(A_{n-1})$-invariant lattice Λ of full rank in $\mathbb{C}^{n-1}$.

Proposition 4.2. *Let W be a crystallographic complex reflection group whose linear part is the symmetric group $\mathfrak{S}_n$ in its irreducible reflection representation as the Weyl group $W(A_{n-1})$. Then W satisfies the Steinberg property.*

Proof. Consider $W' = W(A_{n-1}) \ltimes \Lambda_\alpha$ acting on V' for some $\alpha \in \mathbb{C}$ and let $W = G(1,1,n) \ltimes \Lambda$ for Λ the rank $2(n-1)$ lattice Λ_α regarded as a lattice in $V = \mathbb{C}^n$. We follow the proof of the last lemma to see that any vector in V fixed by a group element of W lies on a reflecting hyperplane for W. Since W is the direct sum of W' and the trivial rank 1 representation of $\mathfrak{S}_n$, the claim follows for W' as well. □

5. Which groups have the Steinberg property?

In this section, we determine those genuine crystallographic complex reflection groups $G(r,1,n) \ltimes \Lambda$ which satisfy Steinberg's theorem. We begin with a small lemma that does not require that W be genuine. The lemma allows us to later reduce to arguments on orbits under the action of 1-dimensional reflection groups. In part (3) below, we consider orbits under the action of $G(r,1,1) \ltimes \Lambda'$ on $\mathbb{C}$, with $G(r,1,1) = \langle \xi \rangle$ acting by multiplication and $\Lambda' \subset \mathbb{C}$ acting by translation (identifying $\mathbb{C}e_1$ with $\mathbb{C}$).

Lemma 5.1. *Let $W = G(r,p,n) \ltimes \Lambda$ for some $G(r,p,n)$-invariant lattice Λ in $\mathbb{C}^n$. Consider any g in W and write $\lambda_1, \ldots, \lambda_n$ for the diagonal entries of $\mathrm{Lin}(g)$. Suppose one of the following holds:*

(1) $\lambda_j \notin \{0,1\}$ for some index j and $g^p \neq 1$ with $r \neq p$ and $x_j(\mathrm{Tran}(g^p))e_1 \in \Lambda$;
(2) $\lambda_j \notin \{0,1\}$ for some index j and $g^p = 1$ with $r \neq p$ and $x_j(\mathrm{Tran}(g))\, e_1 \in \Lambda\, (1-\lambda_j)$;
(3) $\lambda_j, \lambda_\ell \notin \{0,1\}$ for some $j \neq \ell$, the lattice Λ is invariant under $G(r,1,n)$, and

$$\frac{x_j(\mathrm{Tran}(g))}{1-\lambda_j} \quad \textit{and} \quad \frac{x_\ell(\mathrm{Tran}(g))}{1-\lambda_\ell}$$

lie in the same orbit under the action of $G(r,1,1) \ltimes \Lambda'$ on $\mathbb{C}$ for $\Lambda' = x_1\big(\mathbb{C}(e_1 - e_2) \cap \Lambda\big)$.

Then any vector fixed by g lies on a reflecting hyperplane of W.

Proof. We use Lemma 2.1 throughout. Suppose (1) holds and set $\beta = x_j(\mathrm{Tran}(g^p))$. Any vector u in V fixed by g is fixed by g^p and thus satisfies $x_j(u) = \lambda_j^p x_j(u) + \beta$, since $\mathrm{Lin}(g^p) = (\mathrm{Lin}(g))^p$. Hence u is fixed by the affine transformation s defined by

$$s(v) = \mathrm{diag}(1, \ldots, 1, \lambda_j^p, 1, \ldots, 1)\, v + \beta e_j \quad \text{ for } v \in \mathbb{C}^n,$$

with λ_j^p as j-th entry. Then $\mathrm{Lin}(s) \neq 1$ lies in $G(r,p,n)$ and $\mathrm{Tran}(s) = \beta e_j$ lies in Λ since βe_1 lies in Λ and Λ is invariant under $\mathrm{Lin}(W)$. As the central reflecting hyperplane for $\mathrm{Lin}(s)$ is perpendicular to $\mathrm{Tran}(s)$, the transformation s is in fact an affine reflection in W.

Now suppose (2) holds. Any vector fixed by g is fixed by the affine reflection s defined by

$$s(v) = \mathrm{diag}(1, \ldots, 1, \xi^p, 1, \ldots, 1)\, v + \beta e_j \quad \text{ for } v \in \mathbb{C}^n,$$

with ξ^p as j-th entry and $\beta = \frac{1-\xi^p}{1-\lambda_j} x_j(\mathrm{Tran}(g))$. Then $\mathrm{Lin}(s)$ lies in $G(r,p,n)$. We set $\beta' = (1-\xi^p)^{-1}\beta$. Then as $\beta' e_1$ lies in Λ, both $\beta' e_j$ and $\xi^p \beta' e_j$ lie in Λ as well since the lattice Λ is $\mathrm{Lin}(W)$-invariant. Hence their difference, $\mathrm{Tran}(s) = \beta e_j = (1-\xi^p)\beta' e_j$, also lies in Λ and s is an affine reflection in W.

