INVARIANT DERIVATIONS AND DIFFERENTIAL FORMS FOR REFLECTION GROUPS

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To Peter Orlik and Hiroaki Terao on their 80th and 65th birthdays, respectively.

Abstract. Classical invariant theory of a complex reflection group W highlights three beautiful structures:

- the W-invariant polynomials constitute a polynomial algebra, over which
- the W-invariant differential forms with polynomial coefficients constitute an exterior algebra, and
- the relative invariants of any W-representation constitute a free module.

When W is a *duality* (or *well-generated*) group, we give an explicit description of the isotypic component within the differential forms of the irreducible reflection representation. This resolves a conjecture of Armstrong, Rhoades and the first author, and relates to Lie-theoretic conjectures and results of Bazlov, Broer, Joseph, Reeder, and Stembridge, and also Deconcini, Papi, and Procesi. We establish this result by examining the space of W-invariant differential derivations; these are derivations whose coefficients are not just polynomials, but differential forms with polynomial coefficients.

For every complex reflection group W, we show that the space of invariant differential derivations is finitely generated as a module over the invariant differential forms by the basic derivations together with their exterior derivatives. When W is a duality group, we show that the space of invariant differential derivations is free as a module over the exterior subalgebra of W-invariant forms generated by all but the top-degree exterior generator. (The basic invariant of highest degree is omitted.)

Our arguments for duality groups are case-free, i.e., they do not rely on any reflection group classification.

1. Introduction

A celebrated result of Solomon [29] exhibits the set of differential forms invariant under the action of a complex reflection group as an *exterior algebra*. A similar result holds when we consider derivations instead of differential forms, i.e., elements of $S(V^*) \otimes V$ instead of $S(V^*) \otimes \wedge V^*$, for a reflection representation V with symmetric algebra $S(V^*)$. The polynomial degrees of generators of these sets of invariants are positioned into various combinatorial identities expressing the geometry, topology, and representation theory of reflection groups. Recently, a theory of Catalan combinatorics for reflection groups (e.g., see [1]) has prompted questions about the structure of invariant forms for other representations of a reflection group. Of particular interest are the *differential derivations*, elements of $S(V^*) \otimes \wedge V^* \otimes V$. The first author together with Armstrong and Rhoades conjectured a formula [1, Conj. 11.5′] for the Poincaré polynomial of the invariant differential derivations when the reflection group W is real, i.e., a finite Coxeter group. We verify this conjecture and show that the set of

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invariant differential derivations,

$$(S(V^*) \otimes \wedge V^* \otimes V)^W ,$$

is a free module over an exterior algebra constructed from exterior derivatives df_i of all but one of the basic invariants f_i; the last basic invariant of highest polynomial degree is omitted. In fact, we give the explicit structure of the invariant differential derivations for all complex reflection groups W that are duality groups. We also give a basis for nonduality groups. We explain these two main results next.

Invariant theory of reflection groups. Recall that a *reflection* on a finite dimensional vector space $V = \mathbb{C}^\ell$ is a nonidentity general linear transformation that fixes a hyperplane in V pointwise. A *complex reflection group* W is a subgroup of $\mathrm{GL}(V)$ generated by *reflections*; we assume all reflection groups are finite. We fix a $\mathbb{C}$-basis $x_1, \ldots, x_\ell$ of V^* with dual $\mathbb{C}$-basis $y_1, \ldots, y_\ell$ of V and identify the symmetric algebra $S := \mathrm{Sym}(V^*)$ with the polynomial ring $\mathbb{C}[x_1, \ldots, x_\ell]$, which carries a W-action by linear substitutions. Shephard and Todd [26] and Chevalley [8] showed that the W-invariant subalgebra S^W is again *polynomial*:

$$S^W = \mathbb{C}[f_1, \ldots, f_\ell]$$

for certain algebraically independent polynomials $f_1, \ldots, f_\ell$ called *basic invariants*. One can choose $f_1, \ldots, f_\ell$ homogeneous; we assume $\deg(f_1) \le \cdots \le \deg(f_\ell)$ after re-indexing. It follows that for any W-representation U, the space of *relative invariants* $(S \otimes U)^W$ forms a free S^W-module of rank $\dim_{\mathbb{C}}(U)$ (see, e.g., Hochster and Eagon [15, Prop. 16]). Note that we take all tensor products and exterior algebras over $\mathbb{C}$ unless otherwise indicated. We also assume all representations are complex and finite dimensional.

Differential forms and derivations. Two particular cases have received much attention. First, when $U = \wedge V^*$, one may identify $S \otimes U = S \otimes \wedge V^*$ with the S-module of *differential forms with polynomial coefficients* generated by $dx_1, \ldots, dx_\ell$. Solomon's theorem [26, 29] asserts that $(S \otimes \wedge V^*)^W$ is not just a free S^W-module, but, in fact, an exterior algebra over S^W on exterior generators $df_1, \ldots, df_\ell$, where $df_j := \sum_{i=1}^{\ell} \frac{\partial f_j}{\partial x_i} \otimes x_i$ has degree $e_i := \deg(f_i) - 1$:

$$(S \otimes \wedge V^*)^W = \bigwedge_{S^W} \{df_1, \ldots, df_\ell\} .$$

Second, when $U = V$, one may identify $S \otimes U = S \otimes V$ with the set of *derivations* $S \to S$ on V generated by the partial derivatives $\partial/\partial x_1, \ldots, \partial/\partial x_n$. Here, a derivation $\theta = \sum_{i=1}^{\ell} \theta^{(i)} \otimes y_i$ maps f in S to $\theta(f) = \sum_{i=1}^{\ell} \theta^{(i)} \frac{\partial}{\partial x_i}(f)$. One may choose a homogeneous basis $\theta_1, \ldots, \theta_\ell$ for the free S^W-module $(S \otimes V)^W$, i.e.,

$$(S \otimes V)^W = S^W \theta_1 \oplus \ldots \oplus S^W \theta_\ell \tag{1.1}$$

for some *basic derivations* $\theta_j = \sum_{i=1}^{\ell} \theta_i^{(j)} \otimes y_i$ with each $\theta_i^{(j)}$ in S homogeneous of fixed degree, say e_j^*.

Duality groups. Our main results combine these contexts, with special results for duality groups. An irreducible complex reflection group W is a *duality group* if its *coexponents* $e_1^* \ge \cdots \ge e_\ell^*$ and *exponents* $e_1 \le \cdots \le e_\ell$ determine each other via the relation $e_i + e_i^* = h$, where $h := \deg(f_\ell) = e_\ell + 1$ is the largest degree of a basic invariant, called the *Coxeter number* for the duality group W.

Duality groups include all irreducible *real* reflection groups (i.e., finite Coxeter groups) as well as symmetry groups of *regular complex polytopes* [9]. It was observed by Orlik and Solomon [21] in a case-by-case fashion (using the classification [26]) that an irreducible complex reflection group W is a duality group if and only if it is *well-generated*, that is, generated by $\ell = \dim(V)$ reflections, but we will not need this fact in the sequel.

Main theorems. We consider the set M of mixed forms, called *differential derivations*,

$$M := S \otimes \wedge V^* \otimes V.$$

We view M as an $(S \otimes \wedge V^*)$-module via multiplication in the first two tensor positions. Its W-invariant subspace M^W is then a $(S \otimes \wedge V^*)^W$-module. In general, M^W will not be a *free* module. Nevertheless, our first main result asserts that for duality groups, M^W *is* a free R-module, where R is the subalgebra of the invariant forms generated by all df_i but the last,

$$R := \bigwedge_{S^W} \{df_1, \ldots, df_{\ell-1}\},$$

with only df_ℓ omitted. To give an R-basis, we extend the usual exterior derivative operator d on S to a function on $S \otimes \wedge V^* \otimes V$ defining $d(f \otimes \omega \otimes y) = \sum_{1 \le i \le \ell} \frac{\partial f}{\partial x_i} \otimes (x_i \wedge \omega) \otimes y$. We identify $S \otimes V$ with the subspace $S \otimes 1 \otimes V$ of differential derivations,

$$\begin{array}{ccc} S \otimes V & \hookrightarrow & S \otimes \wedge V^* \otimes V \\ f \otimes y & \longmapsto & f \otimes \; 1 \; \otimes y, \end{array}$$

and apply d to derivations using this inclusion. Our first main result is shown with case-free arguments (it does not depend on any classification).

Theorem 1.1. *For W a duality (well-generated) complex reflection group, $(S \otimes \wedge V^* \otimes V)^W$ forms a free R-module on the 2ℓ basis elements $\{\theta_1, \ldots, \theta_\ell,\ d\theta_1, \ldots, d\theta_\ell\}$.*

For arbitrary complex reflection groups, we have no uniform statement like Theorem 1.1. However, we give an explicit free S^W-basis for $(S \otimes \wedge V^* \otimes V)^W$ in the remaining (non-duality) cases, showing this:

Theorem 1.2. *For W any complex reflection group, $(S \otimes \wedge V^* \otimes V)^W$ is generated as a module over the exterior algebra $(S \otimes \wedge V^*)^W = \bigwedge_{S^W} \{df_1, \ldots, df_\ell\}$ by the 2ℓ generators $\{\theta_1, \ldots, \theta_\ell,\ d\theta_1, \ldots, d\theta_\ell\}$.*

To be clear: $(S \otimes \wedge V^* \otimes V)^W$ is not *freely* generated as a module over $\bigwedge_{S^W} \{df_1, \ldots, df_\ell\}$ by $\{\theta_i, d\theta_i\}_{i=1}^{\ell}$.

Example 1.3. A rank $\ell = 1$ reflection group $W \subset \mathrm{GL}(V) = \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ is a *cyclic group* $W = \langle \zeta \rangle \cong \mathbb{Z}/h\mathbb{Z}$ for some $\zeta = e^{\frac{2\pi i}{h}}$ in $\mathbb{C}$. Let $V = \mathbb{C}x$ and $V^* = \mathbb{C}y$ with y dual to x. Under the generator of W, $x \mapsto \zeta^{-1} x$ and $y \mapsto \zeta y$. Then

$$S = \mathbb{C}[x], \quad S^W = \mathbb{C}[f_1] \quad \text{ where } f_1 = f_\ell = x^h \text{ has degree } h.$$

The S^W-module of invariant forms, $(S \otimes \wedge V^*)^W$, is an exterior algebra over S^W generated by $df_1 = hx^{h-1} \otimes x$ of degree $e_1 = h - 1$, that is,

$$\begin{array}{rcccccc} (S \otimes \wedge V^*)^W & = & (S \otimes \wedge^0 V^*)^W & \oplus & (S \otimes \wedge^1 V^*)^W & & \\ & = & \underbrace{S^W(1 \otimes 1)}_{:=R} & \oplus & S^W(x^{h-1} \otimes x) & = & S^W \oplus S^W df_1. \end{array}$$

On the other hand, the S^W-module of invariant derivations is

$$(S \otimes V)^W = S^W(x \otimes y) = S^W \theta_1, \qquad \text{for } \theta_1 = x \otimes y \qquad \text{of degree } e_1^* = 1.$$

In particular, W is a duality group, since $e_1^* + e_1 = 1 + (h-1) = \deg(f_\ell)$. Now consider the invariant differential derivations: An easy check confirms that $M^W = (S \otimes \wedge V^* \otimes V)^W$ is a free module over $R = S^W$ with basis $\{\theta_1, d\theta_1\}$. Here, we have identified $\theta_1 = x \otimes y$ with $x \otimes 1 \otimes y$ and $d\theta_1 = 1 \otimes x \otimes y$. Thus

$$(S \otimes \wedge V^* \otimes V)^W = S^W(x \otimes 1 \otimes y) \oplus S^W(1 \otimes x \otimes y)\,.$$

Outline. Section 2 provides further context and implications of Theorem 1.1 while Section 3 gives its relation to some theorems and conjectures in Lie theory. We collect some tools for establishing helpful reflection group numerology (like Molien's Theorem and the Gutkin-Opdam Lemma) in Section 4 and reap that numerology in Section 5. Section 6 is a slight digression deriving basis conditions reminiscent of Saito's Criterion for free arrangements. A linear independence condition is given in Section 7. We complete the proof of Theorem 1.1 in Section 9 after showing that duality groups exhibit auspicious numerology in Section 8.

The remainder is about nonduality groups and Theorem 1.2. Section 10 addresses rank two reflection groups, while Section 11 considers the Shephard and Todd group G_{31} — the only irreducible non-duality group that is neither of rank two, nor within the Shephard and Todd infinite family of monomial groups $G(r,p,n)$. Section 12 assembles the proof of Theorem 1.2, relegating the case of $G(r,p,n)$ to Appendix 14. When $p=1$ or $p=r$, these are duality groups and covered by Theorem 1.1; when $1<p<r$, they are nonduality groups and we give an alternate basis for the invariant differential derivations in this appendix.

We consider some further questions in Section 13.

2. Implications of Theorem 1.1

To provide context for Theorem 1.1, we first note a few consequences and special cases.

The case of exterior degree zero. Theorem 1.1 implies something that we already knew about

$$(S \otimes \wedge^0 V^* \otimes V)^W = (S \otimes V)^W,$$

namely, that it is a free S^W-module on the basis $\{\theta_j\}_{j \in \{1,\ldots,\ell\}}$; this is true even when W is not a duality group. We will end up using this fact in the proof of the theorem.

The case of top exterior degree. At the opposite extreme, Theorem 1.1 asserts that

$$(S \otimes \wedge^\ell V^* \otimes V)^W \cong (S \otimes \det \otimes V)^W$$

is free as an S^W-module on the basis $\{df_1 \cdots df_{\ell-1}\, d\theta_k\}_{k \in \{1,\ldots,\ell\}}$. This agrees with the polynomial degrees of a basis found in [27] for any reflection group W, as we may view $(S \otimes \det \otimes V)^W$ as the space of invariant derivations for the "twisted reflection representation" $\det \otimes V$, where $\det : W \mapsto \mathbb{C}^*$ is the determinant character of W acting on V. Indeed, if $J = \det\left(\frac{\partial f_i}{\partial x_j}\right)$ is the Jacobian determinant of the basic invariants $f_1, \ldots, f_\ell$ (see Section 6), then the forms $df_1 \cdots df_{\ell-1}\, d\theta_k$ have degrees

$$e_1 + \ldots + e_{\ell-1} + e_k^* - 1 = (e_1 + \ldots + e_\ell) - (e_\ell + 1) + e_k^* = \deg J - (h - e_k^*) = \deg J - e_k$$

when W is a duality group (see [27, Cor. 13(e)]).

