POINCARÉ-BIRKHOFF-WITT THEOREMS

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Abstract. We sample some Poincaré-Birkhoff-Witt theorems appearing in mathematics. Along the way, we compare modern techniques used to establish such results, for example, the Composition-Diamond Lemma, Gröbner basis theory, and the homological approaches of Braverman and Gaitsgory and of Polishchuk and Positselski. We discuss several contexts for PBW theorems and their applications, such as Drinfeld-Jimbo quantum groups, graded Hecke algebras, and symplectic reflection and related algebras.

1. Introduction

In 1900, Poincaré [76] published a fundamental result on Lie algebras that would prove a powerful tool in representation theory: A Lie algebra embeds into an associative algebra that behaves in many ways like a polynomial ring. Capelli [20] proved a special case of this theorem, for the general linear group, ten years earlier. In 1937, Birkhoff [10] and Witt [97] independently formulated and proved versions of the theorem that we use today, although neither author cited this earlier work. The result was called the Birkhoff-Witt Theorem for years and then later the Poincaré-Witt Theorem (see Cartan and Eilenberg [21]) before Bourbaki [14] prompted use of its current name, the *Poincaré-Birkhoff-Witt Theorem*.

The original theorem on Lie algebras was greatly expanded over time by a number of authors to describe various algebras, especially those defined by quadratic-type relations (including Koszul rings over semisimple algebras). Poincaré-Birkhoff-Witt theorems are often used as a springboard for investigating the representation theory of algebras. These theorems are used to

- reveal an algebra as a deformation of another, well-behaved algebra,
- posit a convenient basis (of "monomials") for an algebra, and
- endow an algebra with a canonical homogeneous (or graded) version.

In this survey, we sample some of the various Poincaré-Birkhoff-Witt theorems, applications, and techniques used to date for proving these results. Our survey

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is not intended to be all-inclusive; we instead seek to highlight a few of the more recent contributions and provide a helpful resource for users of Poincaré-Birkhoff-Witt theorems, which we henceforth refer to as *PBW theorems*.

We begin with a quick review in Section 2 of the original PBW Theorem for enveloping algebras of Lie algebras. We next discuss PBW properties for quadratic algebras in Section 3, and for Koszul algebras in particular, before turning to arbitrary finitely generated algebras in Section 4. We recall needed facts on Hochschild cohomology and algebraic deformation theory in Section 5, and more background on Koszul algebras is given in Section 6. Sections 7–8 outline techniques for proving PBW results recently used in more general settings, some by way of homological methods and others via the Composition-Diamond Lemma (and Gröbner basis theory). One inevitably is led to similar computations when applying any of these techniques to specific algebras, but with different points of view. Homological approaches can help to organize computations and may contain additional information, while approaches using Gröbner basis theory are particularly well-suited for computer computation. We focus on some classes of algebras in Sections 9 and 10 of recent interest: Drinfeld-Jimbo quantum groups, Nichols algebras of diagonal type, symplectic reflection algebras, rational Cherednik algebras, and graded (Drinfeld) Hecke algebras. In Section 11, we mention applications in positive characteristic (including algebras built on group actions in the modular case) and other generalizations that mathematicians have only just begun to explore.

We take all tensor products over an underlying field k unless otherwise indicated and assume all algebras are associative k-algebras with unity. Note that although we limit discussions to finitely generated algebras over k for simplicity, many remarks extend to more general settings.

2. Lie algebras and the classical PBW Theorem

All PBW theorems harken back to a classical theorem for universal enveloping algebras of Lie algebras established independently by Poincaré [76], Birkhoff [10], and Witt [97]. In this section, we recall this original PBW theorem in order to set the stage for other PBW theorems and properties; for comprehensive historical treatments, see [47, 94].

A finite dimensional *Lie algebra* is a finite dimensional vector space $\mathfrak{g}$ over a field k together with a binary operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

(i) (antisymmetry) $[v, v] = 0$ and

(ii) (Jacobi identity) $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$

for all $u, v, w \in \mathfrak{g}$. Condition (i) implies $[v, w] = -[w, v]$ for all v, w in $\mathfrak{g}$ (and is equivalent to this condition in all characteristics other than 2).

The *universal enveloping algebra* $U(\mathfrak{g})$ of $\mathfrak{g}$ is the associative algebra generated by the vectors in $\mathfrak{g}$ with relations $vw - wv = [v, w]$ for all v, w in $\mathfrak{g}$, i.e.,

$$U(\mathfrak{g}) = {}^{T(\mathfrak{g})}\big/_{(v \otimes w - w \otimes v - [v,w] \,:\, v, w \in \mathfrak{g})},$$

where $T(\mathfrak{g})$ is the tensor algebra of the vector space $\mathfrak{g}$ over k. It can be defined by a universal property: $U(\mathfrak{g})$ is the (unique up to isomorphism) associative algebra such that any linear map ϕ from $\mathfrak{g}$ to an associative algebra A satisfying $[\phi(v), \phi(w)] = \phi([v, w])$ for all $v, w \in \mathfrak{g}$ factors through $U(\mathfrak{g})$. (The bracket operation on an associative algebra A is given by $[a, b] := ab - ba$ for all $a, b \in A$.) As an algebra, $U(\mathfrak{g})$ is filtered, under the assignment of degree 1 to each vector in $\mathfrak{g}$.

