SEMIINVARIANTS OF FINITE REFLECTION GROUPS

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ABSTRACT. Let G be a finite group of complex $n \times n$ unitary matrices generated by reflections acting on $\mathbb{C}^n$. Let R be the ring of invariant polynomials, and χ be a multiplicative character of G. Let Ω^χ be the R-module of χ-invariant differential forms. We define a multiplication in Ω^χ and show that under this multiplication Ω^χ has an *exterior algebra* structure. We also show how to extend the results to vector fields, and exhibit a relationship between χ-invariant forms and logarithmic forms.

1. INTRODUCTION

In 1989, P. Doyle and C. McMullen [2] solved the fifth degree polynomial with a highly symmetrical dynamical system which preserved the Galois group A_5. In 1997, S. Crass and P. Doyle [1] solved the sixth degree polynomial by again finding a dynamical system with special symmetry—this time A_6 symmetry. Each dynamical system was formed by iterating a map that was equivariant under the projective action of the group. Such maps correspond naturally to semiinvariant differential forms. Because almost nothing was known about these forms, constructing the necessary dynamical systems was a difficult step in both cases.

We introduce here a general theory of semiinvariants. Specifically, we show that for any finite unitary reflection group G and multiplicative character χ of G, the module of χ-invariant differential forms has a natural multiplication which turns the module into an *exterior algebra.* This exterior algebra structure allows us to understand completely the forms that give rise to highly symmetrical dynamical systems, and gives us tools to compute these forms explicitly. We also show how to extend these results to vector fields (or *derivations*), and observe the relationship between semiinvariants and logarithmic forms.

The theory presented here builds on work by R. Stanley, who characterized the module of χ-invariant polynomials in 1977 [8]. It also builds on more recent work by Orlik, Saito, Solomon, Terao and others on invariant derivations and the theory of hyperplane arrangements (see [3], Chapter 6). Note that det-invariant forms have received attention under the name of *anti-invariant forms* in the context of Coxeter groups (see e.g. [7]).

2. NOTATION

Let G be a finite group of complex $n \times n$ unitary matrices generated by reflections acting on $V := \mathbb{C}^n$. Recall that a unitary matrix is a reflection if it has finite order and fixes a hyperplane pointwise in V. Let $S := \mathbb{C}[x_1, \ldots, x_n]$ be the ring of polynomials of V. Let $f_1, \ldots, f_n \in S$ be basic invariants, and $R = \mathbb{C}[f_1, \ldots, f_n]$ be the ring of invariant polynomials. Let χ be a multiplicative character of G. Denote the module of differential p-forms on V by

$$\begin{aligned} \Omega^p &:= \bigoplus_{1 \leq i_1 < \ldots < i_p \leq n} S dx_{i_1} \wedge \ldots \wedge dx_{i_p} \\ &\simeq S \otimes \textstyle\bigwedge^p V^*. \end{aligned}$$

The group G acts contragradiently on V^* and S, and Ω^p is a $\mathbb{C}[G]$-module. Define the R-module of *χ-invariant* differential p-forms as

$$(\Omega^p)^\chi := \{\omega \in \Omega^p : g\omega = \chi(g)\omega \text{ for all } g \in G\}.$$

Let

$$\Omega^\chi := \bigoplus_{0 \leq p} (\Omega^p)^\chi.$$

It is convenient to define $\mathcal{I}^p$ as the set of multiindices of $\{1, ..., n\}$ of length p:

$$\mathcal{I}^p := \{I = \{I_1, ..., I_p\} : 1 \leq I_1 < \ldots < I_p \leq n\}.$$

For a multiindex I, let I^c denote the complementary index. Denote the volume form on V by $vol := dx_1 \wedge \ldots \wedge dx_n$. If f and g are differential forms, we write $f \doteq g$ if $f = cg$ for some c in $\mathbb{C}^*$.

We recall some facts and notation from Arrangements of Hyperplanes ([3], p. 228). Let $\mathcal{A}$ be the hyperplane arrangement defined by G. For each $H \in \mathcal{A}$, define $\alpha_H \in S$ by $\ker(\alpha_H) = H$. Fix some $H \in \mathcal{A}$, and let G_H be the cyclic subgroup of elements in G that fix H pointwise. Let s_H be a generator of G_H and let $o(s_H)$ be the order of s_H. Define $a_H(\chi)$ as the least integer satisfying $0 \leq a_H(\chi) < o(s_H)$ and $\chi(s_H) = \det(s_H)^{-a_H(\chi)}$. Let

$$Q_\chi = \prod_{H \in \mathcal{A}} \alpha_H^{a_H(\chi)}.$$

The polynomial Q_χ is uniquely determined, upto a nonzero scaler multiple, by the group G.