The Hilbert series consequence. Theorem 1.1 has implications for *Hilbert series* analogous to those given by the Shephard-Todd-Chevalley and Solomon theorems with $S = \oplus_{i\geq 0} S_i$ graded by polynomial degree. Just as these classical results immediately imply that

$$\begin{aligned}
\mathrm{Hilb}(S^W;q) &:= \sum_{i\geq 0} q^i \dim S_i^W &&= \prod_{i=1}^{\ell}(1-q^{e_i+1})^{-1},\\
\mathrm{Hilb}\big((S\otimes V)^W;q\big) &:= \sum_{i\geq 0} q^i \dim\,(S_i\otimes V)^W &&= \mathrm{Hilb}(S^W;q)\cdot\sum_{i=1}^{\ell} q^{e_i^*}\,, \qquad \text{and}\\
\mathrm{Hilb}\big((S\otimes \wedge V^*)^W;q,t\big) &:= \sum_{i,j\geq 0} q^i\, t^j \dim(S_i\otimes \wedge^j V^*)^W &&= \mathrm{Hilb}(S^W;q)\cdot\prod_{i=1}^{\ell}(1+q^{e_i}t)\,,
\end{aligned}$$

Theorem 1.1 analogously immediately implies that

$$\begin{aligned}
\mathrm{Hilb}\left((S\otimes\wedge V^*\otimes V)^W;q,t\right) &:= \sum_{i,j\geq 0} q^i\,t^j\,\dim\left(S_i\otimes\wedge^j V^*\otimes V\right)^W\\
&= \mathrm{Hilb}(S^W;q,t)\cdot\sum_{i=1}^{\ell}\,(q^{e_i^*}+q^{e_i^*-1}t)\\
&= \mathrm{Hilb}(S^W;q)\cdot(q+t)\cdot\left(\prod_{i=1}^{\ell-1}\,1+q^{e_i}t\right)\left(\sum_{i=1}^{\ell}\,q^{e_i^*-1}\right).
\end{aligned} \tag{2.1}$$

Our original motivation, in fact, was the special case of (2.1) for real reflection groups W, which appeared as [1, Conj. 11.5′], based on Coxeter-Catalan combinatorics and computer experimentation.

3. The Lie theory connection

We now explain how the case of Theorem 1.1 when W is a *Weyl group*, i.e, a finite crystallographic real reflection group, relates to Lie-theoretic results and work of Bazlov, Broer, Joseph, Reeder, and Stembridge, and also DeConcini, Papi, and Procesi. Let G be a simply-connected, compact simple Lie group with a choice of maximal torus T. Denote by $\mathfrak{g}$ and V the complexification of their corresponding Lie algebras, and let $W := N_G(T)/T$ be the associated Weyl group acting on a real vector space V. Then G acts on $\wedge\mathfrak{g}^*$, while W acts on $S := S(V^*)$ and on its *coinvariant algebra*

$$S/S_+^W \cong H^*(G/T)$$

where the last isomorphism to cohomology, due to Borel, is grade-doubling, and where S_+^W is the ideal generated by invariant polynomials of positive degree. Classical results (see [23]) give isomorphisms

$$(\wedge\mathfrak{g}^*)^G \;\cong\; H^*(G) \;\cong\; (H^*(G/T)\otimes H^*(T))^W \;\cong\; \left(S/S_+^W\otimes\wedge V^*\right)^W \tag{3.1}$$

exhibiting both of these rings as (isomorphic) exterior algebras, with exterior generators $P_1, P_2, \ldots, P_\ell$ where P_i lies in $(\wedge^{2e_i+1}\mathfrak{g}^*)^G$. The isomorphism is again homogeneous after doubling the grading in S/S_+^W.

Reeder [24] conjectured a similar relation between G-invariants and W-invariants, relating two Hilbert series associated with a finite-dimensional G-representation M:

$$P_G(M;t) := \sum_{j\geq 0} t^j \dim(\wedge^j \mathfrak{g}^* \otimes M)^G ,$$
$$P_W(M^T;q,t) := \sum_{i,j\geq 0} q^i\, t^j \dim\left((S/S^W_+)_i \otimes \wedge^j V^* \otimes M^T\right)^W .$$

Conjecture 3.1. [24, Conj. 7.1] *If M is* **small***, meaning its weight space $M_{2\alpha} = 0$ for all roots α, one has*

$$P_G(M;t) = P_W(M^T;q,t)|_{q=t^2}.$$

Various special cases of Conjecture 3.1 were known at the time that it was formulated. For example, when M is the trivial G-representation it follows from (3.1) above. Reeder [24, Cor. 4.2] proved the $t = 1$ specialization of Conjecture 3.1 and credited it also to Kostant: For M small,

$$\dim(\wedge\mathfrak{g}^*\otimes M)^G = P_G(M;t)|_{t=1} = P_W(M^T;q,t)|_{q=t=1} = \dim\left(S/S^W_+ \otimes \wedge V^* \otimes M^T\right)^W . \tag{3.2}$$

The type A special case was also known to follow from the "first-layer" formulas of Stembridge [33]. The type B special case was recently[1] verified in work of DeConcini and Papi, and Stembridge independently[2] verified Conjecture 3.1 case-by-case for all types.

A further bit of motivation comes from a generalization of Chevalley's restriction theorem due to Broer [5]. Chevalley's result asserts that restriction $\mathfrak{g}^* \to V^*$ induces an isomorphism of polynomial rings

$$S(\mathfrak{g}^*)^G \to S^W, \tag{3.3}$$

while Broer [5] showed more generally that, for any small G-module M, restriction also induces an isomorphism (of modules over the polynomial rings in (3.3))

$$(S(\mathfrak{g}^*) \otimes M)^G \to (S \otimes M^T)^W.$$

Broer's result suggested to the authors the following enhanced version of Conjecture 3.1.

Conjecture 3.2. *(Enhanced Reeder Conjecture) For a small G-representation M, there is an isomorphism*

$$(\wedge\mathfrak{g}^* \otimes M)^G \cong \left(S/S^W_+ \otimes \wedge V^* \otimes M^T\right)^W$$

of modules over the exterior algebra in (3.1) *which is degree-preserving after doubling the grading in S/S^W_+.*

While this paper was under review, DeConcini and Papi [10, p. 259] showed that *not all* small G-representations M satisfy Conjecture 3.2, and one asks, "For which small G-representations does the Enhanced Reeder Conjecture 3.2 hold?" DeConcini and Papi [10, Thm. 2.2 and Cor. 6.6] proved the conjecture holds for two particularly important cases of small G-representations, namely, the *adjoint representation*[3] $M = \mathfrak{g}$ and the *little adjoint representation*, which are the $\mathfrak{g}$-irreducibles whose highest weights are the highest root and

[1] DeConcini and Papi, personal communication, 2016.

[2] Stembridge, personal communication, 2016.

[3] It is exactly in the case of the adjoint representation that Bazlov [2] proved Conjecture 3.1, and he credits this special case of the conjecture to Joseph [17].

highest short root, respectively. The adjoint case $M = \mathfrak{g}$ connects our work to the following result of DeConcini, Papi, and Procesi.

Theorem 3.3. [11, Thm 1.1] *Regard $(\wedge\mathfrak{g}^* \otimes \mathfrak{g})^G$ as a module over the exterior subalgebra R of $(\wedge\mathfrak{g}^*)^G$ generated by $P_1, P_2, \ldots, P_{\ell-1}$, via multiplication in the first tensor factor. Then $(\wedge\mathfrak{g}^* \otimes \mathfrak{g})^G$ is free as an R-module, with basis elements $\{f_i, u_i\}_{i=1}^{\ell}$ of degrees* $\deg(f_i) = 2e_i, \deg(u_i) = 2e_i - 1$.

An alternate proof of Theorem 3.3 follows from our Theorem 1.1, using the adjoint special case of Conjecture 3.2, that is, [10, Thm. 2.2], after modding out by S_+^W and bearing in mind that $\{e_i^*\}_{i=1}^{\ell} = \{e_i\}_{i=1}^{\ell}$ for Weyl groups W.

4. Degree sums and the Gutkin-Opdam Lemma

Before determining bases for the invariant differential derivations, we recall some tools for investigating the relevant numerology, most notably a useful lemma for finding the sum of degrees in a basis.

Degree sum. Let k be a field, and let A be a graded k-algebra and integral domain. Consider a free graded A-module $M \cong A^p$ of finite rank, say with homogeneous basis $m_1, \ldots, m_p$. The (unordered) list of degrees $\deg(m_1), \ldots, \deg(m_p)$ are uniquely determined by the quotient of Hilbert series

$$\sum_{i=1}^{p} q^{\deg(m_i)} = \mathrm{Hilb}(M, q)/\mathrm{Hilb}(A, q).$$

Thus we may assign to any such M the *degree sum*

$$\Delta_A(M) := \sum_{i=1}^{p} \deg(m_i) = \left[\frac{\partial}{\partial q} \frac{\mathrm{Hilb}(M, q)}{\mathrm{Hilb}(A, q)}\right]_{q=1}.$$

If one knows this degree sum *a priori*, then one may determine an explicit A-basis for M by just checking independence over the fraction field:

Lemma 4.1. *Let A be a graded k-algebra and integral domain and $M \cong A^p$ a free graded A-module. A set of homogeneous elements $\{n_1, \ldots, n_p\}$ in M with $\sum_{i=1}^{p} \deg(n_i) = \Delta_A(M)$ is an A-basis for M if and only if it is linearly independent over the fraction field $K = \mathrm{Frac}(A)$.*

Proof. The forward implication is clear. For the reverse implication, note that linear independence is equivalent to nonsingularity of the matrix B in $A^{p\times p}$ with $\mathbf{n} = B\mathbf{m}$, for $\mathbf{n} = [n_1, \ldots, n_p]^T$ and $\mathbf{m} = [m_1, \ldots, m_p]^T$. Since $\det(B) \neq 0$, its expansion contains a nonzero term indexed by a permutation π with $b_{i,\pi(i)} \neq 0$ for each $i = 1, 2, ..., p$; after re-indexing, one may assume π is the identity permutation. Hence $\deg(n_i) = \deg(b_{i,i}) + \deg(m_i)$ for $i = 1, \ldots, p$. Since $\sum_i \deg(n_i) = \Delta_A(M) = \sum_i \deg(m_i)$, one has that $\deg(n_i) = \deg(m_i)$ for each i. Thus after re-ordering the rows and columns of B in increasing order of degree, B will be block upper triangular, with each diagonal block an invertible matrix with entries in k. Therefore B gives an A-module automorphism of M sending the A-basis $\mathbf{m}$ to $\mathbf{n}$. $\square$

Modules over the Invariant Ring. As mentioned in the Introduction, a result of Hochster and Eagon implies that for any representation U of a complex reflection group W, the set of all *relative invariants* $M = (S \otimes U)^W$ is a free module of finite rank $p = \dim_{\mathbb{C}} U$ over the graded k-algebra $A = S^W$. We introduce an abbreviation for the above degree sum:

Definition 4.2. Let W be a complex reflection group. For any W-representation U, set
$$\Delta(U) := \Delta_{S^W}\left((S\otimes U)^W\right) = \sum_{1\le i\le p} \deg\psi_i$$
for any S^W-basis $\{\psi_i\}_{i=1}^p$ of $(S\otimes U)^W$.

Local Data. We next review an *a priori* calculation for the degree sum $\Delta(U)$, Lemma 4.3 below, due originally to Gutkin [14], and later rediscovered by Opdam [20, Lemma 2.1]; see also Broué [6, Prop. 4.3.3 and eqn. (4.6)].

The formula for $\Delta(U)$ is expressed in what is sometimes called the *local data* for U at each reflecting hyperplane H of W. The pointwise stabilizer subgroup W_H in W of H is cyclic, say of order e_H; note that e_H is the maximal order of a reflection in W fixing H pointwise. The W_H-irreducible representations are the powers $\{\det^j\}_{j=0}^{e_H-1}$ of the 1-dimensional (linear) character $\det := \det\downarrow_{W_H}^W$ restricted from W to W_H acting on V. It is convenient to introduce the *representation ring*
$$R(W_H) := \mathbb{Z}[v]/(v^{e_H}-1),$$
where v^j represents the class of the 1-dimensional representation $\det^j$, and define a $\mathbb{Z}$-linear functional
$$D_H : R(W_H) \to \mathbb{Z}, \quad v^j \mapsto j\,.$$
Then for any W-representation U, the functional D_H on the restricted representation $U\downarrow_{W_H}^W$ can be expressed in terms of the inner products $\mu_{H,j} := \langle U\downarrow_{W_H}^W, \det^j\rangle_{W_H}$ as
$$D_H\left(U\downarrow_{W_H}^W\right) = \sum_{j=0}^{e_H-1} j\cdot\mu_{H,j}\,.$$

Lemma 4.3. *(Gutkin-Opdam Lemma) Let U be a representation of a complex reflection group W. Then*
$$\Delta(U) = \sum_H D_H\left(U\downarrow_{W_H}^W\right),$$
where the sum runs over all reflecting hyperplanes H for W.

Lemma 4.3 can be deduced (see Broué [6, §4.5.2]) from the following standard variant of Molien's theorem. In its statement, Tr indicates trace.

Lemma 4.4. [6, Lem. 3.28] *For any W-representation U,*
$$\mathrm{Hilb}\left((S\otimes U)^W, q\right) = \tfrac{1}{|W|}\sum_{w\in W}\frac{\mathrm{Tr}_U(w^{-1})}{\det(1-qw)}\,. \tag{4.1}$$

For a complex reflection group W, taking U to be the trivial representation in Lemma 4.1 immediately implies the well-known fact that
$$\prod_{i=1}^{\ell}\frac{1}{1-q^{e_i+1}} = \mathrm{Hilb}(S^W,q) = \tfrac{1}{|W|}\sum_{w\in W}\frac{1}{\det(1-qw)}$$
as $S^W = \mathbb{C}[f_1,\ldots,f_\ell]$ with $\deg(f_i) = e_i+1$. As noted by Shephard and Todd [26, §8], comparing the first two coefficients in the Laurent expansions about $q=1$ on the left and

right immediately gives these facts:

$$|W| = \prod_{i=1}^{\ell}(e_i+1), \tag{4.2}$$

$$N := \#\{\text{reflections in } W\} = \sum_{i=1}^{\ell} e_i\,. \tag{4.3}$$

5. Numerology from the Gutkin-Opdam lemma

Again, W is a complex reflection group. This section harvests numerology from Lemma 4.3.