Original PBW Theorem. *A Lie algebra $\mathfrak{g}$ embeds into its universal enveloping algebra $U(\mathfrak{g})$, and the associated graded algebra of $U(\mathfrak{g})$ is isomorphic to $S(\mathfrak{g})$, the symmetric algebra on the vector space $\mathfrak{g}$.*

Thus the original PBW Theorem compares a universal enveloping algebra $U(\mathfrak{g})$ to an algebra of (commutative) polynomials. Since monomials form a k-basis for a polynomial algebra, the original PBW theorem is often rephrased in terms of a *PBW basis* (with tensor signs between vectors dropped):

PBW Basis Theorem. *Let $v_1, \ldots, v_n$ be an ordered k-vector space basis of the Lie algebra $\mathfrak{g}$. Then $\{v_1^{a_1} \cdots v_n^{a_n} : a_i \in \mathbb{N}\}$ is a k-basis of the universal enveloping algebra $U(\mathfrak{g})$.*

Example 2.1. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ consists of 2×2 matrices of trace 0 with entries in $\mathbb{C}$ under the bracket operation on the associative algebra of all 2×2 matrices. The standard basis of $\mathfrak{sl}_2(\mathbb{C})$ is

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

for which $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. Thus $U(\mathfrak{sl}_2(\mathbb{C}))$ is the associative $\mathbb{C}$-algebra generated by three symbols that we also denote by e, f, h (abusing notation) subject to the relations $ef - fe = h$, $he - eh = 2e$, $hf - fh = -2f$. It has $\mathbb{C}$-basis $\{e^a h^b f^c : a, b, c \in \mathbb{N}\}$.

Proofs of the original PBW Theorem vary (and by how much is open to interpretation). The interested reader may wish to consult, for example, the texts [21], [28], [55], [57], and [95]. Jacobson [56] proved a PBW theorem for restricted enveloping algebras in positive characteristic. Higgins [54] gives references and a comprehensive PBW theorem over more general ground rings. A PBW theorem for Lie superalgebras goes back to Milnor and Moore [72] (see also Kac [59]). Grivel's historical article [47] includes further references on generalizations to other ground rings, to Leibniz algebras, and to Weyl algebras. In Sections 7 and 8 below, we discuss two proof techniques particularly well suited to generalization: a combinatorial approach through the Composition-Diamond Lemma and a homological approach through algebraic deformation theory. First we lay some groundwork on quadratic algebras.

3. Homogeneous quadratic algebras

Many authors have defined the notions of PBW algebra, PBW basis, PBW deformation, or PBW property in order to establish theorems like the original PBW Theorem in more general settings. Let us compare a few of these concepts, beginning in this section with those defined for *homogeneous* quadratic algebras.

Quadratic algebras. Consider a finite dimensional vector space V over k with basis $v_1, \ldots, v_n$. Let T be its tensor algebra over k, i.e., the free k-algebra $k\langle v_1, \ldots, v_n\rangle$ generated by the v_i. Then T is an $\mathbb{N}$-graded k-algebra with

$$T^0 = k,\ T^1 = V,\ T^2 = V \otimes V,\ T^3 = V \otimes V \otimes V,\ \text{etc.}$$

We often omit tensor signs in writing elements of T as is customary in noncommutive algebra, e.g., writing x^3 for $x \otimes x \otimes x$ and xy for $x \otimes y$.

Suppose P is a set of filtered (nonhomogeneous) relations in degree 2,

$$P \subseteq T^0 \oplus T^1 \oplus T^2,$$

and let $I = (P)$ be the 2-sided ideal in T generated by P. The quotient $A = T/I$ is a *nonhomogeneous quadratic algebra*. If P consists of elements of homogeneous degree 2, i.e., $P \subseteq T^2$, then A is a *homogeneous quadratic algebra*. Thus a quadratic algebra is just an algebra whose relations are generated by (homogeneous or nonhomogenous) quadratic expressions.

We usually write each element of a finitely presented algebra $A = T/I$ as a coset representative in T, suppressing mention of the ideal I. Then a *k-basis* for A is a subset of T representing cosets modulo I which form a basis for A as a k-vector space. Some authors say a quadratic algebra has a *PBW basis* if it has the same k-basis as a universal enveloping algebra, i.e., if $\{v_1^{a_1} \cdots v_n^{a_n} : a_i \in \mathbb{N}\}$ is a basis for A as a k-vector space. Such algebras include Weyl algebras, quantum/skew polynomial rings, some iterated Ore extensions, some quantum groups, etc.

Priddy's PBW algebras. Priddy [79] gave a broader definition of PBW basis for homogeneous quadratic algebras in terms of any ordered basis of V (say, $v_1 < v_2 < \cdots < v_n$) in establishing the notion of Koszul algebras. (A quadratic algebra is *Koszul* if the boundary maps in its minimal free resolution have matrix entries that are linear forms; see Section 6.) Priddy first extended the ordering degree-lexicographically to a monomial ordering on the tensor algebra T, where we regard pure tensors in $v_1, \ldots, v_n$ as monomials. He then called a k-vector space basis for $A = T/I$ a *PBW basis* (and the algebra A a *PBW algebra*) if the product of any two basis elements either lay again in the basis or could be expressed modulo I as a sum of larger elements in the basis. In doing so, Priddy [79, Theorem 5.3] gave a class of Koszul algebras which is easy to study:

Theorem 3.1. *If a homogeneous quadratic algebra has a PBW basis, then it is Koszul.*

Polishchuk and Positselski reframed Priddy's idea; we summarize their approach (see [77, Chapter 4, Section 1]) using the notion of leading monomial *LM* of any element of T written in terms of the basis $v_1, \ldots, v_n$ of V. Suppose the set of generating relations P is a subspace of T^2. Consider those monomials that are not divisible by the leading monomial of any generating quadratic relation:

$$\mathcal{B}_P = \{\text{monomials } m \in T : LM(a) \nmid m, \ \forall a \in P\}.$$

Polishchuk and Positselski call $\mathcal{B}_P$ a *PBW basis* of the quadratic algebra A (and A a *PBW algebra*) whenever $\mathcal{B}_P$ is a k-basis of A.