Lastly, say (3) holds. Since the two given quotients are in the same orbit, their weighted difference

$$\alpha \;=\; \tfrac{x_j(\mathrm{Tran(g)})}{1-\lambda_j} \;-\; \xi^m\, \tfrac{x_\ell(\mathrm{Tran(g)})}{1-\lambda_\ell}$$

lies in Λ' for some $m \geq 0$, i.e., $\alpha(e_1 - e_2) \in \Lambda$. Since Λ is $G(r,1,n)$-invariant, $\alpha(e_j - \xi^{-m} e_\ell)$ lies in Λ as well. Let s be the affine transformation defined by

$$s(v) = \sigma(v) + \alpha(e_j - \xi^{-m} e_\ell) \quad \text{for } v \in V,$$

where $\sigma \in G(r,p,n)$ is the weighted transposition $e_j \mapsto \xi^{-m} e_\ell$ and $e_\ell \mapsto \xi^m e_j$. Then s is an affine reflection in W since $\mathrm{Tran}(s)$ is perpendicular to the central reflecting hyperplane H_σ of σ. We argue that any vector fixed by g is fixed by s. Indeed, if $u \in V$ is fixed by g, then

$$x_j(u) = \tfrac{x_j(\mathrm{Tran(g)})}{1-\lambda_j} \quad \text{and} \quad x_\ell(u) = \tfrac{x_\ell(\mathrm{Tran(g)})}{1-\lambda_\ell}\,.$$

Since $\alpha = x_j(u) - \xi^m x_\ell(u)$, the vector u is fixed by s.

In all three cases, we see that any vector fixed by g lies on an affine reflecting hyperplane for W. $\square$

The next example shows how we use use orbits in $\mathbb{C}$ to determine fixed point spaces. We also use this example in the next proof.

Example 5.2. Consider $W = [G(4,1,n)]_2$ with $n > 1$ and some g in W with nontrivial fixed point space V^g in $V = \mathbb{C}^n$. Suppose $\mathrm{Lin}(g)$ has two diagonal entries λ_j and λ_ℓ that are -1. We claim that V^g lies on a reflection hyperplane for W. Since $\mathrm{Lin}(g)$ is monomial,

$$g(e_j) = -e_j + \beta_j \quad \text{and} \quad g(e_\ell) = -e_\ell + \beta_\ell$$

where $\beta_j = x_j(\mathrm{Tran}(g))$ and $\beta_\ell = x_\ell(\mathrm{Tran}(g))$. Here, β_j, β_ℓ lie in the set $\mathbb{Z}[i]/(1-i) = \{(a+bi)/2 :$ a, b both even or both odd$\}$. If β_j or β_ℓ lie in $\mathbb{Z}[i]$, then Lemma 5.1(1) implies the claim. Hence we assume β_j and β_ℓ are both not in $\mathbb{Z}[i]$. Then each can be written in the form $(a+bi)/2$ with a, b both odd and thus identified with the vertex (a,b) of a square in a tessellation of the plane $\mathbb{R}^2$ by squares of side length one, with one square centered at zero. Any vertex in this tessellation can be obtained from any other by an even number of horizontal and vertical translations (by one unit) together with some rotations of 90°. Thus $\beta_j/2$ and $\beta_\ell/2$ lie in the same orbit under the action of $G(4,1,1) \ltimes \Lambda'$ for $\Lambda' = \mathbb{Z}[i]/(1-i)$. Lemma 5.1(3) then implies the claim.

The next proposition begins the analysis of genuine crystallographic groups.

Proposition 5.3. *The groups $[G(r,1,n)]_1$ and $[G(4,1,n)]_2$ for $r \geq 3$, $n \geq 1$ have the Steinberg property.*

Proof. Let W be one of the given groups. The claim follows for $n = 1$, see Subsection 3.1, so we assume $n > 1$. Fix $g \neq 1$ in W with nontrivial fixed point space V^g. We assume g itself is not a reflection, nor any power of g, so $\mathrm{Lin}(g)$ and its powers are also non-reflections by Lemma 2.1. Note that $\mathrm{Lin}(g) \neq 1$.

Diagonal action. First suppose that $\mathrm{Lin}(g)$ is diagonal. Then at least two diagonal entries are nontrivial. In case $W = [G(r,1,n)]_1$, $\mathrm{Tran}(g)$ lies in $(\mathbb{Z}[\xi])^n$ (by inspection of $\Lambda = \mathrm{Tran}(W)$) and Lemma 5.1(1) implies that u lies on a reflecting hyperplane for W. In case $W = [G(4,1,n)]_2$, we may replace g by g^2 if warranted (as $V^g \subset V^{g^2}$) and assume at least two diagonal entries of $\mathrm{Lin}(g)$ are -1. Then by Example 5.2, any vector fixed by g lies on a reflecting hyperplane for W.