Number of Reflecting Hyperplanes. We first see that Lemma 4.3 implies that the coexponents e_i^* sum to the number N^* of reflecting hyperplanes for W.

Example 5.1. Let $U = V$, the reflection representation. Each reflecting hyperplane H has $\mu_{H,j}(V) = 0$ for $j \neq 0, 1$, with $\mu_{H,0} = \ell - 1$ and $\mu_{H,1} = 1$, and hence $D_H(V \downarrow^W_{W_H}) = 1$. Thus in this case, Lemma 4.3 implies that the coexponents $e_i^* := \deg(\theta_i)$ for the S^W-basis $\{\theta_i\}_{i=1}^{\ell}$ of $(S \otimes V)^W$ satisfy the well-known formula (e.g., see [7, p. 130])

$$N^* := \#\{\text{reflecting hyperplanes in } W\} = \sum_H 1 = \Delta(V) = \sum_{i=1}^{\ell} e_i^*\,. \tag{5.1}$$

Graded representations. In order to apply Lemma 4.3 to *graded* W-representations, we consider the *graded* representation ring $R(W_H)[[t]]$, that is, the ring of graded virtual W_H-characters. We also extend D_H to a map $R(W_H)[[t]] \to \mathbb{Z}$ coefficientwise, defining $D_H\left(\sum_{k\geq 0} t^k\chi\right) := \sum_{k\geq 0} t^k D_H(\chi)$. The sum in the following corollary is over all reflecting hyperplanes H of W.

Corollary 5.2. *For any W-representation U,*

$$\sum_{m=0}^{\ell} \Delta(\wedge^m V^* \otimes U)\, t^m = (1+t)^{\ell-1} \sum_H D_H\left((1+v^{e_H-1}t) \sum_{j=0}^{e_H-1} \mu_{H,j}\, v^j \right).$$

Proof. Recall that each $U \downarrow^W_{W_H} = \sum_{j=0}^{e_H-1} \mu_{H,j}\, v^j$ in $R(W_H)$. The restriction $V^* \downarrow^W_{W_H}$ is a sum of $\ell - 1$ copies of the trivial representation and one copy of $\det^{e_H-1}$. Hence the restriction $\wedge V^* \downarrow^W_{W_H}$ will be represented by $(1+t)^{\ell-1}(1+v^{e_H-1}t)$, and $(\wedge V^* \otimes U) \downarrow^W_{W_H}$ will be represented by $(1+t)^{\ell-1}(1+v^{e_H-1}t) \sum_{j=0}^{e_H-1} \mu_{H,j}\, v^j$ in $R(W_H)[[t]]$. The result then follows from Lemma 4.3. □

Solomon's theorem. We illustrate in the next example how Corollary 5.2 gives Solomon's result [29] that the space of W-invariant differential forms is generated by $df_1, \ldots, df_\ell$ as an exterior algebra[4].

Example 5.3. We show that $(S \otimes \wedge^m V^*)^W$ has S^W-basis $\{df_I\}_{I \in \binom{[\ell]}{m}}$ where $\binom{[\ell]}{m}$ denotes the collection of all m-element subsets $I = \{i_1, \ldots, i_\ell\}$ of the set $[\ell] := \{1, 2, \ldots, \ell\}$, with $1 \leq i_1 < \ldots < i_m \leq \ell$, and where

$$df_I := df_{i_1} \wedge \cdots \wedge df_{i_m}.$$

[4] For an alternate geometric proof sketch of this result, see Berest, Etingof, and Ginzburg [3, Remark 1.17].

Apply Corollary 5.2 to the trivial representation U, obtaining

$$\sum_{m=0}^{\ell} \Delta(\wedge^m V^*) t^m = (1+t)^{\ell-1} \sum_H D_H(1+v^{e_H-1}t) = (1+t)^{\ell-1} \sum_H (e_H-1)\,t = Nt(1+t)^{\ell-1}$$

where the last equality uses Equation (4.3). Therefore $\Delta(\wedge^m V^*) = \binom{\ell-1}{m-1}N$. Note that the sum of degrees of the elements in the alleged basis $\{df_I\}_{I\in\binom{[\ell]}{m}}$ matches this:

$$\sum_{I\in\binom{[\ell]}{m}} \sum_{i\in I} e_i = \sum_{i=1}^{\ell} e_i \,\#\left\{I \in \tbinom{[\ell]}{m} : i \in I\right\} = N\tbinom{\ell-1}{m-1} = \Delta(\wedge^m V^*).$$

Hence by Lemma 4.1, it suffices only to check the linear independence of df_I over $K = \mathrm{Frac}(S)$. As observed by Solomon, the m-fold wedge products of the elements $df_1, \ldots, df_\ell$ form a K-basis for $K \otimes \wedge^m V^*$ if and only their top wedge is nonvanishing:

$$df_1 \wedge \cdots \wedge df_\ell = \det\left(\frac{\partial f_i}{\partial x_j}\right) \otimes x_1 \wedge \cdots \wedge x_\ell \neq 0.$$

But this follows immediately from the Jacobi Criterion [16]: The algebraic independence of $f_1, \ldots, f_\ell$ implies that the matrix of coefficients $(\frac{\partial f_i}{\partial x_j})$ of $df_1, \ldots, df_\ell$ is nonsingular.

Numerology of differential derivations. Consider the W-representation $U := \wedge^m V^* \otimes V$ of dimension

$$p := \ell\tbinom{\ell}{m} = \dim_{\mathbb{C}}(\wedge^m V^* \otimes V)\,.$$

We show that $\Delta(\wedge^m V^* \otimes V)$ depends only on N, N^*, and $\ell := \dim_{\mathbb{C}} V$.

Proposition 5.4. *For any complex reflection group W and $1 \le m \le \ell$,*

$$\Delta(\wedge^m V^* \otimes V) = (\ell-1)\tbinom{\ell-1}{m-1}N + \tbinom{\ell-1}{m}N^*\,.$$

Proof. We take $U = V$ in Corollary 5.2:

$$\begin{aligned}\sum_{m=0}^{\ell} \Delta(\wedge^m V^* \otimes V) t^m &= (1+t)^{\ell-1} \sum_H D_H\left((1+v^{e_H-1}t)(\ell-1+v)\right)\\ &= (1+t)^{\ell-1} \sum_H D_H\left(\ell-1+(\ell-1)tv^{e_H-1}+v+t\right)\\ &= (1+t)^{\ell-1} \sum_H \left((\ell-1)t(e_H-1)+1\right) = (1+t)^{\ell-1}\big((\ell-1)Nt+N^*\big),\end{aligned}$$

using Equations (4.3) and (5.1). The proposition now follows from the binomial theorem. ☐

Proposition 5.4 and Lemma 4.1 imply a corollary used repeatedly in the proofs of Theorems 1.1 and 1.2.

Corollary 5.5. *Any set of homogeneous elements $\{\psi_i\}_{i=1,2,\ldots,\ell\binom{\ell}{m}}$ in $(S\otimes\wedge^m V^*\otimes V)^W$ with*

$$\sum_i \deg(\psi_i) = \Delta(\wedge^m V^* \otimes V) = (\ell-1)\tbinom{\ell-1}{m-1}N + \tbinom{\ell-1}{m}N^*$$

forms an S^W-basis for $(S\otimes\wedge^m V^*\otimes V)^W$ if and only if they are linearly independent over $K = \mathrm{Frac}(S)$.

6. Digression: A Saito Criterion

This section, although not needed for the sequel, extends Corollary 5.5 to a condition similar to *Saito's criterion* for free hyperplane arrangements [22, §4.2] for any complex reflection group W. We first recall some facts about the coefficient matrices for the differential forms df_i and basic derivations θ_i.

Jacobian matrix and product of hyperplanes. Recall the defining polynomial Q and the Jacobian polynomial J for a complex reflection group W:

$$Q := \prod_H l_H, \qquad \text{and} \quad J := \prod_H l_H^{e_H-1}.$$

Here, the product is taken over all reflecting hyperplanes $H = \ker l_H$ in V for some choice of linear forms $l_H \in V^*$ with $e_H = |\mathrm{Stab}_W(H)|$. Note that Q and J are only well-defined up to nonzero scalars. Let $\mathrm{Jac}(f)$ and $M(\theta)$, respectively, be the matrices in $S^{\ell\times\ell}$ that express $\{df_i\}_{i\in[\ell]}$ and $\{\theta_i\}_{i\in[\ell]}$ in the S-bases $\{1\otimes x_i\}_{i\in[\ell]}$ and $\{1\otimes y_i\}_{i\in[\ell]}$ for $S\otimes V^*$ and $S\otimes V$, respectively. Steinberg [32] and Orlik and Solomon [21, §2], respectively, showed that

$$J = \det(\mathrm{Jac}(f)) \qquad \text{and} \qquad Q = \det(M(\theta)),$$

so that

$$\begin{aligned} N &= \sum_{i=1}^{\ell} e_i = \deg(J) = \#\{\text{reflections in } W\} \quad \text{and} \\ N^* &= \sum_{i=1}^{\ell} e_i^* = \deg(Q) = \#\{\text{reflecting hyperplanes for } W\}. \end{aligned} \tag{6.1}$$

Note that Terao [34] also showed that invariant derivations θ_i also give an S-basis for the *module of derivations* of the reflection hyperplane arrangement of W; see also [22, §4.1, §6.3].

When W is a duality group, observe that (by definition)

$$h := \deg(f_\ell) = \frac{1}{\ell}\sum_{i=1}^{\ell}(e_i+e_i^*) = \frac{N+N^*}{\ell}. \tag{6.2}$$

Matrix of Coefficients. We capture the coefficients of any set of invariant differential derivations in a matrix of coefficients. Consider the obvious free S-basis for $S\otimes\wedge^m V^*\otimes V$ given by

$$\{dx_I\otimes y_j : 1\le i,j\le \ell\} \qquad \text{with} \qquad dx_I := 1\otimes x_{i_1}\wedge\cdots\wedge x_{i_m} \tag{6.3}$$

for m-subsets $I=\{i_1<\cdots<i_m\}$ of $[\ell]$ and $[\ell]:=\{1,\ldots,\ell\}$. Given a collection $\mathcal{B}\subset(S\otimes\wedge^m V^*\otimes V)^W$, let $\mathrm{Coef}(\mathcal{B})$ denote its coefficient matrix in $S^{p\times p}$ with respect to the S-basis in (6.3).

Lemma 6.1. *For any $\mathcal{B}=\{\psi_i\}_{i=1}^p\subset(S\otimes\wedge^m V^*\otimes V)^W$, the product $J^{(\ell-1)\binom{\ell-1}{m-1}}Q^{\binom{\ell-1}{m}}$ divides $\det\mathrm{Coef}(\mathcal{B})$.*

Proof. Fix a reflecting hyperplane H in V for W and a reflection s in W of maximal order e_H fixing H. Choose coordinates $x_1,\ldots,x_\ell$ of V^* so that $l_H=x_1$ and s acts diagonally with nonidentity eigenvalue ξ:

$$s(y_i) = \begin{cases} \xi^{-1} y_1 & \text{if } i = 1, \\ y_i & \text{if } i \neq 1, \end{cases} \qquad \text{and} \qquad s(x_i) = \begin{cases} \xi x_1 & \text{if } i = 1, \\ x_i & \text{if } i \neq 1\,. \end{cases}$$

Each row of $\mathrm{Coef}(\mathcal{B})$ lists the coefficients $f_{I,j}$ of some invariant differential derivation $\psi_i = \sum f_{I,j}\, dx_I \otimes y_j$ in $(S \otimes \wedge^m V^* \otimes V)$, while each column of $\mathrm{Coef}(\mathcal{B})$ is indexed by a pair (I, j) for I an m-subset of $[\ell]$ and $j \in [\ell]$. Note two observations:

- For each pair (I, j) with $j \neq 1$ but $1 \in I$, the polynomial $x_1^{e_H - 1}$ divides $f_{I,j}$ since
$$s(dx_I \otimes y_j) = \xi(dx_I \otimes y_j) \quad \text{implies that} \quad s(f_{I,j}) = \xi^{-1} f_{I,j}\,;$$
thus $(\ell-1)\binom{\ell-1}{m-1}$ different columns of the matrix $\mathrm{Coef}(\mathcal{B})$ are divisible by $x_1^{e_H-1}$.
- For each pair (I, j) with $j = 1$ but $1 \in I$, the polynomial x_1 divides $f_{I,j}$, since
$$s(dx_I \otimes y_j) = \xi^{-1}(dx_I \otimes y_j) \quad \text{implies that} \quad s(f_{I,j}) = \xi f_{I,j}\,;$$
thus $\binom{\ell-1}{m}$ different columns of the matrix $\mathrm{Coef}(\mathcal{B})$ are divisible by x_1.

Hence $\ell_H = x_1$ when raised to the power $(e_H - 1)(\ell - 1)\binom{\ell-1}{m-1} + \binom{\ell-1}{m}$ divides $\det \mathrm{Coef}(\mathcal{B})$. This holds for each reflecting hyperplane H, and therefore unique factorization implies that $J^{(\ell-1)\binom{\ell-1}{m-1}} Q^{\binom{\ell-1}{m}}$ divides $\det \mathrm{Coef}(\mathcal{B})$. □

Saito-like Criterion. Again, let $K = \mathbb{C}(x_1, \ldots, x_\ell)$ be the fraction field of $S = \mathbb{C}[x_1, \ldots, x_\ell]$.