R. Stanley [8] proved that $(\Omega^0)^\chi = R\,Q_\chi$, and since vol is $(\det^{-1})$-invariant, it follows that

$$(*) \qquad (\Omega^n)^\chi = R\,Q_{\chi \cdot \det}\, vol.$$

R. Steinberg [9] proved that $Q_{\det} = \prod_{H\in\mathcal{A}} \alpha_H^{o(s_H)-1}$ is the determinant of the Jacobian matrix $\left\{\frac{\partial}{\partial x_i} f_j\right\}$, upto a nonzero scalar multiple. Note also that $Q_{\det^{-1}} = \prod_{H\in\mathcal{A}} \alpha_H$ ([3], p. 229).

3. χ-WEDGING

The next lemma will be used to show that Q_χ divides the exterior product of any two χ-invariant forms.

Lemma 1. *Suppose that μ is a χ-invariant p-form. Fix a hyperplane $H \in \mathcal{A}$, and let $a = a_H(\chi)$. Choose coordinates in which $x_1 = \alpha_H$ and s_H is diagonal. If*

$$\mu = \sum_{I\in\mathcal{I}^p} \mu_I \, dx_{I_1} \wedge \ldots \wedge dx_{I_p}$$

in these coordinates, then x_1^{a-1} divides μ_I whenever $I_1 = 1$ and x_1^a divides μ_I whenever $I_1 \neq 1$, for each $I = \{I_1, \ldots, I_p\} \in \mathcal{I}^p$.

Proof. Let $s = s_H$ and ρ be the determinant of s. Then

$$s = \begin{pmatrix} \rho & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

and $s^{-1}dx_1 = \rho\, dx_1$, $s^{-1}dx_2 = dx_2$, $\ldots$, $s^{-1}dx_n = dx_n$.

Let $I = \{I_1, I_2, ..., I_p\} \in \mathcal{I}^p$. If $I_1 = 1$, then

$$\begin{aligned} s^{-1}(\mu_I \;\; dx_{I_1} \wedge \cdots \wedge dx_{I_p}) &= s^{-1}\mu_I \;\; s^{-1}dx_1 \wedge \cdots \wedge s^{-1}dx_{I_p} \\ &= \mu_I \circ s \;\; \rho \;\; dx_1 \wedge \cdots \wedge dx_{I_p}. \end{aligned}$$

If $I_1 \neq 1$, then

$$\begin{aligned} s^{-1}(\mu_I \;\; dx_{I_1} \wedge \cdots \wedge dx_{I_p}) &= s^{-1}\mu_I \;\; s^{-1}dx_{I_1} \wedge \cdots \wedge s^{-1}dx_{I_p} \\ &= \mu_I \circ s \;\; dx_{I_1} \wedge \cdots \wedge dx_{I_p}. \end{aligned}$$

But μ is χ-invariant, so $\rho^a\mu = \det(s)^a\mu = \chi^{-1}(s)\mu = s^{-1}\mu$. Hence if $I_1 = 1$, then $\rho^a\mu_I = \rho \;\; \mu_I \circ s$, i.e. $\rho^{a-1}\mu_I = \mu_I \circ s$. Thus x_1^{a-1} divides μ_I. Similarly, if $I_1 \neq 1$, then $\rho^a\mu_I = \mu_I \circ s$ and x_1^a divides μ_I.

□

Lemma 2. *Q_χ divides the exterior product of any two χ-invariant differential forms.*

Proof. Let μ be a χ-invariant p-form and ω be a χ-invariant q-form. Fix $H \in \mathcal{A}$. Let $s = s_H$ and $a = a_H(\chi)$. Assume that $a \neq 0$. We show that α_H^a

divides $\mu \wedge \omega$ by choosing coordinates from Lemma 1 in which $\alpha_H = x_1$. Let

$$\begin{aligned} \mu &= \sum_{I \in \mathcal{I}^p} \mu_I \, dx_{I_1} \wedge \ldots \wedge dx_{I_p}, \\ \omega &= \sum_{J \in \mathcal{I}^q} \omega_J \, dx_{J_1} \wedge \ldots \wedge dx_{J_q}, \text{ and} \\ \mu \wedge \omega &= \sum_{K \in \mathcal{I}^{p+q}} \gamma_K \, dx_{K_1} \wedge \ldots \wedge dx_{K_{p+q}} \end{aligned}$$

in these coordinates. Then x_1^a divides μ_I whenever $I_q \neq 1$ and x_1^a divides ω_J whenever $J_1 \neq 1$.

