

## Logarithmic forms and anti-invariant forms of reflection groups

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### Abstract.

Let  $W$  be a finite group generated by unitary reflections and  $\mathcal{A}$  be the set of reflecting hyperplanes. We will give a characterization of the logarithmic differential forms with poles along  $\mathcal{A}$  in terms of anti-invariant differential forms. If  $W$  is a Coxeter group defined over  $\mathbf{R}$ , then the characterization provides a new method to find a basis for the module of logarithmic differential forms out of basic invariants.

**Basic definitions.** Let  $V$  be an  $\ell$ -dimensional unitary space. Let  $W \subset \mathbf{GL}(V)$  be a finite group generated by unitary reflections and  $\mathcal{A}$  be the set of reflecting hyperplanes. We say that  $W$  is a *unitary reflection group* and  $\mathcal{A}$  is the corresponding *unitary reflection arrangement*. Let  $S$  be the algebra of polynomial functions on  $V$ . The algebra  $S$  is naturally graded by  $S = \bigoplus_{q \geq 0} S_q$  where  $S_q$  is the space of homogeneous polynomials of degree  $q$ . Thus  $S_1 = V^*$  is the dual space of  $V$ . Let  $\text{Der}_S$  be the  $S$ -module of  $\mathbf{C}$ -derivations of  $S$ . We say that  $\theta \in \text{Der}_S$  is homogeneous of degree  $q$  if  $\theta(S_1) \subseteq S_q$ . Choose for each hyperplane  $H \in \mathcal{A}$  a linear form  $\alpha_H \in V^*$  such that  $H = \ker(\alpha_H)$ . Define  $Q \in S$  by

$$Q = \prod_{H \in \mathcal{A}} \alpha_H.$$

The polynomial  $Q$  is uniquely determined, up to a constant multiple, by the group  $W$ . When convenient we choose a basis  $e_1, \dots, e_\ell$  for  $V$  and let  $x_1, \dots, x_\ell$  denote the dual basis for  $V^*$ . Let  $\langle \cdot, \cdot \rangle : V^* \times V \rightarrow \mathbf{C}$  denote the natural pairing. Thus  $\langle x_i, e_j \rangle = \delta_{ij}$ . For each  $v \in V$  let  $\partial_v \in \text{Der}_S$  be the unique derivation such that  $\partial_v x = \langle x, v \rangle$  for  $x \in V^*$ . Define  $\partial_i \in \text{Der}_S$  by  $\partial_i = \partial_{e_i}$ . Then  $\partial_i x_j = \delta_{ij}$  and  $\text{Der}_S$  is a free  $S$ -module with basis  $\partial_1, \dots, \partial_\ell$ . There is a natural isomorphism  $S \otimes V \rightarrow \text{Der}_S$  of  $S$ -modules given by

$$f \otimes v \mapsto f \partial_v$$

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for  $f \in S$  and  $v \in V$ . Let  $\Omega^1 = \text{Hom}_S(\text{Der}_S, S)$  be the  $S$ -module dual to  $\text{Der}_S$ . Define  $d : S \rightarrow \Omega^1$  by  $df(\theta) = \theta(f)$  for  $f \in S$  and  $\theta \in \text{Der}_S$ . Then  $d(ff') = (df)f' + f(df')$  for  $f, f' \in S$ . Furthermore,  $\Omega^1$  is a free  $S$ -module with basis  $dx_1, \dots, dx_\ell$  and  $df = \sum_{i=1}^\ell (\partial_i f) dx_i$ . There is a natural isomorphism  $S \otimes V^* \rightarrow \Omega^1$  of  $S$ -modules given by

$$f \otimes x \mapsto f dx$$

for  $f \in S$  and  $x \in V^*$ . The modules  $\text{Der}_S$  and  $\Omega^1$  inherit gradings from  $S$  which are defined by  $\deg(f\partial_v) = \deg(f)$  and  $\deg(f dx) = \deg(f)$  if  $f \in S$  is homogeneous. Define  $\Omega^p = \bigwedge_S^p \Omega^1$  ( $p = 1, \dots, \ell$ ). The  $S$ -module  $\Omega^p$  is free with a basis  $\{dx_{i_1} \wedge \dots \wedge dx_{i_p} \mid 1 \leq i_1 < \dots < i_p \leq \ell\}$ . It is naturally isomorphic to  $S \otimes_{\mathbf{C}} \bigwedge^p V^*$ . Let  $\Omega^p(\mathcal{A})$  be the  $S$ -module of logarithmic  $p$ -forms with poles along  $\mathcal{A}$  [Sai3][OrT]:

$$\Omega^p(\mathcal{A}) = \left\{ \frac{\eta}{Q} \mid \eta \in \Omega^p, d\left(\frac{\eta}{Q}\right) \in \frac{1}{Q} \Omega^{p+1} \right\}$$

where  $d$  is the exterior differentiation.