Corollary 6.2. *For a homogeneous subset $\mathcal{B} = \{\psi_i\}_{i=1}^p$ of $(S \otimes \wedge^m V^* \otimes V)^W$, the following are equivalent:*

(a) *$\mathcal{B}$ forms an S^W-basis for $(S \otimes \wedge^m V^* \otimes V)^W$.*
(b) *$\det \mathrm{Coef}(\mathcal{B})$ is nonzero of degree $(\ell-1)\binom{\ell-1}{m-1}N + \binom{\ell-1}{m}N^*$.*
(c) *$\det \mathrm{Coef}(\mathcal{B}) = c \cdot J^{(\ell-1)\binom{\ell-1}{m-1}} Q^{\binom{\ell-1}{m}}$ for some nonzero scalar c in $\mathbb{C}$.*
(d) *$\sum_{i=1}^p \deg(\psi_i) = (\ell-1)\binom{\ell-1}{m-1}N + \binom{\ell-1}{m}N^*$ and $\mathcal{B}$ is K-linearly independent in the space $K \otimes \wedge^m V^* \otimes V$.*

Proof. Corollary 5.5 gives the equivalence of (a) and (d), linear algebra gives the equivalence of (d) and (b), and Lemma 6.1 gives the equivalence of (b) and (c). □

Remark 6.3. An argument with Cramer's rule shows that Corollary 6.2 (c) implies (a) directly without using Corollary 5.5, or appealing to any Hilbert series argument. Indeed, (c) implies that $\mathcal{B}$ *spans* $(S \otimes \wedge^m V^* \otimes V)^W$ over S^W, as we explain next. Label the S-basis elements $dx_I \otimes y_j$ of $S \otimes \wedge^m V^* \otimes V$ as $z_1, \ldots, z_p$ for convenience. Then the matrix $\mathrm{Coef}(\mathcal{B})$ in $S^{p \times p}$ expresses the elements $\mathcal{B} = \{\psi_i\}_{i=1}^p$ in the S-basis $\{z_i\}_{i=1}^p$:

$$\psi_j = \sum_{i=1}^p \mathrm{Coef}(\mathcal{B})_{ij} \cdot z_i\,. \tag{6.4}$$

To show that a typical element $\sum_{i=1}^p s_i z_i$ in $(S \otimes \wedge^m V^* \otimes V)^W$ lies in the S^W-span of $\mathcal{B}$, find k_i in the fraction field K of S (using $\det \mathrm{Coef}(\mathcal{B}) \neq 0$) with

$$\sum_{i=1}^p s_i z_i = \sum_{j=1}^p k_j \psi_j\,. \tag{6.5}$$

We may assume each k_j lies in K^W, else apply the symmetrizer $\frac{1}{|W|}\sum_{g\in W} g(-)$ to (6.5) and use the W-invariance of $\sum_i s_i z_i$ and of each ψ_j. To show the k_j actually lie in S^W, substitute (6.4) into (6.5), giving a matrix equation relating the column vectors $\mathbf{s} = [s_1, \ldots, s_p]^t$ and $\mathbf{k} = [k_1, \ldots, k_p]^t$ in K^p:

$$\mathbf{s} = \mathrm{Coef}(\mathcal{B}) \cdot \mathbf{k}.$$

Cramer's rule then implies that

$$k_i = \frac{\det \mathrm{Coef}(\mathcal{B}_{(i)})}{\det \mathrm{Coef}(\mathcal{B})} \tag{6.6}$$

where the numerator matrix $\mathrm{Coef}(\mathcal{B}_{(i)})$ is obtained from $\mathrm{Coef}(\mathcal{B})$ by replacing its i^{th} column with $\mathbf{s}$. Then since $\mathrm{Coef}(\mathcal{B}_{(i)})$ expresses the elements $\psi_1, \ldots, \psi_{i-1}, \sum_i s_i z_i, \psi_{i+1}, \ldots, \psi_p$ of $(S \otimes \wedge^m V^* \otimes V)^W$ in terms of the basis $\{z_i\}$, Lemma 6.1 implies that its determinant is divisible by the nonzero polynomial $\det \mathrm{Coef}(\mathcal{B})$. Thus the right side of (6.6) lies in S, so that its left side k_i lies in $K^W \cap S = S^W$, as desired.

7. Independence over the fraction field

In this section, we use Springer's theory of regular elements to investigate differential derivations with coefficients in the fraction field $K = \mathbb{C}(x_1, \ldots, x_\ell)$ of $S = \mathbb{C}[x_1, \ldots, x_\ell]$. We will later show that Theorem 1.1 (for duality groups) follows from a more general statement established in this section for arbitrary reflection groups, Theorem 7.3, describing a K-vector space basis for $(K \otimes \wedge^m V^* \otimes V)^W$.

We first give a definition and a lemma. Recall the notation V^{reg} for the complement within V of the union of all reflecting hyperplanes for W, that is, the subset of vectors in V having regular W-orbit. Recall the notation $df_I := df_{i_1} \wedge \cdots \wedge df_{i_m} \in S \otimes \wedge^m V^*$ for subsets $I = \{i_1 < \ldots < i_m\} \subset [\ell] := \{1, 2, \ldots, \ell\}$, and the notation $\binom{[\ell]}{m}$ for the collection of all m-subsets $I \subset [\ell]$.

Definition 7.1. Define a K-linear map

$$K \otimes V^* \xrightarrow{E} K \qquad \text{by} \quad E(dx_i) = E(1 \otimes x_i) = x_i\,.$$

One also has a K-linear map $K \otimes V^* \otimes V \xrightarrow{E \otimes \mathbf{1}_V} K \otimes V\,.$

Note that by *Euler's identity*, for any homogeneous f in S,

$$E(df) = \deg(f) \cdot f. \tag{7.1}$$

Likewise, for $\theta = \sum_{j=1}^{\ell} \theta^{(j)} \otimes y_j$ in $S \otimes V$ with each $\theta^{(j)}$ homogeneous of fixed degree $\deg(\theta)$,

$$(E \otimes \mathbf{1}_V)(d\theta) = \deg(\theta) \cdot \theta. \tag{7.2}$$

We first give a K-basis for differential derivations with coefficients in K. Recall from (1.1) that $\theta_1, \ldots, \theta_\ell$ are any choice of basic derivations, i.e., any choice of homogeneous S^W-basis of $(S \otimes V)^W = (S \otimes \wedge^0 V^* \otimes V)^W$ (identifying $S \otimes V$ with $S \otimes 1 \otimes V$).

Lemma 7.2. *For each $0 \le m \le \ell$, the following set gives a K-basis for $K \otimes \wedge^m V^* \otimes V$:*

$$\widetilde{\mathcal{B}}^{(m)} := \left\{ df_I\, \theta_k \right\}_{I \in \binom{[\ell]}{m},\ k \in [\ell]}\,. \tag{7.3}$$

Furthermore, elements of $S \otimes V^* \otimes V$ can be expressed in the K-basis $\widetilde{\mathcal{B}}^{(1)}$ with coefficients in $(JQ)^{-\ell} S$.

Proof. The matrix that expresses $\widetilde{\mathcal{B}}^{(m)}$ in the usual S-basis $\{dx_I \otimes y_j : I \in \binom{[\ell]}{m},\ j \in [\ell]\}$ of $S \otimes \wedge^m V^* \otimes V$ is the tensor product of the matrices $\wedge^m(\mathrm{Jac}(f)) \otimes M(\theta)$, where $\wedge^m(\mathrm{Jac}(f))$ is the m^{th} exterior power of $\mathrm{Jac}(f)$. The invertibility of $\mathrm{Jac}(f)$ and functoriality of $\wedge^m(-)$ imply the invertibility of $\wedge^m(\mathrm{Jac}(f))$. Then since $M(\theta)$ is also invertible, so is the tensor product $\wedge^m(\mathrm{Jac}(f)) \otimes M(\theta)$, and hence $\widetilde{\mathcal{B}}^{(m)}$ is another K-basis. The last assertion of the proposition then follows, since in the $m = 1$ case,

$$\det(\mathrm{Jac}(f) \otimes M(\theta)) = \det(\mathrm{Jac}(f))^\ell \cdot \det(M(\theta))^\ell = J^\ell Q^\ell. \quad \square$$

We will show that Theorem 1.1 follows from the next theorem.

Theorem 7.3. *Let W be a complex reflection group with homogeneous basic invariants $f_1, \ldots, f_\ell$, and an index i_0 in $1, 2, \ldots, \ell$ that satisfies*

$$V^{\mathrm{reg}} \cap \bigcap_{i \neq i_0} f_i^{-1}\{0\} \neq \varnothing. \tag{7.4}$$

Then for each $m = 0, 1, \ldots, \ell$, the following set gives a K-vector space basis for $K \otimes \wedge^m V^* \otimes V$:

$$\mathcal{B}^{(m)} := \Big\{ df_I \, d\theta_k \Big\}_{I \in \binom{[\ell]\setminus\{i_0\}}{m-1},\, k \in [\ell]} \quad \sqcup \quad \Big\{ df_I \, \theta_k \Big\}_{I \in \binom{[\ell]\setminus\{i_0\}}{m},\, k \in [\ell]}. \tag{7.5}$$

Proof of Theorem 7.3. There is nothing to prove in the case $m = 0$. We consider first the extreme case $m = 1$, then the opposite extreme case $m = \ell$, and finally the intermediate cases with $2 \le m \le \ell - 1$.

The case $m = 1$. Note that the set $\mathcal{B}^{(1)}$ that we want to show is a K-basis for $K \otimes V^* \otimes V$,

$$\mathcal{B}^{(1)} = \{df_i \, \theta_k : i \in [\ell] \setminus \{i_0\}, k \in [\ell]\} \quad \sqcup \quad \{d\theta_k : k \in [\ell]\},$$

has substantial overlap with the known K-basis $\widetilde{\mathcal{B}}^{(1)}$ for $K \otimes V^* \otimes V$ given in Lemma 7.2,

$$\widetilde{\mathcal{B}}^{(1)} = \{df_i \, \theta_k : i, k \in [\ell]\} = \{df_i \, \theta_k : i \in [\ell] \setminus \{i_0\}, k \in [\ell]\} \quad \sqcup \quad \{df_{i_0} \theta_k : k \in [\ell]\}.$$

Thus we need only show that when working in the quotient of $K \otimes V^* \otimes V$ by the K-subspace spanned by

$$\mathcal{B}^{(1)} \cap \widetilde{\mathcal{B}}^{(1)} = \{df_i \, \theta_k : i \in [\ell] \setminus \{i_0\}, k \in [\ell]\},$$

a *nonsingular* matrix in $K^{\ell \times \ell}$ expresses the images of the elements

$$\{d\theta_k : k \in [\ell]\} = \mathcal{B}^{(1)} \setminus \mathcal{B}^{(1)} \cap \widetilde{\mathcal{B}}^{(1)}$$

uniquely in terms of the images of the elements

$$\{df_{i_0} \, \theta_k : k \in [\ell]\} = \widetilde{\mathcal{B}}^{(1)} \setminus \mathcal{B}^{(1)} \cap \widetilde{\mathcal{B}}^{(1)}.$$

Here is how one produces this $\ell \times \ell$ matrix. First use Lemma 7.2 to uniquely write

$$d\theta_k = \sum_{i,j \in [\ell]} r_{i,j,k} \; df_i \, \theta_j \quad \text{for each } k \in [\ell], \tag{7.6}$$

with $r_{i,j,k}$ in $(JQ)^{-\ell} S$. Then the matrix in $K^{\ell \times \ell}$ that we wish to show is nonsingular is $(r_{i_0,j,k})_{j,k \in [\ell]}$.

To this end, apply to (7.6) the map $E \otimes \mathbf{1}_V$ from Definition 7.1, giving a system of equations in $K \otimes 1 \otimes V$:

$$e_k^* \ \theta_k = \sum_{i,j \in [\ell]} r_{i,j,k} \ \deg(f_i) \ f_i \, \theta_j \quad \text{ for each } k \in [\ell].$$

Since $\{\theta_j\}_{j \in [\ell]}$ forms a K-basis for $K \otimes V$, this gives a linear system in K:

$$e_k^* \ \delta_{j,k} = \sum_{i \in [\ell]} r_{i,j,k} \ \deg(f_i) \ f_i \quad \text{ for each } j, k \in [\ell] \,, \tag{7.7}$$

where $\delta_{j,k}$ denotes the Kronecker delta function.

To show that $(r_{i_0,j,k})_{j,k \in [\ell]}$ in $K^{\ell \times \ell}$ is nonsingular, we will evaluate each of its entries at a carefully chosen vector v. By the hypothesis (7.4), one can choose a vector v in V^{reg} with the property that $f_i(v) = 0$ for $i \neq i_0$. Since the coefficients $r_{i,j,k}$ lie in $(JQ)^{-\ell} S$, and since J, Q vanish nowhere on V^{reg}, one may evaluate the linear system (7.7) at v to obtain a linear system over $\mathbb{C}$:

$$e_k^* \ \delta_{j,k} = r_{i_0,j,k}(v) \ \deg(f_{i_0}) \ f_{i_0}(v) \quad \text{ for each } j, k \in [\ell] \,. \tag{7.8}$$

We claim $f_{i_0}(v) \neq 0$: otherwise $f_i(v) = 0$ for every i in $[\ell]$, meaning v is in the common zero locus within V of the homogeneous system of parameters $f_1, \ldots, f_\ell$ in S, forcing the contradiction $v = 0$. As the coexponents e_k^* are also nonzero, (7.8) shows that the specialized matrix $(r_{i_0,j,k}(v))_{j,k \in [\ell]}$ in $\mathbb{C}^{\ell \times \ell}$ is diagonal with nonzero determinant. Hence it is nonsingular, and so is the unspecialized matrix $(r_{i_0,j,k})_{j,k \in [\ell]}$ in $K^{\ell \times \ell}$, as desired.