Gröbner bases. Priddy's definition and the reformulation of Polishchuk and Positselski immediately call to mind the theory of Gröbner bases. Recall that a set $\mathcal{G}$ of nonzero elements generating an ideal I is called a (noncommutative) *Gröbner basis* if the leading monomial of each nonzero element of I is divisible by the leading monomial of some element of $\mathcal{G}$ with respect to a fixed monomial (i.e., term) ordering (see [73] or [64]). (Gröbner bases and Gröbner-Shirshov bases were developed independently in various contexts by Shirshov [90] in 1962, Hironaka [48] in 1964, Buchberger [17] in 1965, Bokut' [11] in 1976, and Bergman [9] in 1978.) A Gröbner basis $\mathcal{G}$ is *quadratic* if it consists of homogeneous elements of degree 2 (i.e., lies in T^2) and it is *minimal* if no proper subset is also a Gröbner basis. A version of the Composition-Diamond Lemma for associative algebras (see Section 8) implies that if $\mathcal{G}$ is a Gröbner basis for I, then

$$\mathcal{B}_{\mathcal{G}} = \{\text{monomials } m \in T : \ LM(a) \nmid m, \ \forall a \in \mathcal{G}\}$$

is a k-basis for $A = T(V)/I$.

Example 3.2. Let A be the $\mathbb{C}$-algebra generated by symbols x, y with a single generating relation $xy = y^2$. Set $V = \mathbb{C}\text{-span}\{x, y\}$ and $P = \{xy - y^2\}$ so that $A = T(V)/(P)$. A Gröbner basis $\mathcal{G}$ for the ideal $I = (P)$ with respect to the degree-lexicographical monomial ordering with $x < y$ is infinite:

$$\begin{aligned}
\mathcal{G} &= \{yx^n y - x^{n+1} y : n \in \mathbb{N}\},\\
\mathcal{B}_P &= \{\text{monomials } m \in T \text{ that are not divisible by } y^2\},\\
\mathcal{B}_{\mathcal{G}} &= \{\text{monomials } m \in T \text{ that are not divisible by } yx^n y \text{ for any } n \in \mathbb{N}\}.
\end{aligned}$$

Hence, A is not a PBW algebra using the ordering $x < y$ since $\mathcal{B}_{\mathcal{G}}$ is a $\mathbb{C}$-basis for A but $\mathcal{B}_P$ is not.

If we instead take some monomial ordering with $x > y$, then $\mathscr{G} = P$ is a Gröbner basis for the ideal $I = (P)$ and $\mathcal{B}_{\mathscr{G}} = \mathcal{B}_P$ is a $\mathbb{C}$-basis of A:

$$\begin{aligned} \mathcal{B}_P = \mathcal{B}_{\mathscr{G}} &= \{\text{monomials } m \in T \text{ that are not divisible by } xy\} \\ &= \{y^a x^b : a, b \in \mathbb{N}\}. \end{aligned}$$

Hence A is a PBW algebra using the ordering $y < x$.

Quadratic Gröbner bases. How do the sets of monomials $\mathcal{B}_P$ and $\mathcal{B}_{\mathscr{G}}$ compare after fixing an appropriate monomial ordering? Suppose $\mathscr{G}$ is a minimal Gröbner basis for $I = (P)$ (which implies that no element of $\mathscr{G}$ has leading monomial dividing that of another). Then $\mathcal{B}_{\mathscr{G}} \subset \mathcal{B}_P$, and the reverse inclusion holds whenever $\mathscr{G}$ is quadratic (since then $\mathscr{G}$ must be a subset of the subspace P). Since each graded piece of A is finite dimensional over k, a PBW basis thus corresponds to a quadratic Gröbner basis:

$$\mathcal{B}_P \text{ is a PBW basis of } A \iff \mathcal{B}_{\mathscr{G}} = \mathcal{B}_P \iff \mathscr{G} \text{ is quadratic.}$$

Thus authors sometimes call any algebra defined by an ideal of relations with a quadratic Gröbner basis a PBW algebra. In any case (see [2], [19], [36], [65]):

Theorem 3.3. *Any quadratic algebra whose ideal of relations has a (noncommutative) quadratic Gröbner basis is Koszul.*

Backelin (see [77, Chapter 4, Section 3]) gave an example of a Koszul algebra defined by an ideal of relations with no quadratic Gröbner basis. Eisenbud, Reeves, and Totaro [31, p. 187] gave an example of a commutative Koszul algebra whose ideal of relations does not have a quadratic Gröbner basis with respect to *any* ordering, even after a change of basis (see also [36]).

We relate Gröbner bases and PBW theorems for *nonhomogeneous* algebras in Section 8.

4. Nonhomogeneous algebras: PBW deformations

Algebras defined by generators and relations are not naturally graded, but merely filtered, and often one wants to pass to some graded or homogeneous version of the algebra for quick information. There is more than one way to do this in general. The original PBW Theorem shows that the universal enveloping algebra of a Lie algebra has one natural homogeneous version. Authors apply this idea to other algebras, saying that an algebra satisfies a *PBW property* when graded versions are isomorphic and call the original algebra a *PBW deformation* of this graded version. We make these notions precise in this section and relate them to the work of Braverman and Gaitsgory and of Polishchuk and Positselski on Koszul algebras in the next section.