Cycle decomposition. Now assume $\mathrm{Lin}(g)$ is not diagonal. Every element of $G(r,1,n)$ is conjugate by an element of the symmetric group $\mathfrak{S}_n$ to a disjoint product of nontrivial ξ-weighted cycles (see [12])

$$c = t_m^{a_m} t_\ell^{a_\ell}\ (m \quad m+1 \quad m+2 \quad \cdots \quad \ell)\,,$$

where each $t_m = \text{diag}(1,\dots,1,\xi,1\dots,1)$, a diagonal matrix with ξ as the m-th entry, and $a_m, a_\ell \in \mathbb{Z}_{\geq 0}$. Here, we identify the cycle $(m\ \ m+1\ \cdots\ \ell)$ in $\mathfrak{S}_n$ with the corresponding permutation matrix in $G(1,1,n)$. We write $\text{Lin}(g)$ as a product of such weighted cycles and fix attention on one such weighted cycle c with $(m\ \ m+1\ \cdots\ \ell)$ nontrivial.

Symmetric group action. Suppose that $\det(t_m^{a_m}\ t_\ell^{a_\ell}) = 1$. Then c is conjugate by $t_m^{-a_\ell}$ to the cycle $(m\ \ m+1\ \cdots\ \ell)$ as $1 = \xi^{a_m+a_\ell}$ and we may assume c is this cycle. Then since $V^g \subset V^c$, Lemma 4.1 implies V^g lies on a reflecting hyperplane for W.

Diagonal power. Now suppose $\det(t_m^{a_m}\ t_\ell^{a_\ell}) = \lambda \neq 1$. Then $c^{m-\ell+1}$ in $G(r,1,n)$ is diagonal, as $m-\ell+1$ is the length of the cycle, and $\text{Lin}(g^{m-\ell+1}) = (\text{Lin}(g))^{m-\ell+1}$ is block diagonal with one block itself a scalar matrix λI of size at least 2×2. The arguments for the case when $\text{Lin}(g)$ is diagonal above then show that any vector fixed by $g^{m-\ell+1}$, and thus any fixed by g, lies on a reflecting hyperplane for W. □

Remark 5.4. The idea for showing that $[G(4,1,n)]_2$ has the Steinberg property in the above proof of Theorem 5.3 does not apply to $[G(3,1,n)]_2$ as the corresponding tessellation of the plane $\mathbb{R}^2$ from Example 5.2 has two orbits of vertices under the 1 dimensional group in Lemma 5.1(3). In fact, the group $[G(3,1,n)]_2$ does *not* have the Steinberg property, and Proposition 6.1 below corrects a claim in [10].

We find that even crystallographic groups of rank 2 may require some special analysis. In the proof of the next proposition, we examine 1-dimensional sublattice structure.

Proposition 5.5. *The group $[G(6,2,2)]_2$ has the Steinberg property.*

Proof. Suppose u in V is fixed by some nonreflection $g \neq 1$ in $W = [G(6,2,2)]_2$. Let $\omega = \xi^2 = e^{2\pi i/3}$. One may check that for any γ in $\mathbb{Z}[\omega] = \mathbb{Z}[\xi]$, W includes reflections about affine hyperplanes

$$x_1 = \gamma,\quad x_2 = \gamma,\quad x_1 - \xi^j x_j = \gamma \ \text{ for } j \text{ odd},\quad x_1 - \xi^j x_j = (1-\omega)\gamma \ \text{ for } j \text{ even}. \tag{5.1}$$

We argue that u lies on one of these hyperplanes by applying the eight linear forms $x_1, x_2, x_1 - \xi^j x_2$ to u and showing one of the resulting values lies in $\mathbb{Z}[\omega]$ or in $(1-\omega)\mathbb{Z}[\omega]$ (for j even).

By Lemma 2.1(c), $\text{Lin}(g)$ is not a reflection and we may assume that $\text{Lin}(g)$ is $\text{diag}(\omega,\omega^2)$, $\text{diag}(-1,-1)$, or $\text{diag}(\omega^2,\omega^2)$ by replacing g by a power of g if necessary. Fix $\alpha,\beta \in \mathbb{Z}[\omega]$ with

$$\text{Tran}(g) = \alpha\begin{pmatrix}-\omega^2\\ -1\end{pmatrix} + \beta(1-\omega^2)\begin{pmatrix}1\\ -1\end{pmatrix}.$$

First, suppose $\text{Lin}(g) = \text{diag}(\omega,\omega^2)$. Then $x_1(u) + \omega x_2(u) = \beta \in \mathbb{Z}[\omega]$ and u lies on a reflecting hyperplane of (5.1).