The case $m = \ell$. To show that $\mathcal{B}^{(\ell)} = \{df_{[\ell] \setminus \{i_0\}} \, d\theta_k\}_{k \in [\ell]}$ is K-linearly independent, consider a dependence

$$0 = \sum_{k \in [\ell]} c_k \ df_{[\ell] \setminus \{i_0\}} \, d\theta_k.$$

Substitute the expressions for $d\theta_k$ from Equation (7.6) to obtain

$$0 = \sum_{i,j,k \in [\ell]} c_k \ r_{i,j,k} \ df_{[\ell] \setminus \{i_0\}} \, df_i \, \theta_j = \sum_{j,k \in [\ell]} c_k \ (-1)^{i_0} \ r_{i_0,j,k} \ df_{[\ell]} \, \theta_j \,.$$

But $\{df_{[\ell]} \, \theta_j\}_{j \in [\ell]}$ is a K-basis for $K \otimes \wedge^\ell(V^*) \otimes V$ by Lemma 7.2, hence,

$$0 = \sum_{k \in [\ell]} c_k \ (-1)^{i_0} \ r_{i_0,j,k} \quad \text{ for each } j \in [\ell] \,.$$

The matrix $(r_{i_0,j,k})_{j,k \in [\ell]}$ was already shown nonsingular in the $m = 1$ case, and hence $c_k = 0$ for each k.

The intermediate cases $2 \leq m \leq \ell - 1$. To show that $\mathcal{B}^{(m)}$ is K-linearly independent, consider a dependence

$$0 = \sum_{\substack{I \in \binom{[\ell] \setminus \{i_0\}}{m-1} \\ k \in [\ell]}} c_{I,k} \ df_I \, d\theta_k \quad + \sum_{\substack{I \in \binom{[\ell] \setminus \{i_0\}}{m} \\ k \in [\ell]}} c_{I,k} \ df_I \, \theta_k \,. \tag{7.9}$$

It suffices to show all coefficients $c_{I,k}$ in the first sum vanish: If so, then the second sum gives a dependence among a subset of the K-basis $\widetilde{\mathcal{B}}^{(m)}$ from Lemma 7.2, and hence its coefficients

$c_{I,k}$ must also vanish. To this end, fix a subset $I_0 \in \binom{[\ell]\setminus\{i_0\}}{m-1}$ and consider its complementary subset within $[\ell] \setminus \{i_0\}$, namely,

$$I_0^c := [\ell] \setminus \{i_0\} \setminus I_0.$$

Since $|I_0^c| = (\ell-1) - (m-1) = \ell - m$, we note that

- $I_0^c \cap I \neq \varnothing$ for each $I \subset [\ell] \setminus \{i_0\}$ with $|I| = m$, and
- $I_0^c \cap I \neq \varnothing$ for each $I \subset [\ell] \setminus \{i_0\}$ with $|I| = m-1$ and $I \neq I_0$.

Consequently, multiplying both sides of Equation (7.9) by $df_{I_0^c}$ causes all terms in the second sum to vanish, as well as most of the terms in the first sum, leaving only

$$0 = \sum_{k\in[\ell]} \pm c_{I_0,k}\ df_{[\ell]\setminus\{i_0\}}\, d\theta_k\,,$$

with sign corresponding to that in $df_{I_0^c} \wedge df_{I_0} = \pm df_{[\ell]\setminus\{i_0\}}$. But then by the case $m = \ell$ already proven, the coefficients $c_{I_0,k} = 0$ all vanish, and thus the coefficients in the first sum of Equation (7.9) vanish, as desired. This completes the proof of Theorem 7.3. $\square$

8. Numerology of duality groups

We now fix our focus on duality groups and the candidate basis for $(S \otimes \wedge^m V^* \otimes V)^W$ given in Theorem 1.1. We check in this section that these sets, comprising the putative basis, have appropriate degree sum. Let

$$\mathcal{B}^{(m)} = \Big\{df_I\, \theta_k\Big\}_{I\in\binom{[\ell-1]}{m},\, k\in[\ell]} \ \sqcup\ \Big\{df_I\, d\theta_k\Big\}_{I\in\binom{[\ell-1]}{m-1},\, k\in[\ell]} \qquad \text{for} \quad 0 \le m \le \ell\,. \tag{8.1}$$

Here one interprets the second set as empty when $m = 0$ and the first set as empty when $m = \ell$.

Lemma 8.1. *For a duality group W, the sum of the degrees of elements in $\mathcal{B}^{(m)}$ above is*

$$\Delta(\wedge^m V^* \otimes V) = (\ell-1)\tbinom{\ell-1}{m-1}N + \tbinom{\ell-1}{m}N^*.$$

Proof. Using the shorthand notation $e_I := \sum_{i\in I} e_i$ for subsets $I \subset [\ell]$, note that

$$\deg(df_I\, \theta_k) = e_I + e_k^* \qquad \text{and} \qquad \deg(df_I\, d\theta_k) = e_I + e_k^* - 1,$$

and therefore the sum of degrees for $\mathcal{B}^{(m)}$ is

$$\sum_{\substack{I\in\binom{[\ell-1]}{m},\\ k\in[\ell]}} (e_I + e_k^*) + \sum_{\substack{I\in\binom{[\ell-1]}{m-1},\\ k\in[\ell]}} (e_I + e_k^* - 1)$$

$$= \sum_{k\in[\ell]} \left(\sum_{I\in\binom{[\ell-1]}{m}} e_k^* + \sum_{I\in\binom{[\ell-1]}{m-1}} (e_k^* - 1) \right) + \sum_{k\in[\ell]} \left(\sum_{I\in\binom{[\ell-1]}{m}} e_I + \sum_{I\in\binom{[\ell-1]}{m-1}} e_I \right).$$

Now the first sum over k can be rewritten as

$$\tbinom{\ell-1}{m}N^* + \tbinom{\ell-1}{m-1}(N^* - \ell) = \tbinom{\ell}{m}N^* - \ell\tbinom{\ell-1}{m-1},$$

while the second sum over k can be rewritten as

$$\ell\left(\sum_{i\in[\ell-1]}\binom{\ell-2}{m-1}e_i+\sum_{i\in[\ell-1]}\binom{\ell-2}{m-2}e_i\right)=\ell\binom{\ell-1}{m-1}(N-(h-1))$$

since $\sum_{i\in[\ell-1]} e_i = (\sum_{i\in[\ell]} e_i) - e_\ell = N-(h-1)$ by Equation (4.3), as $e_\ell+1=\deg(f_\ell)=h$ by definition. Hence the degree sum is

$$\binom{\ell}{m}N^*+\ell N\binom{\ell-1}{m-1}-\ell h\binom{\ell-1}{m-1}=\binom{\ell-1}{m}N^*+(\ell-1)\binom{\ell-1}{m-1}N=\Delta(\wedge^m V^*\otimes V),$$

where the first equality used the duality group equation $N+N^*=h\ell$ (see (6.2)). $\square$

9. Duality groups and proof of Theorem 1.1

We now investigate differential derivations invariant under duality groups and prove Theorem 1.1. We combine the linear independence results from Section 7 with the numerology of the last section. We first check that the hypothesis (7.4) in Theorem 7.3 holds for all duality groups when one chooses the index $i_0=\ell$. We emphasize that although both Lemma 9.1 and its consequence Corollary 9.3 below could easily be checked case-by-case, we give case-free proofs so that Theorem 1.1 relies on no classification of reflection groups.

Lemma 9.1. *Irreducible complex reflection groups have exactly one coexponent equal to* 1*.*

Proof. Since V is a nontrivial irreducible W-representation, and since the polynomial ring $S=\mathrm{Sym}(V^*)$ carries the trivial representation $\mathbf{1}$ in its degree zero component S_0 and the representation V^* in its degree one component S_1, Schur's Lemma implies

$$\begin{array}{lclclcl} \dim(S_0\otimes V)^W &=& \dim(\mathbf{1}\otimes V)^W &=& \dim V^W &=& 0\,,\\ \dim(S_1\otimes V)^W &=& \dim(V^*\otimes V)^W &=& \dim \mathrm{Hom}_{\mathbb{C}}(V,V)^W &=& \dim \mathrm{Hom}_{\mathbb{C}W}(V,V) \;=\; 1. \end{array}$$

Hence among the S^W-basis elements $\theta_1,\ldots,\theta_\ell$ for $(S\otimes V)^W$, there must be none of degree zero, and exactly one of degree one; in fact, the latter must be a multiple of the *Euler derivation* $\theta_E := x_1\otimes y_1+\cdots+x_\ell\otimes y_\ell$. $\square$

Remark 9.2. The above proof shows more generally that even for non-reflection (finite) groups W acting nontrivially and irreducibly on $V=\mathbb{C}^\ell$, there will be, up to scaling, only the Euler derivation θ_E as a W-invariant derivation in $(S\otimes V)^W$ of degree one.

Corollary 9.3. *For any duality group W, there is a unique highest exponent e_ℓ and accompanying unique highest degree $h=\deg(f_\ell)=e_\ell+1$.*

Lemma 9.4. *Duality groups W satisfy hypothesis* (7.4) *in Theorem 7.3 with $i_0=\ell$, that is,*

$$V^{\mathrm{reg}}\ \cap\ \bigcap_{i=1}^{\ell-1} f_i^{-1}\{0\}\neq\varnothing.$$

Proof. (cf. [4, p. 4]) Springer [30, Prop. 3.2(i)] showed that if W is a complex reflection group with basic invariants $f_1,\ldots,f_\ell$ and ζ is any primitive d^{th} root of unity in $\mathbb{C}$, then

$$\bigcap_{\substack{i=1,\ldots,\ell:\\ d\nmid\deg(f_i)}} f_i^{-1}\{0\}=\bigcup_{g\in W}\ker(\zeta\mathbf{1}_V-g). \tag{9.1}$$

On the other hand, Lehrer and Michel [18, Thm. 1.2] showed existence of g in W with $\ker(\zeta\mathbf{1}_V-g)\cap V^{\mathrm{reg}}\neq\varnothing$ if and only if d divides as many degrees $\deg(f_i)=e_i+1$ as codegrees

e_i^*-1. For a duality group W, the equations $e_i+e_i^*=h=\deg(f_\ell)$ imply h divides as many degrees as codegrees. Also, Corollary 9.3 implies that f_ℓ is the *only* basic invariant of degree h, so that (9.1) gives the result. □

We can now deduce the two equivalent statements of our main result. Recall that R is the exterior subalgebra of the W-invariant forms $(S\otimes\wedge V^*)^W=\bigwedge_{S^W}\{df_1,\dots,df_\ell\}$ generated by all df_i except for the last one df_ℓ, that is, $R:=\bigwedge_{S^W}\{df_1,\dots,df_{\ell-1}\}$.

Theorem 1.1. *For W a duality (well-generated) complex reflection group, $(S\otimes\wedge V^*\otimes V)^W$ forms a free R-module on R-basis $\{\theta_1,\dots,\theta_\ell,\ d\theta_1,\dots,d\theta_\ell\}$. Equivalently, $(S\otimes\wedge^m V^*\otimes V)^W$ for $0\le m\le\ell$ has S^W-basis*

$$\mathcal{B}^{(m)}=\Big\{df_I\theta_k\Big\}_{I\in\binom{[\ell-1]}{m},\,k\in[\ell]}\ \sqcup\ \Big\{df_I d\theta_k\Big\}_{I\in\binom{[\ell-1]}{m-1},\,k\in[\ell]}.$$

Proof of Theorem 1.1. Lemma 9.4 and Theorem 7.3 imply that $\mathcal{B}^{(m)}$ has nonsingular coefficient matrix. Indeed, when $i_0=\ell$, the set $\mathcal{B}^{(m)}$ in (7.5) agrees with that in (8.1). Note that this set has cardinality

$$\tbinom{\ell-1}{m-1}\ell+\tbinom{\ell-1}{m}\ell=\tbinom{\ell}{m}\ell=\dim_K\left(K\otimes\wedge^m V^*\otimes V\right)=\operatorname{rank}_{S^W}(S\otimes\wedge^m V^*\otimes V)^W.$$

Lemma 8.1 shows that their degree sum is appropriate, and the theorem then follows from Corollary 5.5. □

Remark 9.5. Theorem 1.1 has an amusingly compact rephrasing: defining by convention $f_0:=1$, the last assertion of the theorem is equivalent to the assertion that $(S\otimes\wedge^m V^*\otimes V)^W$ is a free S^W-module on basis

$$\{\ df_{i_m}\cdots df_{i_2}\cdot d(f_{i_1}\theta_k):\ 0\le i_1<i_2<\cdots<i_m\le\ell-1 \text{ and } k\in[\ell]\,\}. \tag{9.2}$$

The reason is that the elements in (9.2) with $i_1=0$ coincide with the basis elements $\{df_I d\theta_k\}$ in the second part of $\mathcal{B}^{(m)}$, while the elements in (9.2) with $i_1\ge 1$ are almost the same as the basis elements $\{df_I\theta_k\}$ in the first part of $\mathcal{B}^{(m)}$, but differ from them by f_{i_1} times an element in the second part of $\mathcal{B}^{(m)}$.

10. Two dimensional reflection groups

We now consider reflection groups acting on 2-dimensional complex space, i.e., the case when $\ell=2$. We found in Theorem 1.1 an S^W-basis for $(S\otimes\wedge V^*\otimes V)^W$ when W is a duality group. Here, we find a *different* choice of basis that works for *any* rank 2 complex reflection group W, duality or not. Suppose we have basic derivations in $(S\otimes V)^W=(S\otimes 1\otimes V)^W$

$$\begin{aligned}\theta_1&:=x_1\otimes 1\otimes y_1+x_2\otimes 1\otimes y_2\ (=\theta_E), \qquad \text{and}\\ \theta_2&:=\ a\otimes 1\otimes y_1+\ b\otimes 1\otimes y_2,\end{aligned}\tag{10.1}$$

for some a,b in $\mathbb{C}[x_1,x_2]$. With this indexing, $e_1^*=1$ and $e_2^*=\deg(a)=\deg(b)$.