Filtered algebras. Again, consider an algebra A generated by a finite dimensional vector space V over a field k with some defining set of relations P. (More generally, one might consider a module over a group algebra or some other k-algebra.) Let $T = \bigoplus_{i\geq 0} T^i$ be the tensor algebra over V and let $I = (P)$ be the two-sided ideal of relations so that

$$A = T/I\,.$$

If I is homogeneous, then the quotient algebra A is graded. In general, I is nonhomogeneous and the quotient algebra is only filtered, with i-th filtered component $F^i(A) = F^i(T/I) = (F^i(T) + I)/I$ induced from the filtration on T obtained by assigning degree one to each vector in V (i.e., $F^i(T) = T^0 \oplus T^1 \oplus \ldots \oplus T^i$).

Homogeneous versions. One associates to the filtered algebra A two possibly different graded versions. On one hand, we cross out lower order terms in the *generating* set P of relations to obtain a homogeneous version of the original algebra. On the other hand, we cross out lower order terms in each element of the *entire* ideal of relations. Then *PBW conditions* are precisely those under which these two graded versions of the original algebra coincide, as we recall next.

The *associated graded algebra* of A,

$$\mathrm{gr}(A) = \bigoplus_{i\geq 0} F^i(A)\big/F^{i-1}(A)\ ,$$

is a graded version of A which does not depend on the choice of generators P of the ideal of relations I. (We set $F^{-1} = \{0\}$.) The associated graded algebra may be realized concretely by projecting each element in the ideal I onto its leading homogeneous part (see Li [64, Theorem 3.2]):

$$\mathrm{gr}\left(T\big/I\right) \;\cong\; T\big/(\mathrm{LH}(I))\ ,$$

where $\mathrm{LH}(S) = \{\mathrm{LH}(f) : f \in S\}$ for any $S \subseteq T$ and $\mathrm{LH}(f)$ picks off the leading (or highest) homogeneous part of f in the graded algebra T. (Formally, $\mathrm{LH}(f) = f_d$ for $f = \sum_{i=1}^d f_i$ with each f_i in T^i and f_d nonzero.) Those looking for a shortcut may be tempted instead simply to project elements of the generating set P onto their leading homogeneous parts. A natural surjection (of graded algebras) always arises from this homogeneous version of A determined by P to the associated graded algebra of A:

$$T\big/(\mathrm{LH}(P)) \twoheadrightarrow \mathrm{gr}\left(T\big/I\right)\,.$$

PBW deformations. We say the algebra T/I is a *PBW deformation* of its homogeneous version $T/(\mathrm{LH}(P))$ (or satisfies the *PBW property* with respect to P) when the above surjection is also injective, i.e., when the associated graded algebra and the homogeneous algebra determined by P coincide (see [15]):

$$T\big/(\mathrm{LH}(I)) \cong \mathrm{gr}\left(T\big/I\right) \;\cong\; T\big/(\mathrm{LH}(P))\,.$$

In the next section, we explain the connections among PBW deformations, graded (and formal) deformations, and Hochschild cohomology.

In this language, the original PBW Theorem for universal enveloping algebras asserts that the set

$$P = \{v \otimes w - w \otimes v - [v, w] : v, w \in V\}$$

gives rise to a quotient algebra $T/(P)$ that is a PBW deformation of the commutative polynomial ring $S(V)$, for V the underlying vector space of a Lie algebra. Here, each element of V has degree 1 so that the relations are nonhomogeneous of degree 2 and $T/(P)$ is a nonhomogenous quadratic algebra.

We include an example next to show how the PBW property depends on choice of generating relations P defining the algebra T/I. (But note that if A satisfies the PBW property with respect to some generating set P of relations, then the subspace P generates is unique; see [88, Proposition 2.1].)

Example 4.1. We mention a filtered algebra that exhibits the PBW property with respect to one generating set of relations but not another. Consider the (noncommutative) algebra A generated by symbols x and y with defining relations $xy = x$ and $yx = y$:

$$A = {}^{\textstyle k\langle x, y\rangle}\big/_{\textstyle (xy - x, yx - y)}\ ,$$

where $k\langle x, y\rangle$ is the free k-algebra generated by x and y. The algebra A *does not* satisfy the PBW property with respect to the generating relations $xy-x$ and $yx-y$. Indeed, the relations imply that $x^2 = x$ and $y^2 = y$ in A and thus the associated graded algebra $\mathrm{gr}(A)$ is trivial in degree two while the homogeneous version of A is not (as x^2 and y^2 represent nonzero classes). The algebra A *does* exhibit the PBW property with respect to the larger generating set $\{xy - x, yx - y, x^2 - x, y^2 - y\}$ since

$$\mathrm{gr}\, A \cong {}^{\textstyle k\langle x, y\rangle}\big/_{\textstyle (xy, yx, x^2, y^2)}\ .$$

Examples 8.1 and 8.2 explain this recovery of the PBW property in terms of Gröbner bases and the Composition-Diamond Lemma.

5. Deformation Theory and Hochschild cohomology

In the last section, we saw that an algebra defined by nonhomogeneous relations is called a *PBW deformation* when the homogeneous version determined by generating relations coincides with its associated graded algebra. How may one view formally the original nonhomogeneous algebra as a *deformation* of its homogeneous version? In this section, we begin to fit PBW deformations into the theory of algebraic deformations. We recall the theory of deformations of algebras and Hochschild cohomology, a homological tool used to predict deformations and prove PBW properties.