Second, suppose $\text{Lin}(g) = \text{diag}(-1,-1)$. If $x_2(u) \in \mathbb{Z}[\omega]$, then u lies on a reflecting hyperplane of (5.1), so we assume $x_2(u) \notin \mathbb{Z}[\omega]$. As u is fixed by g, we claim applying the five linear forms $x_1, x_2, x_1+x_2, x_1+\omega x_2, x_1+\omega^2 x_2$ to u gives numbers in $(1/2)\mathbb{Z}[\omega]$ which lie in distinct cosets of the subgroup $\mathbb{Z}[\omega]$ of the additive group $(1/2)\mathbb{Z}[\omega]$. Indeed, overlapping cosets would imply that $x_2(u) = (1/3)(1-\omega^2)(1-\omega)x_2(u)$ lay in $(1/3)\mathbb{Z}[\omega] \cap (1/2)\mathbb{Z}[\omega] = \mathbb{Z}[\omega]$. As the subgroup $\mathbb{Z}[\omega]$ has index 4 in the larger group, one of these cosets is trivial. Thus $x_1(u)$, $x_2(u)$, $x_1+x_2(u)$, $x_1+\omega x_2(u)$, or $x_1+\omega^2 x_2(u)$ lies in $\mathbb{Z}[\omega]$ and u lies on some reflecting hyperplane of (5.1).

Third, suppose $\text{Lin}(g) = \text{diag}(\omega^2,\omega^2)$. We apply the three linear forms $x_1 - x_2, x_1 - \omega x_2, x_1 - \omega^2 x_2$ to u and obtain complex numbers $\alpha+2\beta$, $-\omega^2(\alpha+\beta)$, $-\omega\beta$. We argue that one of these numbers lies in $(1-\omega)\mathbb{Z}[\omega]$ and thus u lies on a reflecting hyperplane of (5.1). If β itself lies in $(1-\omega)\mathbb{Z}[\omega]$, then so does $-\omega\beta$, so we assume $\beta \notin \mathbb{Z}[\omega]$. Then $2\beta \notin \mathbb{Z}[\omega]$ as well, since $(1-\omega)\mathbb{Z}[\omega] = (1-\omega^2)\mathbb{Z}[\omega] = \{a+b\omega : a+b \equiv 0 \mod 3\}$. This implies that the complex numbers α, $\alpha+\beta$, and $\alpha+2\beta$ lie in different cosets of the subgroup $(1-\omega)\mathbb{Z}[\omega]$ of the additive group $\mathbb{Z}[\omega]$. As this subgroup has index 3 in the larger group, one of these cosets is trivial, and thus α, $\alpha+\beta$, or $\alpha+2\beta$ lies in $(1-\omega)\mathbb{Z}[\omega]$. But $(1-\omega)\mathbb{Z}[\omega]$ is closed under multiplication by ω, hence $\alpha+2\beta$, $-\omega^2(\alpha+\beta)$, or $-\omega\beta$ must lie in $(1-\omega)\mathbb{Z}[\omega]$ as well. □

Proposition 5.6. *Suppose $W = G \ltimes \Lambda$ is a crystallographic reflection group acting on $\mathbb{C}^n$ for $n \geq 1$ with G equal to $G(4,2,n)$, $G(6,2,n)$, or $G(6,3,n)$ but $W \neq [G(6,3,2)]_2$. Then W has the Steinberg property.*

Proof. The claim follows for $n = 1$, see Subsection 3.1, so we assume $n > 1$. Note that the affine reflecting hyperplanes for groups $[G(4,2,n)]_2$ and $[G(4,1,n)]_1$ coincide and $[G(4,2,n)]_2 \subset [G(4,1,n)]_1$ (see [7]). Similarly, the affine reflecting hyperplanes for $[G(4,2,n)]_1$ and $[G(4,1,n)]_2$ also coincide and $[G(4,2,n)]_1 \subset [G(4,1,n)]_2$. The groups $[G(4,2,n)]_1$ and $[G(4,2,n)]_2$ thus inherit the Steinberg property from $[G(4,1,n)]_1$ and $G[(4,1,n)]_2$, which have the property by Proposition 5.3. (Note that $[G(4,1,n)]_1$ and $[G(4,1,n)]_2$ do not have the same set of hyperplanes.) The group $W = [G(6,2,2)]_2$ has the Steinberg property by Proposition 5.5.

Fix some non-reflection $g \neq 1_W$ in W with nontrivial fixed point space V^g and note $\mathrm{Lin}(g)$ is also not a reflection by Lemma 2.1(3).