Hence, for $I \in \mathcal{I}^p$ and $J \in \mathcal{I}^q$, the polynomial $\mu_I \omega_J$ is divisible by x_1^a given that not both I_1 and J_1 are 1. Since each γ_K is either zero or a sum of terms of the form $\pm \mu_I \omega_J$ where the multiindices I and J are disjoint, x_1^a divides each γ_K and hence $\mu \wedge \omega$. Thus, $\mu \wedge \omega$ is divisible by $\alpha_H^a = \alpha_H^{a_H(\chi)}$. Since H was arbitrary, Q_χ divides $\mu \wedge \omega$.

□

Lemma 2 prompts us to define the following multiplication in Ω^χ: For differential forms μ and ω, define the *χ-wedge* of μ and ω as

$$\mu \curlywedge \omega := \frac{\mu \wedge \omega}{Q_\chi}.$$

If μ and ω are χ-invariant forms, $\mu \curlywedge \omega$ is again χ-invariant. Thus, Lemma 2 implies

Corollary 1. *The R-module Ω^χ is closed under χ-wedging.*

The following proposition gives a condition (similar to Saito's Criterion) for n 1-forms to generate Ω^χ. The proof is similar to Solomon's original argument [6] that $df_1, \ldots, df_n$ generate the module of invariant differential forms.

Proposition 1. *Let $\omega_1, \ldots, \omega_n$ be χ-invariant 1-forms. The forms $\omega_{I_1} \curlywedge \ldots \curlywedge \omega_{I_p}$, for $I \in \mathcal{I}^p$ and $p \geq 0$, generate Ω^χ over R if and only if*

$$\omega_1 \curlywedge \ldots \curlywedge \omega_n \doteq Q_{\chi \cdot \det} \, vol.$$

Proof. Assume that $\omega_1 \curlywedge \ldots \curlywedge \omega_n \doteq Q_{\chi \cdot \det} \, vol$. The p-forms $\omega_{I_1} \curlywedge \ldots \curlywedge \omega_{I_p}$, $I \in \mathcal{I}^p$, are χ-invariant by Corollary 1.

Since $\omega_1 \curlywedge \ldots \curlywedge \omega_n \neq 0$, $\omega_1 \wedge \ldots \wedge \omega_n \neq 0$, and the forms $\omega_{I_1} \wedge \ldots \wedge \omega_{I_p}$, $I \in \mathcal{I}^p$, are linearly independent over $F := \mathbb{C}(x_1, \ldots, x_n)$. If not, there exist rational functions r_I with

$$0 = \sum_{I \in \mathcal{I}^p} r_I \, \omega_{I_1} \wedge \cdots \wedge \omega_{I_p}.$$

Fix $J \in \mathcal{I}^p$ and $J^c \in \mathcal{I}^{n-p}$. Then

$$\begin{aligned} 0 &= \left(\sum_{I \in \mathcal{I}^p} r_I \, \omega_{I_1} \wedge \cdots \wedge \omega_{I_p} \right) \wedge \omega_{J_1^c} \wedge \cdots \wedge \omega_{J_{n-p}^c} \\ &= \pm \, r_J \, \omega_1 \wedge \cdots \wedge \omega_n, \end{aligned}$$

and r_J must be zero. Hence the forms

$$\omega_{I_1} \curlywedge \ldots \curlywedge \omega_{I_p} = (Q_\chi)^{1-p} \, \omega_{I_1} \wedge \ldots \wedge \omega_{I_p}, \quad I \in \mathcal{I}^p,$$

are also linearly independent over F, and thus span

$$\Omega^p(V) := \bigoplus_{I \in \mathcal{I}^p} F dx_{I_1} \wedge \ldots \wedge dx_{I_p}$$

since $\Omega^p(V)$ has dimension $\binom{n}{p}$.