Graded deformations. Let t be a formal parameter. A *graded deformation* of a graded k-algebra A is a graded associative $k[t]$-algebra A_t (for t in degree 1) which is isomorphic to $A[t] = A \otimes_k k[t]$ as a $k[t]$-module with

$$A_t|_{t=0} \cong A.$$

If we specialize t to an element of k in the algebra A_t, then we may no longer have a graded algebra, but a filtered algebra instead.

PBW deformations may be viewed as graded deformations: Each PBW deformation is a graded deformation of its homogeneous version with parameter t specialized to some element of k. Indeed, given a finitely generated algebra $A = T/(P)$, we may insert a formal parameter t of degree 1 throughout the defining relations P to make each relation homogeneous and extend scalars to $k[t]$; the result yields a graded algebra B_t over $k[t]$ with $A = B_t|_{t=1}$ and $B = B_t|_{t=0}$, the homogeneous version of A. One may verify that if A satisfies the PBW property, then this interpolating algebra B_t also satisfies a PBW condition over $k[t]$ and that B_t and $B[t]$ are isomorphic as $k[t]$-modules. Thus as B_t is an associative graded algebra, it defines a graded deformation of B.

Suppose A_t is a graded deformation of a graded k-algebra A. Then up to isomorphism, A_t is just the vector space $A[t]$ together with some associative multiplication given by

$$a * b = ab + \mu_1(a \otimes b)t + \mu_2(a \otimes b)t^2 + \cdots , \tag{5.1}$$

where ab is the product of a and b in A and for each i, and each μ_i is a linear map from $A \otimes A$ to A of degree $-i$, extended to be $k[t]$-linear. The degree condition on the maps μ_i are forced by the fact that A_t is graded for t in degree 1. (One sometimes considers a *formal* deformation, defined over formal power series $k[[t]]$ instead of polynomials $k[t]$.)

The condition that the multiplication $*$ in $A[t]$ be associative imposes conditions on the functions μ_i which are often expressed using Hochschild cohomology. For example, comparing coefficients of t in the equation $(a * b) * c = a * (b * c)$, we see that μ_1 must satisfy

$$a\mu_1(b \otimes c) + \mu_1(a \otimes bc) = \mu_1(ab \otimes c) + \mu_1(a \otimes b)c \tag{5.2}$$

for all $a, b, c \in A$. We see below that this condition implies that μ_1 is a Hochschild 2-cocycle. Comparing coefficients of t^2 yields a condition on μ_1, μ_2 called the *first obstruction*, comparing coefficients of t^3 yields a condition on μ_1, μ_2, μ_3 called the *second obstruction*, and so on. (See [38].)

Hochschild cohomology. Hochschild cohomology is a generalization of group cohomology well suited to noncommutative algebras. It gives information about an algebra A viewed as a bimodule over itself, thus capturing right and left multiplication, and predicts possible multiplication maps μ_i that could be used to define a deformation of A. One may define the Hochschild cohomology of a k-algebra concretely as Hochschild cocycles modulo Hochschild coboundaries by setting

$$\text{Hochschild } i\text{-cochains} = \{\text{linear functions } \phi : \underbrace{A \otimes \cdots \otimes A}_{i-\text{times}} \to A\}$$

(i.e., multilinear maps $A \times \cdots \times A \to A$) with linear boundary operator

$$\delta^*_{i+1} : i\text{-cochains} \to (i+1)\text{-cochains}$$

given by

$$\begin{aligned}(\delta^*_{i+1}\phi)(a_0 \otimes \cdots \otimes a_i) = \\ a_0\phi(a_1 \otimes \cdots \otimes a_i) + \sum_{0 \le j \le i-1} (-1)^{j+1}\phi(a_0 \otimes \cdots \otimes a_{j-1} \otimes a_j a_{j+1} \otimes a_{j+2} \otimes \cdots \otimes a_i) \\ + (-1)^{i+1}\phi(a_0 \otimes \cdots \otimes a_{i-1})a_i \,.\end{aligned}$$

We identify A with $\{0\text{-cochains}\}$. Then

$$\mathrm{HH}^i(A) := \mathrm{Ker}\, \delta^*_{i+1}/\mathrm{Im}\ \delta^*_i \ .$$

We are interested in other concrete realizations of Hochschild cohomology giving isomorphic cohomology groups. Formally, we view any k-algebra A as a bimodule over itself, i.e., a right A^e-module where A^e is its enveloping algebra, $A \otimes A^{op}$, for A^{op} the opposite algebra of A. The Hochschild cohomology of A is then just

$$\mathrm{HH}^\bullet(A) = \mathrm{Ext}^\bullet_{A^e}(A, A).$$

This cohomology is often computed using the A-bimodule bar resolution of A:

$$\cdots \longrightarrow A^{\otimes 4} \xrightarrow{\delta_2} A^{\otimes 3} \xrightarrow{\delta_1} A^{\otimes 2} \xrightarrow{\delta_0} A \longrightarrow 0, \tag{5.3}$$

where δ_0 is the multiplication in A, and, for each $i \ge 1$,

$$\delta_i(a_0 \otimes \cdots \otimes a_{i+1}) = \sum_{j=0}^{i} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1}$$

for $a_0, \ldots, a_{i+1}$ in A. We take the homology of this complex after dropping the initial term A and applying $\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(-, A)$ to obtain the above description of Hochschild cohomology in terms of Hochschild cocycles and coboundaries, using the identification

$$\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A \otimes A^{\otimes i} \otimes A, A) \cong \mathrm{Hom}_k(A^{\otimes i}, A).$$

6. Koszul algebras

We wish to extend the original PBW Theorem for universal enveloping algebras to other nonhomogeneous quadratic algebras. When is a given algebra a PBW deformation of another well-understood and well-behaved algebra? Can we replace the polynomial algebra in the original PBW theorem by any homogeneous quadratic algebra, provided it is well-behaved in some way? We turn to Koszul algebras as a wide class of quadratic algebras generalizing the class of polynomial algebras. In this section, we briefly recall the definition of a Koszul algebra.