First suppose $W = [G(4,2,2)]_3$. Then $\mathrm{Lin}(g^m) = \mathrm{diag}(-1,-1)$ for $m = 1$ or 2 and

$$\mathrm{Tran}(g^m) = \alpha \begin{pmatrix} i \\ -1 \end{pmatrix} + \beta(1+i) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

for some $\alpha, \beta \in \mathbb{Z}[i]$. If g fixes u in V, then $x_1(u) + i x_2(u) = \beta$ and u is fixed by the affine reflection s with $\mathrm{Lin}(s)$ sending e_1 to $-ie_2$ and e_2 to ie_1 and $\mathrm{Tran}(s) = \beta e_1 - i\beta e_2$. The reflection s lies in W since $\mathrm{Tran}(s) = (-\beta)(i,-1) + \beta(1+i)(1,-1)$ lies in Λ. Thus $[G(4,2,2)]_3$ has the Steinberg property.

Now suppose $W = [G(6,p,n)]_1 = G(6,p,n) \ltimes \Lambda$ for some lattice Λ in Table 1. First assume $\mathrm{Lin}(g)$ is diagonal. If $g^p \neq 1$, then Lemma 5.1(1) applies, so we assume $g^p = 1$. Then $\mathrm{Lin}(g)$ has two nontrivial diagonal entries λ_j and λ_ℓ which are p-th roots-of-unity as $(\mathrm{Lin}(g))^p = \mathrm{Lin}(g^p) = 1$. Consider

$$\alpha_j = \tfrac{x_j(\mathrm{Tran}(g))}{1-\lambda_j} \quad \text{and} \quad \alpha_\ell = \tfrac{x_\ell(\mathrm{Tran}(g))}{1-\lambda_\ell} \quad \text{in } \mathbb{C}.$$

One may check directly (using that $r = 6$ and $p = 2$ or $p = 3$) that α_j, α_ℓ must both lie in the set $X = \mathbb{Z}[\omega]/(1-\xi^{6/p})$. There are two orbits in X under the action of $G(6,1,1) \ltimes \mathbb{Z}[\omega]$ on $\mathbb{C}$ with $\mathbb{Z}[\omega]$ one of the orbits. If α_j or α_ℓ lie in different orbits, then either α_j or α_ℓ lies in $\mathbb{Z}[\omega] = x_1(\mathbb{C}e_1 \cap \Lambda)$ and Lemma 5.1(2) applies. Otherwise, both α_j or α_ℓ lie in the same orbit and Lemma 5.1(3) applies with $\Lambda' = x_1(\mathbb{C}(e_1 - e_2) \cap \Lambda) = \mathbb{Z}[\omega]$. We conclude that any vector fixed by g lies on a reflecting hyperplane for W when $\mathrm{Lin}(g)$ is diagonal. When $\mathrm{Lin}(g)$ is not diagonal, we use arguments as in the proof of Proposition 5.3. (Note that we may work up to conjugation by any element in $G(6,1,n)$ since $G(6,p,n)$ is normal in $G(6,1,n)$ and $\mathrm{Tran}(W)$ is $G(6,1,n)$-invariant; applying $G(6,1,n)$ to any reflecting hyperplane for W will produce another reflecting hyperplane for W although not necessarily in the same W-orbit.) □

6. GENUINE GROUPS FAILING STEINBERG'S THEOREM

We now determine genuine crystallographic affine reflection groups in the 4-parameter infinite family failing Steinberg's theorem. There are also genuine crystallographic groups W with $\mathrm{Lin}(W) \neq G(r,p,n)$ which fail to satisfy the Steinberg property, $[G_4]_1$ for example, see Cote [3]. We set $\omega = e^{2\pi i/3}$ in Table 2.

The next proposition also holds for low values of r and n if we exclude the group $W = [G(2,2,3)]_1^\alpha$. Such groups have linear parts which are finite Coxeter groups and appear in Section 7.

Proposition 6.1. *The groups $[G(r,r,n)]_1$ for $r \geq 3$ and $n \geq 3$, $[G(3,1,n)]_2$ for $n \geq 2$, and $[G(6,3,2)]_2$ do not have the Steinberg property.*

Proof. Table 2 records elements in each group W which fix a point in V not on a reflecting hyperplane for W. For $W = [G(3,1,2)]_2$ or $[G(6,3,2)]_2$, one may check directly that the fixed point space V^g of the element g given in the table does not lie on an reflecting hyperplanes for W. For $W = [G(r,r,3)]_1$, one may check that $V^g = \{u\}$ for a vector u in $V = \mathbb{C}^3$ with $x_1(u)$, $x_2(u)$, and $x_3(u)$ lying in different orbits under the action of $G(r,1,1) \ltimes \mathbb{Z}[\xi]$. But each reflecting hyperplane for $[G(r,r,3)]_1$ is the zero set of a