Choose an arbitrary χ-invariant p-form μ. Then there exist rational functions $t_I \in F$ with

$$\mu = \sum_{I \in \mathcal{I}^p} t_I \, \omega_{I_1} \curlywedge \cdots \curlywedge \omega_{I_p}.$$

Fix $J \in \mathcal{I}^p$ and its complementary index J^c. We will show that $t_J \in R$.

By Corollary 1, the n-form $(\omega_{J_1^c} \curlywedge \cdots \curlywedge \omega_{J_{n-p}^c}) \curlywedge \mu$ is χ-invariant. Thus by Equation $(*)$ above, there exists a polynomial $f \in R$ with

$$\left(\omega_{J_1^c} \curlywedge \cdots \curlywedge \omega_{J_{n-p}^c} \right) \curlywedge \mu = f \, Q_{\chi \cdot \det} \, vol.$$

On the other hand,

$$\begin{aligned} &\left(\omega_{J_1^c} \curlywedge \cdots \curlywedge \omega_{J_{n-p}^c} \right) \curlywedge \mu \\ &\qquad = \left(\omega_{J_1^c} \curlywedge \cdots \curlywedge \omega_{J_{n-p}^c} \right) \curlywedge \sum_{I \in \mathcal{I}^p} t_I \, \omega_{I_1} \curlywedge \cdots \curlywedge \omega_{I_p} \\ &\qquad = \left(Q_\chi^{1-n} \right) \left(\omega_{J_1^c} \wedge \cdots \wedge \omega_{J_{n-p}^c} \right) \wedge \sum_{I \in \mathcal{I}^p} t_I \, \omega_{I_1} \wedge \cdots \wedge \omega_{I_p} \\ &\qquad = \left(Q_\chi^{1-n} \right) \, \pm \, t_J \, \omega_1 \wedge \cdots \wedge \omega_n \\ &\qquad = \pm \, t_J \, \omega_1 \curlywedge \cdots \curlywedge \omega_n \\ &\qquad \doteq \pm \, t_J \, Q_{\chi \cdot \det} \, vol. \end{aligned}$$

Thus $f \, Q_{\chi \cdot \det} \doteq \pm t_J \, Q_{\chi \cdot \det}$. Hence, $t_J \in R$. Since J was arbitrary, μ is in the R-span of $\{\omega_{I_1} \curlywedge \ldots \curlywedge \omega_{I_p}, \, I_p \in \mathcal{I}^p\}$.

The converse follows from Equation $(*)$ above.

$\square$

4. Condition satisfied

Since Ω^p has rank $\binom{n}{p}$, the R-module $(\Omega^p)^\chi$ is also free of rank $\binom{n}{p}$ (this follows from Lemma 6.45 of [3], p. 232). We will show that the generators of $(\Omega^1)^\chi$ satisfy the condition given in Proposition 1, but first we must gather some preliminary facts.

We recall some results about invariant vector fields. There exist n invariant vector fields, called *basic derivations*, that generate the module of invariant vector fields over R (see [3], Section 6.3). Using Saito's Criterion, H. Terao showed that the coefficient matrix of the basic derivations has determinate $Q_{\det^{-1}}$ upto a nonzero scaler multiple (see [3], p. 238). Using the minors of this coefficient matrix, we construct $(\det^{-1})$-invariant 1-forms, $\mu_1, \dots, \mu_n$, that satisfy

$$\mu_1 \wedge \cdots \wedge \mu_n \quad = \quad Q^{n-1}_{\det^{-1}} \, vol.$$

The forms $\mu_1, \dots, \mu_n$ thus generate $\Omega^{\det^{-1}}$ over R by Proposition 1. We will use these forms to give an argument for arbitrary χ.