Theorem 10.1. *For a complex reflection group W acting on $\mathbb{C}^2$, and $0\le m\le 2$, the following sets $\mathcal{B}^{(m)}$ give free S^W-bases for $(S\otimes\wedge^m V^*\otimes V)^W$:*

$$\begin{aligned}\mathcal{B}^{(0)}&:=\{\theta_1,\theta_2\}\\ \mathcal{B}^{(1)}&:=\{df_1\,\theta_1,df_2\,\theta_1,d\theta_1,d\theta_2\}\\ \mathcal{B}^{(2)}&:=\{df_1\,d\theta_1,df_2\,d\theta_1\}.\end{aligned}\tag{10.2}$$

Proof. The $m = 0$ case is immediate. For $m = 1, 2$, the basic derivations as in (10.1) give

$$\mathrm{Coef}(\mathcal{B}^{(2)}) = \begin{array}{c|cc} & df_1 d\theta_1 & df_2 d\theta_1 \\ \hline 1\otimes x_1\wedge x_2 \otimes y_1 & -\frac{\partial f_1}{\partial x_2} & -\frac{\partial f_2}{\partial x_2} \\ 1\otimes x_1\wedge x_2 \otimes y_2 & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} \end{array}, \quad \mathrm{Coef}(\mathcal{B}^{(1)}) = \begin{array}{c|cccc} & df_1\theta_1 & df_2\theta_1 & d\theta_1 & d\theta_2 \\ \hline 1\otimes x_1 \otimes y_1 & x_1\frac{\partial f_1}{\partial x_1} & x_1\frac{\partial f_2}{\partial x_1} & 1 & \frac{\partial a}{\partial x_1} \\ 1\otimes x_2 \otimes y_2 & x_2\frac{\partial f_1}{\partial x_2} & x_2\frac{\partial f_2}{\partial x_2} & 1 & \frac{\partial b}{\partial x_2} \\ 1\otimes x_1 \otimes y_2 & x_2\frac{\partial f_1}{\partial x_1} & x_2\frac{\partial f_2}{\partial x_1} & 0 & \frac{\partial b}{\partial x_1} \\ 1\otimes x_2 \otimes y_1 & x_1\frac{\partial f_1}{\partial x_2} & x_1\frac{\partial f_2}{\partial x_2} & 0 & \frac{\partial a}{\partial x_2} \end{array}.$$

One now computes that $\mathcal{B}^{(1)}, \mathcal{B}^{(2)}$ have the right degree sums and satisfy the hypotheses of Corollary 5.5:

$$\det \mathrm{Coef}(\mathcal{B}^{(2)}) = \tfrac{\partial f_1}{\partial x_1}\tfrac{\partial f_2}{\partial x_2} - \tfrac{\partial f_1}{\partial x_2}\tfrac{\partial f_2}{\partial x_1} = \det \mathrm{Jac}(f_1, f_2) = J = J^{(\ell-1)\binom{\ell-1}{m-1}} Q^{\binom{\ell-1}{m}} \text{ for } \ell = 2 = m,$$

$$\begin{aligned}\det \mathrm{Coef}(\mathcal{B}^{(1)}) &= \left(\tfrac{\partial f_1}{\partial x_1}\tfrac{\partial f_2}{\partial x_2} - \tfrac{\partial f_1}{\partial x_2}\tfrac{\partial f_2}{\partial x_1}\right)\left(x_1\left(x_1\tfrac{\partial b}{\partial x_1} + x_2\tfrac{\partial b}{\partial x_2}\right) - x_2\left(x_2\tfrac{\partial a}{\partial x_2} + x_1\tfrac{\partial a}{\partial x_1}\right)\right) \\ &= J\,(x_1 e_2^* b - x_2 e_2^* a) \\ &= e_2^*\,J\det(M(\theta_1,\theta_2)) = e_2^*\,JQ = e_2^*\,J^{(\ell-1)\binom{\ell-1}{m-1}} Q^{\binom{\ell-1}{m}} \text{ for } \ell = 2, m = 1. \quad \square\end{aligned}$$

This gives an immediate Hilbert series corollary when $\ell = 2$.

Corollary 10.2. *For a complex reflection group W acting on $\mathbb{C}^2$,*

$$\frac{\mathrm{Hilb}((S\otimes \wedge V^*\otimes V)^W; q, t)}{\mathrm{Hilb}(S^W; q)} = t^0(q + q^{e_2^*}) + t^1(1 + q^{e_2^*-1} + q^{e_1+1} + q^{e_2+1}) + t^2(q^{e_1} + q^{e_2}).$$

In the case of a duality group W with $\ell = 2$, one can check that this agrees with description (2.1), bearing in mind that $e_1^* = 1$ and $e_2^* + e_1 = h = e_2 + 1$ with the above conventions.

11. The reflection group G_{31}

The group $W = G_{31}$ is an irreducible complex reflection group of rank 4 containing 60 reflections, each of order 2 (so $N^* = N = 60$), although it is not the complexification of a Coxeter group. It is not a duality group; the exponents are $(7, 11, 19, 23)$ and the coexponents are $(1, 13, 17, 29)$. Using a computer to complete a Molien-style summation as in Lemma 4.1 for $W = G_{31}$ (taking $U = V = \mathbb{C}^4$), one obtains

$$\frac{\mathrm{Hilb}((S\otimes \wedge V^*\otimes V)^W; q, t)}{\mathrm{Hilb}(S^W; q)} = (1+q^7t)(1+q^{11}t)(q+t)(1+q^{12})(1+q^{19}t+q^{16}+q^{23}t)\,. \tag{11.1}$$

It is not hard to see that this is inconsistent with a description of $(S\otimes \wedge^m V^*\otimes V)^W$ exactly as in Theorem 1.1. However, rewriting the right side of (11.1) as

$$(1+q^7t)(1+q^{11}t)\cdot(q+t)\left[(1+q^{12}+q^{16}+q^{28}) + (q^{19}t + q^{23}t + q^{19+12} + q^{23+12}t)\right]$$

suggests a modified statement. Let

$$R' := \bigwedge_{S^W}\{df_1, df_2\}$$

as a subalgebra of $(S \otimes \wedge V^*)^W = \bigwedge_{S^W}\{df_1, df_2, df_3, df_4\}$.

Theorem 11.1. *For $W = G_{31}$, the R'-module $(S \otimes \wedge V^* \otimes V)^W$ is free with R'-basis*

$$\left\{\theta_i \ , d\theta_i\right\}_{i=1,2,3,4} \ \sqcup \ \left\{ \begin{array}{cccc} df_3\,\theta_1, & df_4\,\theta_1, & df_3\,\theta_2, & df_4\,\theta_2, \\ df_3\,d\theta_1, & df_4\,d\theta_1, & df_3\,d\theta_2, & df_4\,d\theta_2 \end{array} \right\}.$$

Proof. One can check that the elements listed above that lie in $S \otimes \wedge^m V^* \otimes V$ have degrees adding to

$$\Delta(\wedge^m V^* \otimes V) = 60\left(3\binom{3}{m-1} + \binom{3}{m}\right), \quad \text{for} \quad 0 \leq m \leq 4.$$

Thus the theorem follows from Corollary 5.5 after one checks that each matrix of coefficients for $0 \leq m \leq 4$ is nonsingular. We did this in `Mathematica`, using explicit choices of basic invariant polynomials f_1, f_2, f_3, f_4 of degrees $8, 12, 20, 24$ and basic derivations $\theta_1, \theta_2, \theta_3, \theta_4$ of degrees $1, 13, 17, 29$, constructed as prescribed by Orlik and Terao (using Maschke's [19] invariants F_8, F_{12}, F_{20}, with $F_{24} = \det \mathrm{Hessian}(F_8)$; see also Dimca and Sticlaru [12] and also [22, p. 285]). □

12. Proof of Theorem 1.2

We first recall the statement of the theorem.

Theorem 1.2. *For W any complex reflection group, $(S \otimes \wedge V^* \otimes V)^W$ is generated as a module over the exterior algebra $(S \otimes \wedge V^*)^W = \bigwedge_{S^W}\{df_1, \ldots, df_\ell\}$ by the 2ℓ generators $\{\theta_1, \ldots, \theta_\ell,\ d\theta_1, \ldots, d\theta_\ell\}$.*

Proof. The general statement follows from the case where W is irreducible. For irreducible W, we proceed case-by-case, taking advantage of the fact that the irreducible non-duality (that is, not well-generated) reflection groups fall into three camps:

- The 2-dimensional groups ($\ell = 2$).
- The exceptional group G_{31} (with $\ell = 4$).
- The infinite family of monomial groups $G(r, p, \ell)$ for $1 < p < r$.

Reflection groups of dimension 2 were considered in Section 10; Theorem 10.1 gives a basis. The group G_{31} was considered in Section 11; Theorem 11.1 gives a basis. The groups $G(r, p, \ell)$ are considered in the appendix, as some direct computation is required to prove the pattern in this general case; Theorem 14.2 gives a basis. In each case, we provided an explicit S^W-module basis for $(S \otimes \wedge V^m \otimes V)^W$ whose elements all have either the form $df_I\theta_k$ or $df_I d\theta_k$ for various subsets $I \subset [\ell]$ and k in $[\ell]$. □

13. Remarks and questions

What about $U = \wedge^k V$? One might wonder whether for complex reflection groups W, or even just duality groups, one can factor the Hilbert series more generally for $\left(S \otimes \wedge V^* \otimes \wedge^k V\right)^W$

when k takes values besides $k = 0, 1$. One can manipulate Molien-style computations using this consequence of Lemma 4.1:

$$\begin{aligned} \mathrm{Hilb}\left((S \otimes \wedge V^* \otimes \wedge V)^W ;\ q, t, u \right) &:= \sum_{i,j,k} \left(\dim S_i \otimes \wedge^j V^* \otimes \wedge^k V \right)^W q^i\, t^j\, u^k \\ &= \tfrac{1}{|W|} \sum_{w \in W} \frac{\det(1 + uw^{-1}) \det(1 + tw)}{\det(1 - qw)} \ . \end{aligned}$$

Things seem not to factor so nicely unless $k \in \{0, 1, \ell{-}1, \ell\}$, but at least we have a reciprocity:

Proposition 13.1. *Let W be a complex reflection group and set*

$$\tau(q, t, u) := \mathrm{Hilb}\big((S \otimes \wedge V^* \otimes \wedge V)^W; q, t, u \big) .$$

Then τ satisfies the reciprocity

$$\tau(q, t, u) = t^\ell\, u^\ell\, \tau(q, t^{-1}, u^{-1}) .$$

Proof. Let $\mathbb{C}_{\det}$ be a 1-dimensional W-module carrying the determinant character of W acting on V, and likewise for $\mathbb{C}_{\det^{-1}}$. The W-equivariant perfect pairings

$$\wedge^j V^* \otimes \wedge^{\ell - j} V^* \longrightarrow \wedge^\ell V^* \cong \mathbb{C}_{\det^{-1}} \quad \text{and} \quad \wedge^k V \otimes \wedge^{\ell - k} V \longrightarrow \wedge^\ell V \cong \mathbb{C}_{\det}$$

imply that

$$S \otimes \wedge^j V^* \otimes \mathbb{C}_{\det^{-1}} \cong S \otimes \wedge^{\ell - j} V \quad \text{and} \quad \wedge^k V \otimes \mathbb{C}_{\det} \cong \wedge^{\ell - k} V^*$$

as W-modules (see [27], proof of Corollary 4), since $V \cong V^{**}$ as W-modules. The result then follows from the isomorphisms of W-modules

$$S \otimes \wedge^j V^* \otimes \wedge^k V \cong (S \otimes \wedge^j V^* \otimes \mathbb{C}_{\det^{-1}}) \otimes (\wedge^k V \otimes \mathbb{C}_{\det}) \cong (S \otimes \wedge^{\ell - j} V) \otimes (\wedge^{\ell - k} V^*) . \quad \square$$

A similar argument confirms the following.

Proposition 13.2. *Let W be a complex reflection group and set*

$$\tau(\chi, q, t, u) := \mathrm{Hilb}\big((S \otimes \wedge V^* \otimes \wedge V \otimes \mathbb{C}_\chi)^W; q, t, u \big) = \tfrac{1}{|W|} \sum_{w \in W} \chi^{-1}(w)\, \frac{\det(1 + uw^{-1}) \det(1 + tw)}{\det(1 - qw)}$$

for any character $\chi : W \to \mathbb{C}^*$ afforded by a 1-dimensional W-module $\mathbb{C}_\chi$. Then τ satisfies the reciprocity

$$\tau(\chi; q, t, u) = t^\ell\, u^\ell\, \tau(\chi; q, t^{-1}, u^{-1}) .$$

Remark 13.3. The last two results generalize an observation for real reflection groups from [13, Eqn. (1.24)].

Example 13.4. For the Weyl group $W = W(F_4)$, with exponents $(1, 5, 7, 11)$, and $V^* \cong V$, a computation in `Mathematica` gives

$$\begin{aligned} &\mathrm{Hilb}\left((S \otimes \wedge V \otimes \wedge V)^W ; q, t, u \right) / \ \mathrm{Hilb}(S^W, q) \\ &= \quad u^0 (1 + qt)(1 + q^5 t)(1 + q^7 t)(1 + q^{11} t) \\ &\quad + u^1 (q + t)(1 + q^4 + q^6 + q^{10})(1 + qt)(1 + q^5 t)(1 + q^7 t) \\ &\quad + u^2 (q + t)(1 + qt)(1 + q^4) \Big((q^5 + q^7 - q^9 + q^{11} + q^{13})(1 + t^2) + (1 + q^6 + q^8 + q^{10} + q^{12} + q^{18}) t \Big) \\ &\quad + u^3 (1 + qt)(1 + q^4 + q^6 + q^{10})(q + t)(q^5 + t)(q^7 + t) \\ &\quad + u^4 (q + t)(q^5 + t)(q^7 + t)(q^{11} + t) . \end{aligned}$$

The coefficient of u^2 does not seem to factor further, but Proposition 13.1 explains the duality between the coefficients of u^k and of $u^{\ell-k}$.

14. Appendix: The case of $G(r,p,\ell)$

The Shephard and Todd infinite family of reflection groups $G(r,p,\ell)$ includes the Weyl groups of types B_ℓ, D_ℓ, the dihedral groups, and symmetric groups. To define these groups, fix an integer $r \geq 1$. Then $G(r,1,\ell)$ is the set of $\ell \times \ell$ monomial matrices (i.e., matrices with a single nonzero entry in each row and column) whose nonzero entries are complex r-th roots of unity. The group $G(r,1,\ell)$ is the wreath product of the symmetric group of order $\ell!$ and a cyclic group:

$$G(r,1,\ell) \cong \mathrm{Sym}_\ell \wr \mathbb{Z}/r\mathbb{Z} \cong \mathrm{Sym}_\ell \ltimes (\mathbb{Z}/r\mathbb{Z})^\ell .$$

Each group $G(r,1,\ell)$ acts on $V = \mathbb{C}^\ell$ as a reflection group generated by complex reflections of order 2 and order r. In fact, the group $G(r,1,\ell)$ is the symmetry group of the complex cross-polytope in $\mathbb{C}^\ell$, a *regular complex polytope* as studied by Shephard [25] and Coxeter [9].