TABLE 2. Genuine Crystallographic groups failing Steinberg's theorem

Group W	$g \in W$ **with** $V^g \not\subset$ **union of reflecting hyperplanes**	
$[G(3,1,2)]_2$	$g(v) = \begin{pmatrix} \omega & \\ & \omega \end{pmatrix} v + \begin{pmatrix} 1/(1-\omega) \\ -1/(1-\omega) \end{pmatrix}$	for $v \in \mathbb{C}^2$
$[G(3,3,3)]_1$	$g(v) = \begin{pmatrix} \omega & & \\ & \omega & \\ & & \omega \end{pmatrix} v + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$	for $v \in \mathbb{C}^3$
$[G(4,4,3)]_1$	$g(v) = \begin{pmatrix} i & & \\ & -1 & \\ & & i \end{pmatrix} v + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$	for $v \in \mathbb{C}^3$
$[G(6,3,2)]_2$	$g(v) = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} v + \begin{pmatrix} 1 \\ \omega-1 \end{pmatrix}$	for $v \in \mathbb{C}^2$
$[G(6,6,3)]_1$	$g(v) = \begin{pmatrix} \omega & & \\ & -1 & \\ & & -\omega^2 \end{pmatrix} v + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$	for $v \in \mathbb{C}^3$

polynomial $x_j = \xi^m x_k + \beta$ for $m \in \mathbb{Z}$ and $\beta \in \mathbb{Z}[\xi]$, and thus any point on a reflecting hyperplane has two coordinates lying in the same orbit of $G(r,1,1) \ltimes \mathbb{Z}[\xi]$ acting on $\mathbb{C}$. Thus the claim follows for $n = 3$, and for larger n by extending the given g by an identity transformation to act on $\mathbb{C}^n$. $\square$

The results in this section and last together give our main result, restated from the introduction:

Theorem 6.2. *Suppose $W = G(r,p,n) \ltimes \Lambda$ is a genuine crystallographic complex reflection group acting on $\mathbb{C}^n$ for some lattice Λ with $r,p,n \geq 1$. Then W has the Steinberg property if and only if W is* $[G(r,1,1)]_1$, $[G(3,1,n)]_1$, $[G(4,1,n)]_{1,2}$, $[G(6,1,n)]_1$, $[G(4,2,n)]_{1,2}$, $[G(4,2,2)]_3$, $[G(6,2,n)]_1$, $[G(6,2,2)]_2$, *or* $[G(6,3,n)]_1$.

7. Complexification of finite Coxeter groups

Now for the nongenuine case. We consider crystallographic reflection groups $G(r,p,n) \ltimes \Lambda$ acting on $V = \mathbb{C}^n$ where $G(r,p,n)$ is the complexification of a finite Coxeter group acting on $\mathbb{R}^n$. We exclude complexifications of affine Weyl groups, as they are not crystallographic, although the groups $G(r,p,n)$ that give rise to nongenuine crystallographic groups are all Weyl groups. The explicit lattices Λ appear in Table 3 with, again, ξ a primitive r-th root-of-unity in $\mathbb{C}$. We include groups whose linear part is the Weyl group $W(A_{n-1})$, the irreducible reflection representation of $G(1,1,n) \cong \mathfrak{S}_n$. We also include groups with linear part $G(r,r,2)$ for $r > 2$ since $G(r,r,2)$ is the complexification of the dihedral groups D_{2r} of order $2r$ after change-of-basis. We use Popov's notation with each α a complex parameter in the *modular strip*

$$\Omega = \{z \in \mathbb{C} : -(1/2) \leq \Re(z) < 1/2, 1 \leq |z| \text{ for } \Re(z) \leq 0, 1 < |z| \text{ for } \Re(z) > 0\}.$$

Note that up to equivalence, there are only 3 parameters of crystallographic groups with linear part the Weyl group $W(B_2) = G(2,1,2)$ and Popov choose to label these $[G(2,1,2)]_k^\alpha$ for $k = 1,2,3$ and α in Ω. As this notation conflicts with his notation for higher dimensional groups, we use indices $k = 1,2,4$ instead so that the lattices for $[G(2,1,2)]_k^\alpha$ and $[G(2,1,n)]_k^\alpha$ for $n > 2$ are always analogous. Note that the group $[G(2,1,2)]_3^\alpha$ is equivalent to $[G(2,1,2)]_4^\alpha$ and $[G(2,1,2)]_5^\alpha$ is equivalent to $G[(2,1,2)]_1^\alpha$, thus we restrict to $n \geq 3$ for $[G(2,1,n)]_5^\alpha$ in Table 3.

Theorem 7.1. *Let $W = G(r,p,n) \ltimes \Lambda$ be a crystallographic reflection group whose linear part* $\mathrm{Lin}(W)$ *is the complexification of a finite Coxeter group. Then W has the Steinberg property if and only if W is* $[W(A_{n-1})]_1^\alpha$, $[G(2,1,n)]_1^\alpha$, *or* $[G(2,2,3)]_1$.