The unitary reflection group  $W$  acts contragradiently on  $V^*$  and thus on  $S$ . The modules  $\text{Der}_S$  and  $\Omega^p$  ( $p = 1, \dots, \ell$ ) also have  $W$ -module structures so that the above isomorphisms are  $W$ -module isomorphisms. If  $M$  is an  $\mathbf{C}[W]$ -module let  $M^W = \{x \in M \mid wx = x \text{ for all } w \in W\}$  denote the space of invariant elements in  $M$ . Let  $M^{\det^{-1}} = \{x \in M \mid wx = \det(w)^{-1}x \text{ for all } w \in W\}$  denote the space of anti-invariant elements in  $M$ . Let  $R = S^W$  be the invariant subring of  $S$  under  $W$ . By a theorem of Shephard, Todd, and Chevalley [Bou, V.5.3, Theorem 3] there exist algebraically independent homogeneous polynomials  $f_1, \dots, f_l \in R$  such that  $R = \mathbf{C}[f_1, \dots, f_l]$ . They are called *basic invariants*. Elements of  $S^{\det^{-1}}$  and  $(\Omega^p)^{\det^{-1}}$  are called *anti-invariants* and *anti-invariant  $p$ -forms* respectively. It is well-known that  $S^{\det^{-1}} = RQ$ .

**The main theorem.** The following theorem gives the relationship between logarithmic forms and anti-invariant forms.

**Theorem 1.** For  $1 \leq p \leq \ell$ ,

$$\Omega^p(\mathcal{A}) = \frac{1}{Q} (\Omega^p)^{\det^{-1}} \otimes_R S.$$

*Proof.* Let  $x_1, \dots, x_\ell$  be an orthonormal basis for  $V^*$ . Let  $\theta_1, \dots, \theta_\ell$  be an  $R$ -basis for  $\text{Der}_S^W$ . Then, by [OrT, Theorem 6.59],  $\theta_1, \dots, \theta_\ell$  is known to be an  $S$ -basis for the module  $D(\mathcal{A})$  of  $\mathcal{A}$ -derivations, where

$$D(\mathcal{A}) = \{\theta \in \text{Der}_S \mid \theta(Q) \in QS\}.$$

By the contraction  $\langle \cdot, \cdot \rangle$  of a 1-form and a derivation, the  $S$ -modules  $D(\mathcal{A})$  and  $\Omega^1(\mathcal{A})$  are  $S$ -dual each other [OrT, Theorem 4.75]. Let  $\{\omega_1, \dots, \omega_\ell\} \subset \Omega^1(\mathcal{A})$  be dual to  $\{\theta_1, \dots, \theta_\ell\}$ . In other words,  $\langle \theta_i, \omega_j \rangle = \delta_{ij}$  (Kronecker's delta). Then  $\{\omega_1, \dots, \omega_\ell\}$  is an  $S$ -basis for  $\Omega^1(\mathcal{A})$ . Then each  $\omega_i$  is obviously  $W$ -invariant and

$$\omega_i \in \left(\frac{1}{Q}\Omega^1\right)^W = \frac{1}{Q}(\Omega^1)^{\det^{-1}}.$$

Therefore we have

$$\Omega^1(\mathcal{A}) \subseteq \frac{1}{Q}(\Omega^1)^{\det^{-1}} \otimes_R S.$$

By [OrT, Proposition 4.81], the set  $\{\omega_{i_1} \wedge \dots \wedge \omega_{i_p} \mid 1 \leq i_1 < \dots < i_p \leq \ell\}$  is a basis for  $\Omega^p(\mathcal{A})$ . In particular,  $\omega_{i_1} \wedge \dots \wedge \omega_{i_p} \in \frac{1}{Q}\Omega^p$ . Since  $\omega_{i_1} \wedge \dots \wedge \omega_{i_p}$  is  $W$ -invariant,  $Q(\omega_{i_1} \wedge \dots \wedge \omega_{i_p}) \in (\Omega^p)^{\det^{-1}}$ . This shows that

$$\Omega^p(\mathcal{A}) \subseteq \frac{1}{Q}(\Omega^p)^{\det^{-1}} \otimes_R S.$$

Conversely let  $\omega \in \frac{1}{Q}(\Omega^p)^{\det^{-1}}$ . Then  $Q\omega \in \Omega^p \subseteq \Omega^p(\mathcal{A})$ . Thus  $Q\omega$  can be uniquely expressed as

$$Q\omega = \sum_{i_1 < \dots < i_p} f_{i_1 \dots i_p} \omega_{i_1} \wedge \dots \wedge \omega_{i_p} \quad (f_{i_1 \dots i_p} \in S).$$

Act  $w \in W$  on both sides, and we get

$$\det(w)^{-1}Q\omega = w(Q)\omega = \sum_{i_1 < \dots < i_p} w(f_{i_1 \dots i_p}) \omega_{i_1} \wedge \dots \wedge \omega_{i_p}.$$

Therefore, by the uniqueness of the expression, we have

$$\det(w)^{-1}f_{i_1 \dots i_p} = w(f_{i_1 \dots i_p}) \quad (w \in W)$$

and  $f_{i_1 \dots i_p} \in S^{\det^{-1}} = RQ$ . This implies that each  $f_{i_1 \dots i_p}/Q$  lies in  $S$  and that

$$\omega = \sum_{i_1 < \dots < i_p} \left(\frac{f_{i_1 \dots i_p}}{Q}\right) \omega_{i_1} \wedge \dots \wedge \omega_{i_p} \in \Omega^p(\mathcal{A}).$$

Thus we have shown the inclusion

$$\frac{1}{Q}(\Omega^p)^{\det^{-1}} \otimes_R S \subseteq \Omega^p(\mathcal{A}).$$

Q.E.D.