For integers $p \geq 1$ dividing r, the group $G(r,p,\ell)$ consists of those matrices in $G(r,1,\ell)$ whose product of nonzero entries is an (r/p)-th root-of-unity. Both $G(r,1,\ell)$ and $G(r,r,\ell)$ are duality groups, and hence covered by Theorem 1.1. When $1 < p < r$, the group $G(r,p,\ell)$ is a nonduality-group.

We record here a convenient choice of basic invariant polynomials, derivations for $G(r,p,\ell)$.

Proposition 14.1. *Let $W = G(r,p,\ell)$ with $1 \leq p < r$ and p dividing r. One may choose basic W-invariant polynomials $\{f_i\}_{i=1}^\ell$ in S and derivations $\{\theta_i\}_{i=1}^\ell$ in $S \otimes 1 \otimes V$ as follows:*

$$f_k = x_1^{rk} + \cdots + x_\ell^{rk} \quad \textit{for } k = 1,2,\ldots,\ell-1, \textit{ and } f_\ell = (x_1 \cdots x_\ell)^{\frac{r}{p}},$$
$$\theta_k = x_1^{(k-1)r+1} \otimes 1 \otimes y_1 + \cdots + x_\ell^{(k-1)r+1} \otimes 1 \otimes y_\ell \quad \textit{for } k = 1,2,\ldots,\ell .$$

In particular, W has

exponents	$(e_1,\ldots,e_{\ell-1},e_\ell) = (r-1, 2r-1, \ldots, (\ell-1)r-1, \frac{\ell r}{p}-1)$,
coexponents	$(e_1^*,\ldots,e_\ell^*) = (1, r+1, 2r+1, \ldots, (\ell-1)r+1)$,
number of reflections	$N = \binom{\ell}{2} r + \ell\left(\frac{r}{p}-1\right)$, *and*
number of hyperplanes	$N^* = \binom{\ell}{2} r + \ell$.

Proof. The W-invariant derivations $\{\theta_i\}_{i=1}^\ell$ above are the usual choice, for example, as in Orlik and Terao [22, Prop. 6.77]. (Or use the $m=0$ case of Corollary 5.5 to verify the θ_i are basic derivations, since their associated coefficient matrix is an easy variant of a Vandermonde matrix.) The W-invariant polynomials $\{f_i\}_{i=1}^\ell$ above are closely related to a more usual choice $\{f_i'\}_{i=1}^\ell$ of basic W-invariants

$$e_1(x_1^r,\ldots,x_\ell^r),\ e_2(x_1^r,\ldots,x_\ell^r),\ \ldots,\ e_{\ell-1}(x_1^r,\ldots,x_\ell^r),\ (x_1\cdots x_\ell)^{\frac{r}{p}},$$

in which $e_k(x_1,\ldots,x_\ell)$ is the k^{th} elementary symmetric polynomial in $x_1,\ldots,x_\ell$, the sum of all square-free monomials of degree k; e.g., see Smith [28, §7.4, Ex. 1]. However, $\{f_i\}_{i=1}^{\ell-1}$ and $\{f_i'\}_{i=1}^{\ell-1}$ generate the same subalgebra of polynomials when working over a field of characteristic zero because the collection of power sums and elementary symmetric functions can be expressed as polynomials in each other; e.g., see [31, Thm. 7.4.4, Cor. 7.7.2, Prop. 7.7.6]. $\square$

Theorem 14.2. *Let $W = G(r, p, \ell)$ with $1 \le p < r$ and p dividing r. Then for $1 \le m \le \ell$, the set*

$$\mathcal{B}^{(\ell,m)} = \{df_I\,\theta_k\}_{I \in \binom{[\ell]}{m},\ k\in[\ell-m]} \quad \sqcup \quad \{df_I\,d\theta_k\}_{I\in\binom{[\ell]}{m-1},\ k\in[\ell-m+1]}$$

gives a free basis for the S^W-module of invariants $\big(S \otimes \wedge^m V^* \otimes V\big)^W$.

Proof. We will apply Corollary 5.5 to $\mathcal{B}^{(\ell,m)}$. There are several steps.

Step 1. First note that $\mathcal{B}^{(\ell,m)}$ has the correct cardinality:

$$\begin{aligned} |\mathcal{B}^{(\ell,m)}| &= (\ell-m)\tbinom{\ell}{m} + (\ell-m+1)\tbinom{\ell}{m-1} \\ &= (\ell-m)\tbinom{\ell}{m} + m\tbinom{\ell}{m} \\ &= \ell\tbinom{\ell}{m} = \mathrm{rank}_{S^W}(S\otimes \wedge V^* \wedge V)^W . \end{aligned}$$

Step 2. We check that the sum of the degrees of the elements in $\mathcal{B}^{(\ell,m)}$ is $\Delta(\wedge^m V^* \otimes V)$, which is straightforward albeit tedious. Again using the shorthand notation $e_I := \sum_{i\in I} e_i$ for $I \subset [\ell]$, this sum is

$$\sum_{\substack{I\in\binom{[\ell]}{m} \\ k\in[\ell-m]}} (e_I + e_k^*) + \sum_{\substack{I\in\binom{[\ell]}{m-1} \\ k\in[\ell-m+1]}} (e_I + e_k^* - 1)\,,$$

which one can rewrite as

$$\begin{aligned} &\tbinom{\ell}{m} \sum_{k\in[\ell-m]} e_k^* + \tbinom{\ell}{m-1} \sum_{k\in[\ell-m+1]} (e_k^* - 1) + \\ &\quad (\ell-m)\left(\sum_{I\in\binom{[\ell-1]}{m}} e_I + \sum_{I\in\binom{[\ell-1]}{m-1}} (e_I + e_\ell) \right) + (\ell-m+1)\left(\sum_{I\in\binom{[\ell-1]}{m-1}} e_I + \sum_{I\in\binom{[\ell-1]}{m-2}} (e_I + e_\ell) \right). \end{aligned}$$

Bearing in mind that $e_i = ir - 1$ for $1 \le i \le \ell - 1$, we employ a shorthand notation

$$g(n,m) := \sum_{\substack{I\in\binom{[k]}{m} \\ i\in I}} e_i = \sum_{\substack{I\in\binom{[k]}{m} \\ i\in I}} (ir-1) = \tbinom{k-1}{m-1} \sum_{i\in I} (ir-1) = \tbinom{k-1}{m-1}\left(r\tbinom{k+1}{2} - k\right) \tag{14.1}$$

to rewrite the degree sum as

$$\begin{aligned} &\tbinom{\ell}{m}\left(r\tbinom{\ell-m}{2} + \ell - m\right) + \tbinom{\ell}{m-1}\left(r\tbinom{\ell-m+1}{2}\right) + \\ &\quad (\ell-m)\left(g(\ell-1,m) + g(\ell-1,m-1) + \tbinom{\ell-1}{m-1}e_\ell\right) + \\ &\qquad (\ell-m+1)\left(g(\ell-1,m-1) + g(\ell-1,m-2) + \tbinom{\ell-1}{m-2}e_\ell\right) . \end{aligned}$$

(Here, we use the fact that $e_i^* = (i-1)r + 1$ for all i.) Finally, substituting in the right side of (14.1) for all $g(n,m)$, and $\frac{\ell r}{p} - 1$ for e_ℓ, we obtain

$$\begin{aligned} &(\ell-1)\tbinom{\ell-1}{m-1}\left(\tbinom{\ell}{2}r + \ell\left(\tfrac{\ell r}{p} - 1\right)\right) + \tbinom{\ell-1}{m}\left(\tbinom{\ell}{2}r + \ell\right) \\ &\qquad = (\ell-1)\tbinom{\ell-1}{m-1}N + \tbinom{\ell-1}{m}N^* = \Delta(\wedge^m V \otimes V^*). \end{aligned}$$

The first equality here was checked by hand and corroborated in computer algebra packages.

Step 3. At this stage, to apply Corollary 5.5, we need only show that the set $\mathcal{B}^{(\ell,m)}$ is K-linearly independent in $K \otimes \wedge^m V^* \otimes V$. We will use this to reduce to the case where $p = 1$, that is, $W = G(r, 1, \ell)$.

Note that the formulas for $\theta_k, d\theta_k, df_k$ in $W = G(r, p, \ell)$ depend on the parameter p in only one place, namely, in the definition of df_ℓ:

$$\begin{array}{rcll} df_k &=& kr \sum_{j=1}^{\ell} x_j^{kr-1} \otimes x_j & \text{for } 1 \le k \le \ell - 1\,, \\ df_\ell &=& \frac{r}{p}(x_1 \cdots x_\ell)^{\frac{r}{p}} \sum_{j=1}^{\ell} x_i^{-1} \otimes x_i\,, & \\ \theta_k &=& \sum_{j=1}^{\ell} x_j^{(k-1)r+1} \otimes 1 \otimes y_j & \text{for } 1 \le k \le \ell\,, \\ d\theta_k &=& ((k-1)r+1) \sum_{j=1}^{\ell} x_j^{(k-1)r} \otimes x_j \otimes y_j & \text{for } 1 \le k \le \ell. \end{array} \tag{14.2}$$

In checking whether the elements of $\mathcal{B}^{(\ell,m)}$ are K-linearly independent, we are free to scale them by elements of the (rational function) field $K = \mathbb{C}(x_1, \ldots, x_\ell)$. Hence, in (14.2), we may divide each element $d\theta_k$ by $(k-1)r + 1$ for $1 \le k \le \ell$, we may also divide each element df_k by kr for $1 \le k \le \ell - 1$, and lastly we may divide df_ℓ by $\frac{r}{p}(x_1 \cdots x_\ell)^{\frac{r}{p}}$. Hence Theorem 14.2 is equivalent to asserting K-linearly independence of the set

$$\mathcal{B}^{(\ell,m)} = \{df_I\, \theta_k\}_{I \in \binom{[\ell]}{m},\ k \in [\ell-m]} \quad \sqcup \quad \{df_I\, d\theta_k\}_{I \in \binom{[\ell]}{m-1},\ k \in [\ell-m+1]}$$

for $\theta_k, d\theta_k, df_k$ for $1 \le k \le \ell$ redefined (from 14.2) to give a simple and uniform family:

$$\begin{array}{rcl} df_k &:=& \sum_{j=1}^{\ell} x_j^{(k-1)r-1} \otimes x_j, \\ \theta_k &:=& \sum_{j=1}^{\ell} x_j^{(k-1)r+1} \otimes 1 \otimes y_j, \\ d\theta_k &:=& \sum_{j=1}^{\ell} x_j^{(k-1)r} \otimes x_j \otimes y_j. \end{array}$$

Note that we have also employed a cyclic shift of the indexing, that is, the old df_ℓ has been replaced by the new df_1, the old df_1 by the new df_2, etc.

This new K-linear independence assertion does not involve the parameter p. Therefore Theorem 14.2 for $W = G(r, p, \ell)$ with $1 \le p < r$ follows upon proving it for $W = G(r, 1, \ell)$, that is, with $p = 1$.

Step 4. We rescale the K-basis elements $\{1 \otimes dx_I \otimes y_k\}_{I \in \binom{\ell}{m}, k \in [\ell]}$ in $K \otimes \wedge^m V^* \otimes V$ as follows:

$$dx_I \otimes y_k \longmapsto x_k^{-1} x_I \otimes dx_I \otimes y_k$$

(recall that $dx_I \otimes y_k = 1 \otimes x_{i_1} \wedge \cdots \wedge x_{i_m} \otimes y_k$ for $I = \{i_1 < \cdots < i_m\}$). Then Theorem 14.2 is equivalent to the assertion that $\mathcal{B}^{(\ell,m)}$ is K-linearly independent after redefining

$$\begin{array}{rcll} df_k &:=& \sum_{j=1}^{\ell} x_j^{(k-1)r} \otimes x_j &= \sum_{j=1}^{\ell} z_j^{k-1} \otimes x_j, \\ \theta_k &:=& \sum_{j=1}^{\ell} x_j^{(k-1)r} \otimes 1 \otimes y_j &= \sum_{j=1}^{\ell} z_j^{k-1} \otimes 1 \otimes y_j, \\ d\theta_k &:=& \sum_{j=1}^{\ell} x_j^{(k-1)r} \otimes x_j \otimes y_j &= \sum_{j=1}^{\ell} z_j^{k-1} \otimes x_j \otimes y_j, \end{array} \tag{14.3}$$

for $k = 1, 2, \ldots, \ell$, where we have set $z_j := x_j^r$ in K.

Step 5. Consider the matrix $B^{(\ell,m)}$ with entries in $\mathbb{C}(z_1, \ldots, z_\ell)$ whose columns express each element of $\mathcal{B}^{(\ell,m)}$, defined via (14.3), in terms of $\{dx_I \otimes y_k\}$ for $I \in \binom{[\ell]}{m}$ and $k \in [\ell]$. Since each $d\theta_k$ is a K-linear combination of terms of the form $1 \otimes x_j \otimes y_j$, the expansion of each

$df_I\, d\theta_k$ in $\mathcal{B}^{(\ell,m)}$ has nonzero coefficient of $dx_I \otimes y_k$ only when $k \in I$. This leads to a block upper-triangular decomposition:

$$\begin{array}{ccc} & & \begin{array}{cc} \{df_I\, d\theta_k\} & \{df_I\, \theta_k\} \end{array} \\ B^{(\ell,m)} = & \begin{array}{r} \{dx_I \otimes y_k : k \in I\} \\ \\ \{dx_I \otimes y_k : k \notin I\} \end{array} & \begin{pmatrix} C^{(\ell,m)} & * \\ & \\ 0 & D^{(\ell,m)} \end{pmatrix}. \end{array}$$

By convention here, $C^{(\ell,0)}$ and $D^{(\ell,\ell)}$ are 0×0 matrices.