Proof. We assume $n > 1$, as the claim holds for $n = 1$ (see Subsection 3.1). The groups $[W(A_{n-1})]_1^\alpha$ have the Steinberg property by Proposition 4.2. If $W = [G(2,1,n)]_1^\alpha$, arguments as in the proof of Proposition 5.3 show that W has the Steinberg property.

TABLE 3. Crystallographic reflection groups W = Coxeter Group $\ltimes \Lambda$

Group W	**dim**	$G(r,p,n)$**-invariant lattice** Λ	**Steinberg's thm**
$[W(A_{n-1})]_1^\alpha$	$n-1\geq 2$	$\Sigma_{j=2}^n(\mathbb{Z}+\mathbb{Z}\alpha)(e_{j-1}-e_j)$	✓
$[G(2,1,1)]_1^\alpha$	$n=1$	$(\mathbb{Z}+\mathbb{Z}\alpha)e_1$	✓
$[G(2,1,n)]_1^\alpha$	$n\geq 2$	$(\mathbb{Z}+\mathbb{Z}\alpha)e_1+\Sigma_{j=2}^n(\mathbb{Z}+\mathbb{Z}\alpha)(e_{j-1}-e_j)$	✓
$[G(2,1,n)]_2^\alpha$	$n\geq 2$	$(\mathbb{Z}+\mathbb{Z}\alpha)e_1+\Sigma_{j=2}^n\left(\mathbb{Z}+\mathbb{Z}\frac{1+\alpha}{2}\right)(e_{j-1}-e_j)$	✗
$[G(2,1,n)]_3^\alpha$	$n\geq 2$	$(\mathbb{Z}+\mathbb{Z}\alpha)e_1+\Sigma_{j=2}^n\left(\frac{1}{2}\mathbb{Z}+\mathbb{Z}\alpha\right)(e_{j-1}-e_j)$	✗
$[G(2,1,n)]_4^\alpha$	$n\geq 2$	$(\mathbb{Z}+\mathbb{Z}\alpha)e_1+\Sigma_{j=2}^n\left(\mathbb{Z}+\mathbb{Z}\frac{\alpha}{2}\right)\mathbb{Z}(e_{j-1}-e_j)$	✗
$[G(2,1,n)]_5^\alpha$	$n\geq 3$	$(\mathbb{Z}+\mathbb{Z}\alpha)e_1+\Sigma_{j=2}^n(1/2)(\mathbb{Z}+\mathbb{Z}\alpha)(e_{j-1}-e_j)$	✗
$[G(2,2,3)]_1^\alpha$	$n=3$	$(\mathbb{Z}+\mathbb{Z}\alpha)(-e_1-e_2)+\Sigma_{j=2}^3(\mathbb{Z}+\mathbb{Z}\alpha)(e_{j-1}-e_j)$	✓
$[G(2,2,n)]_1^\alpha$	$n\geq 4$	$(\mathbb{Z}+\mathbb{Z}\alpha)(-e_1-e_2)+\Sigma_{j=2}^n(\mathbb{Z}+\mathbb{Z}\alpha)(e_{j-1}-e_j)$	✗
$[G(6,6,2)]_1^\alpha$	$n=2$	$(\mathbb{Z}+\mathbb{Z}\alpha)(\xi e_1-e_2)+(\mathbb{Z}+\mathbb{Z}\alpha)(1-\xi^2)(e_1-e_2)$	✗
$[G(6,6,2)]_2^\alpha$	$n=2$	$(\mathbb{Z}+\mathbb{Z}\alpha)(\xi e_1-e_2)+\left(\mathbb{Z}+\mathbb{Z}\frac{\alpha}{3}\right)(1-\xi^2)(e_1-e_2)$	✗
$[G(6,6,2)]_3^\alpha$	$n=2$	$(\mathbb{Z}+\mathbb{Z}\alpha)(\xi e_1-e_2)+\left(\mathbb{Z}+\mathbb{Z}\frac{1+\alpha}{3}\right)(1-\xi^2)(e_1-e_2)$	✗
$[G(6,6,2)]_4^\alpha$	$n=2$	$(\mathbb{Z}+\mathbb{Z}\alpha)(\xi e_1-e_2)+\left(\mathbb{Z}+\mathbb{Z}\frac{2+\alpha}{3}\right)(1-\xi^2)(e_1-e_2)$	✗