We also note the relationship between $Q_{\chi\cdot\det}$ and Q_χ: Fix $H \in \mathcal{A}$ with $a_H(\chi) \neq 0$. The exponent $a_H(\chi \cdot \det)$ is the least nonnegative integer satisfying

$$\begin{aligned}
\det(s_H)^{-a_H(\chi\cdot\det)} &= (\chi \cdot \det)(s_H) \\
&= \chi(s_H)\ \det(s_H) \\
&= \det(s_H)^{-a_H(\chi)}\ \det(s_H) \\
&= \det(s_H)^{-(a_H(\chi)-1)}.
\end{aligned}$$

Hence, $a_H(\chi \cdot \det) = a_H(\chi) - 1$. Now fix $H \in \mathcal{A}$ with $a_H(\chi) = 0$. Then

$$\begin{aligned}
\det(s_H)^{-a_H(\chi\cdot\det)} &= (\chi \cdot \det)(s_H) \\
&= \chi(s_H)\ \det(s_H) \\
&= \det(s_H) \\
&= \det(s_H)^{-(o(s_H)-1)},
\end{aligned}$$

and $a_H(\chi \cdot \det) = o(s_H) - 1$. Thus,

$$\begin{aligned}
Q_{\chi\cdot\det} \quad &= \quad \prod_{H\in\mathcal{A}} \alpha_H^{a_H(\chi\cdot\det)} \\
&= \quad \prod_{\substack{H\in\mathcal{A} \\ \chi(s_H)\neq 1}} \alpha_H^{a_H(\chi)-1} \prod_{\substack{H\in\mathcal{A} \\ \chi(s_H)=1}} \alpha_H^{o(s_H)-1}.
\end{aligned}$$

Proposition 2. *If* $\omega_1, \dots, \omega_n$ *generate* $(\Omega^1)^\chi$ *over* R*, then*

$$\omega_1 \curlywedge \dots \curlywedge \omega_n \doteq Q_{\chi\cdot\det} \, vol.$$

Proof. Let M be the coefficient matrix of $\omega_1, \dots, \omega_n$, i.e. $\omega_1 \wedge \ldots \wedge \omega_n = \det M\, vol$. Suppose that $\det M = 0$. Then one row of M is a linear combination of the other rows over $F = \mathbb{C}(x_1, \dots, x_n)$. Multiplying by a least common multiple yields a relation over S: $\sum_{i=1}^n s_i\, \omega_i = 0$. To get a relation over R, apply a group element g, multiply by $\chi^{-1}(g)$, and then sum over G:

$$\begin{aligned} 0 &= \sum_{g\in G} \sum_{i=1}^n \chi^{-1}(g)\, g s_i\, g\omega_i \\ &= \sum_{i=1}^n \sum_{g\in G} \chi^{-1}(g)\, g s_i\, \chi(g)\, \omega_i \\ &= \sum_{i=1}^n \left(\sum_{g\in G} g s_i \right) \omega_i . \end{aligned}$$

This contradicts the fact that $(\Omega^1)^\chi$ is free over R with basis $\omega_1, \dots, \omega_n$. Hence, $\det M \neq 0$.

By Corollary 1, $\omega_1 \curlywedge \cdots \curlywedge \omega_n$ is a χ-invariant n-form. Thus (from Equation $(*)$) there exists a nonzero $f \in R$ with

$$(Q_\chi)^{1-n} \det M\, vol = (Q_\chi)^{1-n}\, \omega_1 \wedge \ldots \wedge \omega_n = \omega_1 \curlywedge \cdots \curlywedge \omega_n = f\, Q_{\chi\cdot\det}\, vol.$$

Hence, $\det M = f\, Q_{\chi\cdot\det}\, (Q_\chi)^{n-1}$.

We show that f is constant by finding two polynomials that share no factors, yet are each divisible by f. Since each df_i is invariant, each $Q_\chi\, df_i$ is χ-invariant and hence a combination of $\omega_1, \dots, \omega_n$ over R. There exists a matrix of coefficients, N, with entries in S, such that

$$\begin{aligned} Q_\chi df_1 \wedge \cdots \wedge Q_\chi df_n &= \det M\, \det N\, vol \\ &= f\, Q_{\chi\cdot\det}\, (Q_\chi)^{n-1} \det N\, vol. \end{aligned}$$

But, $df_1 \wedge \cdots \wedge df_n \doteq Q_{\det}$, so

$$Q_\chi df_1 \wedge \cdots \wedge Q_\chi df_n \doteq (Q_\chi)^n\, Q_{\det}\, vol.$$

Hence,

$$f\, Q_{\chi\cdot\det}\, \det(N) \doteq Q_\chi\, Q_{\det}$$

and since $\det N \in S$, f divides $Q_\chi\, Q_{\det}\, (Q_{\chi\cdot\det})^{-1}$.