6.1. **Koszul complex.** The algebra S is a *Koszul algebra* if the underlying field k admits a linear S-free resolution, i.e., one with boundary maps given by matrices whose entries are linear forms. Equivalently, S is a Koszul algebra if the following complex of left S-modules is acyclic:

$$\cdots \longrightarrow K_3(S) \longrightarrow K_2(S) \longrightarrow K_1(S) \longrightarrow K_0(S) \longrightarrow k \longrightarrow 0 \tag{6.1}$$

where $K_0(S) = S$, $K_1(S) = S \otimes V$, $K_2(S) = S \otimes R$, and for $i \geq 3$,

$$K_i(S) = S \otimes \left(\bigcap_{j=0}^{i-2} V^{\otimes j} \otimes R \otimes V^{\otimes (i-2-j)} \right).$$

The differential is that inherited from the bar resolution of k as an S-bimodule,

$$\cdots \xrightarrow{\partial_4} S^{\otimes 4} \xrightarrow{\partial_3} S^{\otimes 3} \xrightarrow{\partial_2} S^{\otimes 2} \xrightarrow{\partial_1} S \xrightarrow{\varepsilon} k \longrightarrow 0, \tag{6.2}$$

where ε is the augmentation ($\varepsilon(v) = 0$ for all v in V) and for each $i \geq 1$,

$$\partial_i(s_0 \otimes \cdots \otimes s_i) = (-1)^i \varepsilon(s_i) s_0 \otimes \cdots \otimes s_{i-1} + \sum_{j=0}^{i-1} (-1)^j s_0 \otimes \cdots \otimes s_j s_{j+1} \otimes \cdots \otimes s_i.$$

(Note that for each i, $K_i(S)$ is an S-submodule of $S^{\otimes(i+1)}$.)

Bimodule Koszul complex. Braverman and Gaitsgory gave an equivalent definition of Koszul algebra via the bimodule Koszul complex: Let

$$\widetilde{K}_i(S) = K_i(S) \otimes S, \tag{6.3}$$

an S^e-module (equivalently S-bimodule) where $S^e = S \otimes S^{op}$. Then $\widetilde{K}_\bullet(S)$ embeds into the bimodule bar resolution (5.3) whose i-th term is $S^{\otimes(i+2)}$, and S is Koszul if and only if $\widetilde{K}_\bullet(S)$ is a bimodule resolution of S. Thus we may obtain the Hochschild cohomology $\mathrm{HH}^\bullet(S)$ of S (which contains information about its deformations) by applying $\mathrm{Hom}_{S^e}(-, S)$ either to the Koszul resolution $\widetilde{K}_\bullet(S)$ or to the bar resolution (5.3) of S as an S^e-module (after dropping the initial nonzero terms of each) and taking homology. We see in the next section how these resolutions and the resulting cohomology are used in homological proofs of a generalization of the PBW Theorem from [15, 77, 78].

7. Homological methods and deformations of Koszul algebras

Polishchuk and Positselski [77, 78] and Braverman and Gaitsgory [15] extended the idea of the original PBW Theorem for universal enveloping algebras to other nonhomogeneous quadratic algebras by replacing the polynomial algebra in the theorem by an arbitrary Koszul algebra. They stated conditions for a version of the original PBW Theorem to hold in this greater generality and gave homological proofs. (Polishchuk and Positselski [77] in fact gave two proofs, one homological that goes back to Positselski [78] and another using distributive lattices.) We briefly summarize these two homological approaches in this section and discuss generalizations.

Theorem of Polishchuk and Positselski, Braverman and Gaitsgory. As in the last sections, let V be a finite dimensional vector space over a field k and let T be its tensor algebra over k with i-th filtered component $F^i(T)$. Consider a subspace P of $F^2(T)$ defining a nonhomogeneous quadratic algebra

$$A = T\big/(P)\ .$$

Let $R = \mathrm{LH}(P) \cap T^2$ be the projection of P onto the homogeneous component of degree 2, and set

$$S = T\big/(R)\,,$$

a homogeneous quadratic algebra (the homogeneous version of A as in Section 4). Then A is a PBW deformation of S when $\mathrm{gr}\, A$ and S are isomorphic as graded algebras.

Braverman and Gaitsgory and also Polishchuk and Positselski gave a generalization of the PBW Theorem [15, 77, 78] as follows:

Theorem 7.1. *Let A be a nonhomogeneous quadratic algebra, $A = T/(P)$, and $S = T/(R)$ its corresponding homogeneous quadratic algebra. Suppose S is a Koszul algebra. Then A is a PBW deformation of S if, and only if, the following two conditions hold:*
(I) $P \cap F^1(T) = \{0\}$, *and*
(J) $(F^1(T) \cdot P \cdot F^1(T)) \cap F^2(T) = P$.