Thus it remains to show that $C^{(\ell,m)}$ and $D^{(\ell,m)}$ are invertible. We reduce this to showing invertibility of only $D^{(\ell,m)}$, since we claim that for $m = 1, 2, \ldots, \ell$, the matrices $C^{(\ell,m)}$ and $D^{(\ell,m-1)}$ differ only by row-scalings. To justify this claim, consider a pair (J, j_0) with $J = \{j_1 < \cdots < j_{m-1}\}$ for an $(m-1)$-subset of $[\ell]$ and $j \in [\ell - m + 1]$. Then (J, j_0) indexes both a column in $C^{(\ell,m)}$, that lists the expansion coefficients in

$$\begin{aligned} df_J\, d\theta_{j_0} &= \left(\sum_i z_i^{j_1 - 1} \otimes x_i\right) \cdots \left(\sum_i z_i^{j_{m-1}-1} \otimes x_i\right) \left(\sum_i z_i^{j_0 - 1} \otimes x_i \otimes y_i\right) \\ &= \sum_{(i_1,\ldots,i_{m-1},i_0)} z_{i_1}^{j_1-1} \cdots z_{i_{m-1}}^{j_{m-1}-1} z_{i_0}^{j_0-1} \otimes x_{i_1} \wedge \cdots \wedge x_{i_{m-1}} \wedge x_{i_0} \otimes y_{i_0}\,, \end{aligned} \tag{14.4}$$

and a column in $D^{(\ell,m-1)}$, that lists the expansion coefficients in

$$\begin{aligned} df_J\, \theta_{j_0} &= \left(\sum_i z_i^{j_1 - 1} \otimes x_i\right) \cdots \left(\sum_i z_i^{j_{m-1}-1} \otimes x_i\right) \left(\sum_i z_i^{j_0 - 1} \otimes 1 \otimes y_i\right) \\ &= \sum_{(i_1,\ldots,i_{m-1},i_0)} z_{i_1}^{j_1-1} \cdots z_{i_{m-1}}^{j_{m-1}-1}\, z_{i_0}^{j_0-1} \otimes x_{i_1} \wedge \cdots \wedge x_{i_{m-1}} \otimes y_{i_0}\,. \end{aligned} \tag{14.5}$$

On the other hand, a pair (I, k) where I is an m-subset of $[\ell]$ and $k \in I$ will index both a row for $dx_I \otimes y_k$ in $C^{(\ell,m)}$ and a row for $dx_{I \setminus \{k\}} \otimes y_k$ in $D^{(\ell,m-1)}$. If one assumes that $I = \{i_0, i_1, \ldots, i_{m-1}\}$ in the two expansions (14.4), (14.5) above, then one can see that these two rows will differ by a sign; this sign is the product of the signs of two permutations, namely, those permutations that sort the ordered sequences $(i_1, \ldots, i_{m-1}, i_0)$ and $(i_1, \ldots, i_{m-1})$ into the usual integer orders on I and $I \setminus \{k\}$, respectively.

Step 6. It remains to show invertibility for $0 \le m \le \ell$ of the square matrix $D^{(\ell,m)}$, that is, the submatrix whose columns give the expansion coefficients in $\mathbb{C}(z_1, \ldots, z_\ell)$ for each element of

$$\{df_I\, \theta_k : I \in \tbinom{[\ell]}{m},\ k \in [\ell - m]\}$$

in terms of the basis elements

$$\{dx_I \otimes y_k : I \in \tbinom{[\ell]}{m}, k \notin I\},$$

ignoring coefficients on all other K-basis elements $dx_I \otimes y_k$.

In fact, we will show that $\det D^{(\ell,m)}$ has coefficient ± 1 on its *lexicographically-largest* monomial, that is, the monomial $z_\ell^{a_\ell} z_{\ell-1}^{a_{\ell-1}} \cdots z_2^{a_2} z_1^{a_1}$ that achieves the maximum exponent a_ℓ, and among all such monomials with maximum a_ℓ, also maximizes $a_{\ell-1}$, and so on.

We argue via induction on ℓ by considering the following block decomposition of $D^{(\ell,m)}$:

$$\begin{array}{c|ccc}
 & \{df_I\theta_k : \ell \in I\} & \{df_I\theta_{\ell-m} : \ell \notin I\} & \{df_I\theta_k : \ell \notin I, k \neq \ell - m\} \\
\hline
\{dx_I \otimes y_k : \ell \in I\} & \alpha & * & * \\
\{dx_I \otimes y_\ell\} & \delta & \beta & * \\
\{dx_I \otimes y_k : \ell \notin I \sqcup \{k\}\} & \phi & \epsilon & \gamma
\end{array}.$$

We note two degenerate cases when $m = \ell$ or $m = \ell - 1$: if $m = \ell$ then $D^{(\ell,\ell)}$ is 0×0, as pointed out in Step 5, leaving nothing to prove; if $m = \ell - 1$, the sets indexing the rightmost block of columns and the bottommost block of rows are empty, so that $D^{(\ell,\ell-1)} = \left(\begin{smallmatrix} \alpha & * \\ \delta & \beta \end{smallmatrix}\right)$.

As all rows $dx_I \otimes y_k$ have $k \notin I$, the highest powers of x_ℓ in entries of $\det D^{(\ell,m)}$ are

- at most $z_\ell^{\ell-1}$ in the top block of rows $(\alpha, *, *)$, and $z_\ell^{\ell-1}$ occurs only in the block α,
- at most $z_\ell^{\ell-1-m}$ in the middle rows $(\delta, \beta, *)$, and $z_\ell^{\ell-1-m}$ occurs only in blocks δ, β,
- only z_ℓ^0 in the bottom block of rows (ϕ, ϵ, γ), that is, no z_ℓ's occur at all there.

We examine the terms in the permutation expansion of $\det D^{(\ell,m)}$ that achieve the highest power of z_ℓ. Since $1 \leq m \leq \ell - 1$, without loss of generality, $z_\ell^{\ell-1}$ is a strictly higher power than $z_\ell^{\ell-1-m}$, and thus these terms must use only entries from the block α in the topmost block of rows $(\alpha, *, *)$, that is, they must be terms from the product $\det \alpha \cdot \det \left(\begin{smallmatrix} \beta & * \\ \epsilon & \gamma \end{smallmatrix}\right)$. In the degenerate case $m = \ell - 1$, they must be terms from $\det \alpha \cdot \det \beta$. In the nondegenerate cases $1 \leq m \leq \ell - 2$, they must be terms from $\det \alpha \cdot \det \beta \cdot \det \gamma$ since $z_\ell^{\ell-1-m}$ is a strictly higher power than z_ℓ^0; furthermore, these terms must always pick up entries from α divisible by $z_\ell^{\ell-1}$ and entries from β divisible by $z_\ell^{\ell-1-m}$.

Upon examining α, β, γ more closely, one finds that

$$\begin{aligned}
\gamma &= D^{(\ell-1,m)}, \\
\beta &= z_\ell^{\ell-1-m} \cdot \wedge^m \mathrm{VM}^{(\ell-1)}, \\
\alpha &= z_\ell^{\ell-1} D^{(\ell-1,m-1)} + O(z_\ell^{\ell-2})
\end{aligned} \tag{14.6}$$

where $\wedge^m A$ is the m^{th} *exterior power* of the matrix A, and $\mathrm{VM}^{(n)} := [z_i^{j-1}]_{i,j=1,2,\ldots,n}$ is an $n \times n$ *Vandermonde matrix.* For example, when $\ell = 4, m = 2$, the matrix β is

$$\begin{array}{c|ccc}
 & df_3 df_2 \theta_2 & df_3 df_1 \theta_2 & df_2 df_1 \theta_2 \\
\hline
1 \otimes x_3 \wedge x_2 \otimes y_4 & (z_3^2 z_2^1 - z_2^2 z_3^1) z_4^1 & (z_3^2 z_2^0 - z_2^2 z_3^0) z_4^1 & (z_3^1 z_2^0 - z_2^1 z_3^0) z_4^1 \\
1 \otimes x_3 \wedge x_1 \otimes y_4 & (z_3^2 z_1^1 - z_1^2 z_3^1) z_4^1 & (z_3^2 z_1^0 - z_1^2 z_3^0) z_4^1 & (z_3^1 z_1^0 - z_1^1 z_3^0) z_4^1 \\
1 \otimes x_2 \wedge x_1 \otimes y_4 & (z_2^2 z_1^1 - z_1^2 z_2^1) z_4^1 & (z_2^2 z_1^0 - z_1^2 z_2^0) z_4^1 & (z_2^1 z_1^0 - z_1^1 z_2^0) z_4^1
\end{array} = z_4^{4-1-2} \cdot \wedge^2 \mathrm{VM}^{(3)},$$

and the matrix α (with terms with the highest power $z_4^3 = z_\ell^{\ell-1}$ underlined) is

$$\begin{array}{rc}
 & \begin{array}{cccccc} df_4 df_3 \theta_2 & df_4 df_2 \theta_2 & df_4 df_1 \theta_2 & df_4 df_3 \theta_1 & df_4 df_2 \theta_1 & df_4 df_1 \theta_1 \end{array} \\
\begin{array}{r} 1 \otimes x_4 \wedge x_3 \otimes y_2 \\ 1 \otimes x_4 \wedge x_3 \otimes y_1 \\ 1 \otimes x_4 \wedge x_2 \otimes y_3 \\ 1 \otimes x_4 \wedge x_2 \otimes y_1 \\ 1 \otimes x_4 \wedge x_1 \otimes y_3 \\ 1 \otimes x_4 \wedge x_1 \otimes y_2 \end{array} &
\begin{pmatrix}
(\underline{z_4^3 z_3^2} - z_4^2 z_3^3) z_2^1 & (\underline{z_4^3 z_3^1} - z_4^1 z_3^3) z_2^1 & (\underline{z_4^3 z_3^0} - z_4^0 z_3^3) z_2^1 & (\underline{z_4^3 z_3^2} - z_4^2 z_3^3) z_2^0 & (\underline{z_4^3 z_3^1} - z_4^1 z_3^3) z_2^0 & (\underline{z_4^3 z_3^0} - z_4^0 z_3^3) z_2^0 \\
(\underline{z_4^3 z_3^2} - z_4^2 z_3^3) z_1^1 & (\underline{z_4^3 z_3^1} - z_4^1 z_3^3) z_1^1 & (\underline{z_4^3 z_3^0} - z_4^0 z_3^3) z_1^1 & (\underline{z_4^3 z_3^2} - z_4^2 z_3^3) z_1^0 & (\underline{z_4^3 z_3^1} - z_4^1 z_3^3) z_1^0 & (\underline{z_4^3 z_3^0} - z_4^0 z_3^3) z_1^0 \\
(\underline{z_4^3 z_2^2} - z_4^2 z_2^3) z_3^1 & (\underline{z_4^3 z_2^1} - z_4^1 z_2^3) z_3^1 & (\underline{z_4^3 z_2^0} - z_4^0 z_2^3) z_3^1 & (\underline{z_4^3 z_2^2} - z_4^2 z_2^3) z_3^0 & (\underline{z_4^3 z_2^1} - z_4^1 z_2^3) z_3^0 & (\underline{z_4^3 z_2^0} - z_4^0 z_2^3) z_3^0 \\
(\underline{z_4^3 z_2^2} - z_4^2 z_2^3) z_1^1 & (\underline{z_4^3 z_2^1} - z_4^1 z_2^3) z_1^1 & (\underline{z_4^3 z_2^0} - z_4^0 z_2^3) z_1^1 & (\underline{z_4^3 z_2^2} - z_4^2 z_2^3) z_1^0 & (\underline{z_4^3 z_2^1} - z_4^1 z_2^3) z_1^0 & (\underline{z_4^3 z_2^0} - z_4^0 z_2^3) z_1^0 \\
(\underline{z_4^3 z_1^2} - z_4^2 z_1^3) z_3^1 & (\underline{z_4^3 z_1^1} - z_4^1 z_1^3) z_3^1 & (\underline{z_4^3 z_1^0} - z_4^0 z_1^3) z_3^1 & (\underline{z_4^3 z_1^2} - z_4^2 z_1^3) z_3^0 & (\underline{z_4^3 z_1^1} - z_4^1 z_1^3) z_3^0 & (\underline{z_4^3 z_1^0} - z_4^0 z_1^3) z_3^0 \\
(\underline{z_4^3 z_1^2} - z_4^2 z_1^3) z_2^1 & (\underline{z_4^3 z_1^1} - z_4^1 z_1^3) z_2^1 & (\underline{z_4^3 z_1^0} - z_4^0 z_1^3) z_2^1 & (\underline{z_4^3 z_1^2} - z_4^2 z_1^3) z_2^0 & (\underline{z_4^3 z_1^1} - z_4^1 z_1^3) z_2^0 & (\underline{z_4^3 z_1^0} - z_4^0 z_1^3) z_2^0
\end{pmatrix}
\end{array}.$$

Note that here $\alpha = z_4^{4-1} \cdot D^{(3,1)} + O(z_4^{4-2})$, as asserted in (14.6).

As the lex-largest monomial in $\det D^{(\ell,m)}$ has the same coefficient as in $\det \alpha \cdot \det \beta \cdot \det \gamma$, the descriptions in (14.6) imply that this monomial is a power of z_ℓ times the product of the lex-largest monomials in

$$\det D^{(\ell-1,m)}, \ \det D^{(\ell-1,m-1)}, \ \det \wedge^m \mathrm{VM}^{(\ell-1)}.$$

By induction on ℓ, the coefficient on the lex-largest monomials in $\det D^{(\ell-1,m)}$ and $\det D^{(\ell-1,m-1)}$ are both ± 1. For $\det \wedge^m \mathrm{VM}^{(\ell-1)}$, the Sylvester-Franke Theorem says $\det \wedge^m A = (\det A)^{\binom{n-1}{m-1}}$, and since one has coefficient ± 1 on the lex-largest monomial $z_{\ell-1}^{\ell-2} \cdots z_2^1 z_1^0$ in $\det \mathrm{VM}^{(\ell-1)}$, the same holds for $\det \wedge^m \mathrm{VM}^{(\ell-1)}$. Thus this also holds for $\det D^{(\ell,m)}$, completing the proof. $\square$

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