Since each μ_i (introduced above) is $(\det^{-1})$-invariant, each $Q_{\chi\cdot\det}\, \mu_i$ is χ-invariant, and thus a R-combination of $\omega_1, \dots, \omega_n$. There exists a matrix of coefficients, N', with coefficients in S, such that

$$\begin{aligned} Q_{\chi\cdot\det}\mu_1 \wedge \cdots \wedge Q_{\chi\cdot\det}\mu_n &= \det M\, \det N'\, vol \\ &= f\, Q_{\chi\cdot\det}\, (Q_\chi)^{n-1} \det N'\, vol. \end{aligned}$$

But we choose the μ_i so that

$$Q_{\chi\cdot\det}\mu_1 \wedge \cdots \wedge Q_{\chi\cdot\det}\mu_n = (Q_{\chi\cdot\det})^n\, (Q_{\det^{-1}})^{n-1}\, vol.$$

Hence,

$$f\, Q_{\chi\cdot\det}\ (Q_\chi)^{n-1}\ \det N' = (Q_{\chi\cdot\det})^n\, (Q_{\det^{-1}})^{n-1}$$

and since $\det N' \in S$, $(Q_{\chi\cdot\det} Q_{\det^{-1}})^{n-1}\ (Q_\chi)^{1-n}$ is divisible by f.

We show that the two polynomials

$$Q_\chi\, Q_{\det}\, (Q_{\chi\cdot\det})^{-1} \text{ and } (Q_{\chi\cdot\det} Q_{\det^{-1}})^{n-1}\ (Q_\chi)^{1-n}$$

have no common factors by writing them both in terms of the α_H. We expand the factors:

$$\begin{aligned}
Q_\chi &= \prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)\neq 1}} \alpha_H^{a_H(\chi)}, \\
Q_{\det} &= \prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)\neq 1}} \alpha_H^{o(s_H)-1} \prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)=1}} \alpha_H^{o(s_H)-1}, \\
Q_{\chi\cdot\det} &= \prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)\neq 1}} \alpha_H^{a_H(\chi)-1} \prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)=1}} \alpha_H^{o(s_H)-1}, \\
Q_{\det^{-1}} &= \prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)\neq 1}} \alpha_H \prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)=1}} \alpha_H\,.
\end{aligned}$$

The first polynomial, $Q_\chi\, Q_{\det}\, (Q_{\chi\cdot\det})^{-1}$, simplifies to

$$\prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)\neq 1}} \alpha_H^{o(s_H)},$$

and the second polynomial, $(Q_{\chi\cdot\det}\, Q_{\det^{-1}})^{n-1}\ (Q_\chi)^{1-n}$, simplifies to

$$\left(\prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)=1}} \alpha_H^{o(s_H)}\right)^{n-1}.$$

Since f divides both polynomials, f must be constant. Thus, $\omega_1, \ldots, \omega_n$ satisfy the criterion of Proposition 1.

$\square$

Corollary 2. *There exist n 1-forms $\omega_1, \ldots, \omega_n$ such that Ω^χ is generated over R by the forms $\omega_{I_1} \curlywedge \ldots \curlywedge \omega_{I_p}$, $I \in \mathcal{I}^p$, $p \geq 0$. Thus Ω^χ has the structure of an exterior algebra.*

5. EXAMPLE: G_{26}

For an example, let us take a three dimensional complex reflection group, G_{26}. This group is the symmetry group of a regular complex polyhedron, and is number 26 in Shephard and Todd's enumeration of finite irreducible unitary groups generated by reflections [4]. The group G_{26} consists of 1,296 complex 3×3 matrices and is generated by reflections of order two and three. The associated collineation group (which results from moding out by the scaler matrices) is the Hessian group of order 216.

The group is generated by the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix}, \quad \text{and} \quad \frac{i}{\sqrt{3}} \begin{pmatrix} \alpha & \alpha^2 & \alpha^2 \\ \alpha^2 & \alpha & \alpha^2 \\ \alpha^2 & \alpha^2 & \alpha \end{pmatrix},$$

where α is a primitive cube root of unity.