Now suppose $W=[G(2,2,3)]_1^\alpha$ and consider a nonidentity element $g\in W$ fixing a point in V. We assume g itself is not a reflection and thus $\mathrm{Lin}(g)$ is also not a reflection by Lemma 2.1(3). Then some power of $\mathrm{Lin}(g)$ must be conjugate by an element of $G(2,2,3)$ to $\mathrm{diag}(-1,-1,1)$ or to a 3-cycle in $\mathfrak{S}_3$. If conjugate to a 3-cycle, we appeal to Lemma 4.1. Thus we assume $\mathrm{Lin}(g)=\mathrm{diag}(-1,-1,1)$. Notice

$$\mathrm{Tran}(g)=\beta\begin{pmatrix}-1\\-1\\0\end{pmatrix}+\gamma\begin{pmatrix}1\\-1\\0\end{pmatrix}+\delta\begin{pmatrix}0\\1\\-1\end{pmatrix}\quad\text{for some}\quad\beta,\gamma,\delta\in(\mathbb{Z}+\mathbb{Z}\alpha).$$

By Lemma 2.1(1), g has finite order, so $\delta=0$. If $g(u)=u$, then $x_1(u)=\frac{\gamma-\beta}{2}$ and $x_2(u)=\frac{-\gamma-\beta}{2}$ and u lies on the reflecting hyperplane $\ker(x_1+x_2+\beta)$ of W. (This hyperplane is fixed by the reflection in W with linear part sending e_1 to $-e_2$, e_2 to $-e_1$, and e_3 to e_3 and translational part $(\beta,\beta,0)\in\Lambda$.)

The remaining nongenuine crystallographic complex reflection groups do not have the Steinberg property. For the groups $W=[G(2,1,n)]_k^\alpha$ with $k\neq 1$ and $\alpha\in\Omega$, observe that each reflecting hyperplane has the form $H=\ker(x_j-\beta)$ for some $\beta\in(1/2)(\mathbb{Z}+\mathbb{Z}\alpha)$ or the form $H=\ker(x_j\pm x_\ell-\beta)$ for some β in $x_1(\Lambda\cap\mathbb{C}(e_1-e_2))$. The reflecting hyperplanes for $[G(2,2,n)]_1^\alpha$ all have the form $H=\ker(x_j\pm x_\ell-\beta)$ for some $\beta\in(\mathbb{Z}+\mathbb{Z}\alpha)$. It is straightforward to check that the group elements g in Table 4 each fix a point that is not on one of these hyperplanes, after extending g by the identity to define a transformation on $\mathbb{C}^n$. For the groups $W=[G(6,6,2)]_k^\alpha$, one can similarly check directly that the fixed point set of the element g in W in Table 4 does not lie on any reflecting hyperplane. □

TABLE 4. Crystallographic groups failing Steinberg's theorem

Group W	$g \in W$ **with** $V^g \not\subset$ **union of reflecting hyperplanes**	
$[G(2,1,2)]_2^{\alpha}$	$g(v) = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} v + \begin{pmatrix} (3+\alpha)/2 \\ -(1+\alpha)/2 \end{pmatrix}$	for $v \in \mathbb{C}^2$
$[G(2,1,3)]_3^{\alpha}$	$g(v) = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} v + \begin{pmatrix} (2\alpha+1)/2 \\ -1/2 \end{pmatrix}$	for $v \in \mathbb{C}^3$
$[G(2,1,2)]_4^{\alpha}$	$g(v) = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} v + \begin{pmatrix} (2+\alpha)/2 \\ -\alpha/2 \end{pmatrix}$	for $v \in \mathbb{C}^2$
$[G(2,1,3)]_5^{\alpha}$	$g(v) = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} v + \begin{pmatrix} (3+\alpha)/2 \\ -\alpha/2 \\ -1/2 \end{pmatrix}$	for $v \in \mathbb{C}^3$
$[G(2,2,4)]_1^{\alpha}$	$g(v) = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} v + \begin{pmatrix} 1 \\ 1+\alpha \\ -\alpha \\ 0 \end{pmatrix}$	for $v \in \mathbb{C}^4$
$[G(6,6,2)]_1^{\alpha}$	$g(v) = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} v + \begin{pmatrix} \xi+\alpha(1+\xi) \\ -1-\alpha(1+\xi) \end{pmatrix}$	for $v \in \mathbb{C}^2$
$[G(6,6,2)]_2^{\alpha}$	$g(v) = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} v + \begin{pmatrix} \xi+\alpha(1+\xi)/3 \\ -1-\alpha(1+\xi)/3 \end{pmatrix}$	for $v \in \mathbb{C}^2$
$[G(6,6,2)]_3^{\alpha}$	$g(v) = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} v + \begin{pmatrix} \xi+(1+\alpha)(1+\xi)/3 \\ -1-(1+\alpha)(1+\xi)/3 \end{pmatrix}$	for $v \in \mathbb{C}^2$
$[G(6,6,2)]_4^{\alpha}$	$g(v) = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} v + \begin{pmatrix} \xi+(2+\alpha)(1+\xi)/3 \\ -1-(2+\alpha)(1+\xi)/3 \end{pmatrix}$	for $v \in \mathbb{C}^2$

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