Taking the  $W$ -invariant parts of the both sides in Theorem 1, we have

**Corollary 2.** For  $1 \leq p \leq \ell$ ,

$$(\Omega^p(\mathcal{A}))^W = \frac{1}{Q}(\Omega^p)^{\det^{-1}}.$$

The following theorem is a special case of a theorem obtained by Shepler [She].

**Theorem 3** (Shepler). For  $1 \leq p \leq \ell$ ,

$$(\Omega^p)^{\det^{-1}} = Q^{1-p} \bigwedge_R^p (\Omega^1)^{\det^{-1}}.$$

*Proof.* In the proof of Theorem 1, we have already shown that the both sides have the same  $R$ -basis

$$\{Q(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}) \mid 1 \leq i_1 < \cdots < i_p \leq \ell\}.$$

Q.E.D.

**The Coxeter case.** From now on we assume that  $W$  is a *Coxeter group*. In other words, for an  $\ell$ -dimensional Euclidean space  $V$ ,  $W \subset \mathbf{GL}(V)$  is a finite group generated by orthogonal reflections and  $W$  acts irreducibly on  $V$ . The objects like  $S$ ,  $R$ , and  $\Omega^p$  are defined over  $\mathbf{R}$ . Note that  $\det(w)$  is either  $+1$  or  $-1$  for any  $w \in W$  and thus  $\det = \det^{-1}$ .

Recall the definition of the  $\mathbf{R}$ -linear map  $\hat{d} : S \rightarrow \Omega^1$  in [SoT, Proposition 6.1]:

$$\hat{d}f = \sum_{i=1}^{\ell} (\partial_i f) d(Q(Dx_i)).$$

Here  $D$  is a Saito derivation introduced in [Sai2][SYS]. The following proposition is Proposition 6.1 in [SoT]:

**Proposition 4** (Solomon-Terao). Let  $f_1, \dots, f_\ell$  be basic invariants. Then

$$(\Omega^1)^{\det} = R\hat{d}f_1 \oplus \cdots \oplus R\hat{d}f_\ell.$$

From Theorem 3 and Proposition 4 we get

**Corollary 5.** For  $1 \leq p \leq \ell$ ,

$$(\Omega^p)^{\det} = \bigoplus_{1 \leq i_1 < \cdots < i_p \leq \ell} RQ^{1-p}(\hat{d}f_{i_1} \wedge \cdots \wedge \hat{d}f_{i_p}).$$

Using Theorem 1, we have

**Corollary 6.** For  $1 \leq p \leq \ell$ ,

$$\Omega^p(\mathcal{A}) = \bigoplus_{1 \leq i_1 < \cdots < i_p \leq \ell} SQ^{-p}(\hat{d}f_{i_1} \wedge \cdots \wedge \hat{d}f_{i_p}).$$

This corollary gives a new method using the new differential operator  $\hat{d}$  to calculate a basis for the module of logarithmic forms.

Taking the  $W$ -invariant parts of the both sides in Corollary 6, we also have

**Corollary 7.** For  $1 \leq p \leq \ell$ ,

$$(\Omega^p(\mathcal{A}))^W = \bigoplus_{1 \leq i_1 < \cdots < i_p \leq \ell} RQ^{-p}(\hat{d}f_{i_1} \wedge \cdots \wedge \hat{d}f_{i_p}).$$

**Example 8** ( $B_2$ ). When  $W$  is the Coxeter group of type  $B_2$ , we can choose

$$f_1 = \frac{1}{2}(x_1^2 + x_2^2), \quad f_2 = \frac{1}{4}(x_1^4 + x_2^4).$$

Then, as seen in [SoT, §5.2], the operator  $\hat{d}$  in Proposition 4 satisfies

$$\hat{d}x_1 = -dx_2, \quad \hat{d}x_2 = dx_1.$$

Thus

$$\hat{d}f_1 = -x_1 dx_2 + x_2 dx_1, \quad \hat{d}f_2 = -x_1^3 dx_2 + x_2^3 dx_1.$$

Then  $\hat{d}f_1$  and  $\hat{d}f_2$  form an  $R$ -basis for  $(\Omega^1)^{\det}$  and  $\hat{d}f_1/Q$  and  $\hat{d}f_2/Q$  form an  $S$ -basis for  $\Omega^1(\mathcal{A})$  as Corollaries 5 and 6 assert.

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