The character table for this group reveals six multiplicative characters, each a power of the determinate character. Choose $\chi = \det^3$. Note that

$$Q_{\det^3} = (x^3 - y^3)(x^3 - z^3)(y^3 - z^3),$$

and

$$Q_{\det^4} = x^2y^2z^2(x^9 + 3x^6(y^3 + z^3) + (y^3 + z^3)^3 + 3x^3(y^6 - 7y^3z^3 + xz^6))^2.$$

The following 1-forms are $\det^3$-invariant:

$$\begin{aligned} \omega_1 \quad = \quad & x^2(y - z)(y^2 + yz + z^2)(\ \ 2x^3 - y^3 - z^3)\,dx \\ & -y^2(x - z)(x^2 + xz + z^2)(-x^3 + 2y^3 - z^3)\,dy \\ & -z^2(x - y)(x^2 + xy + y^2)(\ \ x^3 + y^3 - 2z^3)\,dz, \end{aligned}$$

$$\begin{aligned} \omega_2 \quad = \quad & x^2(x^3 - y^3)(x^3 - z^3)(y^3 - z^3)(\ \ x^3 - 5y^3 - 5z^3)\,dx \\ & y^2(x^3 - y^3)(x^3 - z^3)(y^3 - z^3)(-5x^3 + y^3 - 5z^3)\,dy \\ & z^2(x^3 - y^3)(x^3 - z^3)(y^3 - z^3)(-5x^3 - 5y^3 + z^3)\,dz, \end{aligned}$$

$$\begin{aligned} \omega_3 \quad = \quad & x^2(x^3 - y^3)(x^3 - z^3)(y^3 - z^3)(x^9 + 3y^9 + 61y^6z^3 + 61y^3z^6 + 3z^9 \\ & \qquad +9x^6(y^3 + z^3) + x^3(-13y^6 + 122y^3z^3 - 13z^6))\,dx + \\ & y^2(x^3 - y^3)(x^3 - z^3)(y^3 - z^3)(3x^9 + y^9 + 9y^6z^3 - 13y^3z^6 + 3z^9 \\ & \qquad +x^6(-13y^3 + 61z^3) + x^3(9y^6 + 122y^3z^3 + 61z^6))\,dy + \\ & z^2(x^3 - y^3)(x^3 - z^3)(y^3 - z^3)(3x^9 + 3y^9 - 13y^6z^3 + 9y^3z^6 + z^9 \\ & \qquad +x^6(61y^3 - 13z^3) + x^3(61y^6 + 122y^3z^3 + 9z^6))\,dz. \end{aligned}$$

The polynomial $Q_{\det^3}$ divides $\omega_1 \wedge \omega_2$, $\omega_2 \wedge \omega_3$, and $\omega_1 \wedge \omega_3$. The determinate of the coefficient matrix of ω_1, ω_2, and ω_3 is $(-16)Q_{\det^4}\,Q^2_{\det^3}$, hence ω_1, ω_2, and ω_3 χ-wedge to a multiple of $Q_{\chi \cdot \det} = Q_{\det^4}$. Proposition 1 then implies that ω_1, ω_2, and ω_3 generate the entire module of $\det^3$-invariants over the ring of invariants via $\det^3$-wedging.

6. Logarithmic forms

We have so far only discussed regular differential forms; we now consider rational differential forms. The S-module of *logarithmic p-forms with poles along $\mathcal{A}$* (see also [3], p. 124) is defined as

$$\Omega^p(\mathcal{A}) := \{\frac{\omega}{Q_{\det^{-1}}} : \omega \in \Omega^p \text{ and } \omega \wedge d\alpha_H \in \alpha_H\, \Omega^{p+1} \text{ for all } H \in \mathcal{A}\}.$$

Ziegler [10] extends this definition to *multiarrangements of hyperplanes*, hyperplane arrangements in which each hyperplane has a positive integer multiplicity. We apply his definitions to our context of reflection groups and semiinvariants: Let $\mathcal{A}_\chi$ be the multiarrangement consisting of hyperplanes $H \in \mathcal{A}$ each with multiplicity $\alpha_H(\chi)$, i.e. the multiarrangement defined by Q_χ. We define (as in [10]) the module of *logarithmic p-forms* of $\mathcal{A}_\chi$:

$$\Omega^p(\mathcal{A}_\chi) := \{\frac{\omega}{Q_\chi} : \omega \in \Omega^p \text{ and } \omega \wedge d\alpha_H \in \alpha_H^{a_H(\chi)}\, \Omega^{p+1} \text{ for all } H \in \mathcal{A}\}.$$