We have chosen the notation of Braverman and Gaitsgory. The necessity of conditions (I) and (J) can be seen by direct algebraic manipulations. Similarly, direct computation shows that if (I) holds, then (J) is equivalent to (i), (ii), and (iii) of Theorem 7.2 below. Braverman and Gaitsgory used algebraic deformation theory to show that these conditions are also sufficient. Polishchuk and Positselski used properties of an explicit complex defined using the Koszul dual of S. The conditions (i), (ii), (iii) facilitate these connections to homological algebra, and they are easier in practice to verify than checking (J) directly. But in order to state these conditions, we require a canonical decomposition for elements of P: Condition (I) of Theorem 7.1 implies that every element of P can be written as the sum of a nonzero element of R (of degree 2), a linear term, and a constant term, i.e., there exist linear functions $\alpha : R \to V$, $\beta : R \to k$ for which

$$P = \{r - \alpha(r) - \beta(r) \mid r \in R\}.$$

One may then rewrite Condition (J) and reformulate Theorem 7.1 as follows.

Theorem 7.2. *Let A be a nonhomogeneous quadratic algebra, $A = T/(P)$, and $S = T/(R)$ its corresponding homogeneous quadratic algebra. Suppose S is a Koszul algebra. Then A is a PBW deformation of S if, and only if, the following conditions hold:*

(I) $P \cap F^1(T) = \{0\}$,
(i) $\mathrm{Im}(\alpha \otimes \mathrm{id} - \mathrm{id} \otimes \alpha) \subseteq R$,
(ii) $\alpha \circ (\alpha \otimes \mathrm{id} - \mathrm{id} \otimes \alpha) = -(\beta \otimes \mathrm{id} - \mathrm{id} \otimes \beta)$,
(iii) $\beta \circ (\alpha \otimes \mathrm{id} - \mathrm{id} \otimes \alpha) = 0$,

where the maps $\alpha \otimes \mathrm{id} - \mathrm{id} \otimes \alpha$ and $\beta \otimes \mathrm{id} - \mathrm{id} \otimes \beta$ are defined on the subspace $(R \otimes V) \cap (V \otimes R)$ of T.

We explain next how the original PBW Theorem is a consequence of Theorem 7.2. Indeed, Polishchuk and Positselski [77, Chapter 5, Sections 1 and 2] described the "self-consistency conditions" (i), (ii), and (iii) of the theorem as generalizing the Jacobi identity for Lie brackets.

Example 7.3. Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra, $A = U(\mathfrak{g})$ its universal enveloping algebra, and $S = S(\mathfrak{g})$. Then R has $\mathbb{C}$-basis all $v \otimes w - w \otimes v$ for v, w in V, and $\alpha(v \otimes w - w \otimes v) = [v, w]$, $\beta \equiv 0$. Condition (I) is equivalent to antisymmetry of the bracket. Condition (J) is equivalent to the Jacobi identity, with (i), (ii) expressing the condition separately in each degree in the tensor algebra ($\beta \equiv 0$ in this case). More generally, there are examples with $\beta \not\equiv 0$, for instance, the Sridharan enveloping algebras [92].

Homological proofs. We now explain how Braverman and Gaitsgory and Polishchuk and Positselski used algebraic deformation theory and Hochschild cohomology to prove that the conditions of Theorem 7.2 are sufficient. Braverman and Gaitsgory constructed a graded deformation S_t interpolating between S and A (i.e., with $S = S_t|_{t=0}$ and $A = S_t|_{t=1}$), implying that $\mathrm{gr}(A) \cong S$ as graded algebras. They constructed the deformation S_t as follows.

- They identified α with a choice of first multiplication map μ_1 and β with a choice of second multiplication map μ_2, via the canonical embedding of the bimodule Koszul resolution (6.3) into the bar resolution (5.3) of S. (In order to do this, one must extend α, β (respectively, μ_1, μ_2) to be maps on a larger space via an isomorphism $\mathrm{Hom}_k(R, S) \cong \mathrm{Hom}_{S^e}(S \otimes R \otimes S, S)$ (respectively, $\mathrm{Hom}_k(S \otimes S, S) \cong \mathrm{Hom}_{S^e}(S^{\otimes 4}, S)$.)

- Condition (i) is then seen to be equivalent to μ_1 being a Hochschild 2-cocycle (i.e., satisfies Equation (5.2)).
- Condition (ii) is equivalent to the vanishing of the first obstruction.
- Condition (iii) is equivalent to the vanishing of the second obstruction.
- All other obstructions vanish automatically for a Koszul algebra due to the structure of its Hochschild cohomology (see [15]).
- Thus there exist maps μ_i for $i > 2$ defining an associative multiplication $*$ (as in Equation (5.1)) on $S[t]$.

Positselski [78, Theorem 3.3] (see also [77, Proposition 5.7.2]) gave a different homological proof of Theorem 7.2. Let B be the Koszul dual $S^! := \mathrm{Ext}^*_S(k,k)$ of S. Then $S \cong B^! := \mathrm{Ext}^*_B(k,k)$. Polishchuk defined a complex whose terms are the same as those in the bar resolution of B but with boundary maps modified using the functions $\alpha : R \to V$, $\beta : R \to k$ by first identifying β with an element h of B^2 and α with a dual to a derivation d on B. The conditions (i), (ii), and (iii) on α, β correspond to conditions on d, h, under which Positselski called B a CDG-algebra. The idea is that CDG-algebra structures on B are dual to PBW deformations of S. Positselski's proof relies on the Koszul property of S (equivalently of B) to imply collapsing of a spectral sequence with $E^2_{p,q} = \mathrm{Ext}^{-q,p}_B(k,k)$. The sequence converges to the homology of the original complex for B. Koszulness implies the only nonzero terms occur when $p + q = 0$, and we are left with the homology of the total complex in degree 0. By its definition this is simply the algebra A, and it follows that $\mathrm{gr}\, A \cong B^! \cong S$.