Let

$$\Omega(\mathcal{A}_\chi) := \bigoplus_{p\geq 0} \Omega^p(\mathcal{A}_\chi).$$

Corollary 3.

$$\Omega^\chi \subset Q_\chi\, \Omega(\mathcal{A}_\chi).$$

Proof. Choose ω in $(\Omega^p)^\chi$ and fix $H \in \mathcal{A}$. Using Lemma 1, choose coordinates in which $x_1 = \alpha_H$, $\omega = \sum_{I\in\mathcal{I}^p} \omega_I\, dx_{I_1} \wedge \ldots \wedge dx_{I_p}$, and $x_1^{a_H(\chi)}$ divides ω_I if $1 \notin I$. Then $d\alpha_H = dx_1$, and $\omega \wedge d\alpha_H = \omega \wedge dx_1 = \sum_{I,1\notin I} \omega_I \wedge dx_1$, which is divisible by $x_1^{a_H(\chi)}$. Hence, $\omega \wedge d\alpha_H \in \alpha_H^{a_H(\chi)}\, \Omega^{p+1}$. As H was arbitrary, $\frac{\omega}{Q_\chi} \in \Omega(\mathcal{A}_\chi)$.

□

This relationship is stronger when $\chi = \det^{-1}$. In this case, the forms that generate Ω^χ via χ-wedging over R also generate $\Omega(\mathcal{A})$ over S (see [5] for an independent proof).

On a similar note, we have

Proposition 3. *$\Omega(\mathcal{A}_\chi)$ is closed under the exterior product.*

Proof. Let ω/Q_χ and μ/Q_χ be in $\Omega(\mathcal{A}_\chi)$. Fix H in $\mathcal{A}$ and let $a_H(\chi) = a$. Choose coordinates such that $x_1 = \alpha_H$, and write $\omega = \sum_{I\in\mathcal{I}^p} \omega_I\, dx_{I_1} \wedge \ldots \wedge dx_{I_p}$ and $\mu = \sum_{J\in\mathcal{I}^q} \mu_J\, dx_{J_1} \wedge \ldots \wedge dx_{J_q}$ in these coordinates. Since $\omega \wedge dx_1 = \omega \wedge d\alpha_H \in \alpha_H^a \Omega = x_1^a \Omega$, ω_I is divisible by x_1^a as long as $1 \notin I$. Similarly, μ_J is divisible by x_1^a whenever $1 \notin J$. As in the proof of Lemma 2, it follows that Q_χ divides $\omega \wedge \mu$. Whenever $1 \notin I$ *and* $1 \notin J$, x_1^{2a} divides $\omega_I \mu_J$, and thus

$$\frac{\omega \wedge \mu}{Q_\chi} \wedge dx_1$$

is also divisible by x_1^a. Hence α_H^a divides $(1/Q_\chi)\,\omega\wedge\mu\wedge d\alpha_H$, and as H was arbitrary, $(\omega/Q_\chi)\wedge(\mu/Q_\chi)$ is in $\Omega(\mathcal{A}_\chi)$.

□

7. Remarks

Analogous results hold for vector fields, or *derivations*. Let Υ^χ be the module of χ-invariants in the exterior algebra of derivations. Because the group action differs here, Lemma 1 is slightly different, with $a+1$ taking the place of $a-1$ when $I_1 = 1$. The case where $I_1 \neq 1$ is the same as in the original lemma, and hence Q_χ also divides the exterior product of two elements in Υ^χ (the proof is analogous to the case of Ω^χ). The criterion for n derivations to generate Υ^χ via χ-wedging is also slightly different: they must χ-wedge to $Q_{\chi\cdot\det^{-1}}\,\frac{\partial}{\partial x_1}\wedge\ldots\wedge\frac{\partial}{\partial x_n}$ instead of $Q_{\chi\cdot\det}\,dx_1\wedge\ldots\wedge dx_n$. This follows from the fact that $dx_1\wedge\ldots\wedge dx_n$ is $(\det^{-1})$-invariant while $\frac{\partial}{\partial x_1}\wedge\ldots\wedge\frac{\partial}{\partial x_n}$ is det-invariant. Finally, we note that the correspondence between differential p-forms (in Ω^p) and $(n-p)$-forms in Υ (the exterior algebra of derivations) induces a module isomorphism between Ω^χ and $\Upsilon^{\chi\cdot\det}$.

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