Generalizations and extensions. Theorem 7.2 describes nonhomogeneous quadratic algebras whose quadratic versions are Koszul. What if one replaces the underlying field by an arbitrary ring? Etingof and Ginzburg [33] noted that Braverman and Gaitsgory's proof of Theorem 7.2 is in fact valid more generally for Koszul rings over semisimple subrings as defined by Beilinson, Ginzburg, and Soergel [6]. They chose their semisimple subring to be the complex group algebra $\mathbb{C}G$ of a finite group G acting symplectically and their Koszul ring to be a polynomial algebra $S(V)$. They were interested in the case $\alpha \equiv 0$ for their applications to symplectic reflection algebras (outlined in Section 10 below). Halbout, Oudom, and Tang [49] state a generalization of Theorem 7.2 in this setting that allows nonzero α (i.e., allows relations defining the algebra A to set commutators of vectors in V to a combination of group algebra elements and vectors). A proof using the Koszul ring theory of Beilinson, Ginzburg, and Soergel and the results of Braverman and Gaitsgory is outlined in our paper [86] for arbitrary group algebras over the complex numbers. We also included a second proof there for group algebras over arbitrary fields (of characteristic not 2) using the Composition-Diamond Lemma (described in the next section), which has the advantage that it is characteristic free. We adapted the program of Braverman and Gaitsgory to arbitrary nonhomogeneous quadratic algebras and Koszul rings defined over non-semisimple rings in [88], including group rings kG where the characteristic of k divides the order of the group G.

The theory of Braverman and Gaitsgory was further generalized to algebras that are N-Koszul (all relations homogeneous of degree N plus a homological condition) over semisimple or von Neumann regular rings by a number of authors (see [8, 34, 53]). Cassidy and Shelton [22] generalized the theory of Braverman and Gaitsgory in a different direction, to graded algebras over a field satisfying a particular homological finiteness condition (not necessarily having all relations in a single fixed degree).

8. The Composition-Diamond Lemma and Gröbner basis theory

PBW theorems are often proven using diamond or composition lemmas and the theory of (noncommutative) Gröbner bases. Diamond lemmas predict existence of a canonical normal form in a mathematical system. Often one is presented with various ways of simplifying an element to obtain a normal form. If two different ways of rewriting the original element result in the same desired reduced expression, one is reminded of diverging paths meeting like the sides of the shape of a diamond. Diamond lemmas often originate from Newman's Lemma [74] for graph theory. Shirshov (see [90] and [91]) gave a general version for Lie polynomials in 1962 which Bokut' (see [11] and [12]) extended to associative algebras in 1976, using the term "Composition Lemma." Around the same time (Bokut' cites a preprint by Bergman), Bergman [9] developed a similar result which he instead called the Diamond Lemma.

Both the Diamond Lemma and Composition Lemma are easy to explain but difficult to state precisely without the formalism absorbed by Gröbner basis theory. In fact, the level of rigor necessary to carefully state and prove these results can be the subject of debate. Bergman himself writes that the lemma "has been considered obvious and used freely by some ring-theorists... but others seem unaware of it and write out tortuous verifications." (Some authors are reminded of life in a lunatic asylum (see [52]) when making the basic idea rigorous.) We leave careful definitions to any one of numerous texts (for example, see [3], [18], or [65]) and instead present the intuitive idea behind the result developed by Shirshov, Bokut', and Bergman.

The Result of Bokut' (and Shirshov). We first give the original result of Bokut' (see [11, Proposition 1 and Corollary 1]), who used a degree-lexicographical monomial ordering (also see [13]).

Original Composition Lemma . *Suppose a set of relations P defining a k-algebra A is "closed under composition." Then the set of monomials that do not contain the leading monomial of any element of P as a subword is a k-basis of A.*

Before explaining the notion of "closed under composition," we rephrase the results of Bokut' in modern language using Gröbner bases to give a PBW-like basis as in Section 3 (see [45], or [73], for example). Fix a monomial ordering on a free k-algebra T and again write $LM(p)$ for the leading monomial of any p in T. We include the converse of the lemma which can be deduced from the work of Shirshov and Bokut' and was given explicitly by Bergman, who used arbitrary monomial orderings.

Gröbner basis version of Composition Lemma. *The set P is a (noncommutative) Gröbner basis of the ideal I it generates if and only if*

$$\mathcal{B}_P = \{\textit{monomials } m \textit{ in } T: \ m \textit{ not divisible by any } LM(p), \ p \in P\}$$

is a k-basis for the algebra $A = T/I$.

Example 8.1. Let A be the k-algebra generated by symbols x and y and relations $xy = x$ and $yx = y$ (Example 4.1):

$$A = {}^{\textstyle k\langle x, y\rangle}\big/_{\textstyle (xy - x, yx - y)} \,.$$

Let P be the set of defining relations, $P = \{xy - x, yx - y\}$, and consider the degree-lexicographical monomial ordering with $x > y$. Then P is *not* a Gröbner basis of the ideal it generates since $x^2 - x = x(yx - y) - (xy - x)(x - 1)$ lies in the ideal (P) and has leading monomial x^2, which does not lie in the ideal generated by the leading monomials of the elements of P. Indeed, $\mathcal{B}_P$ contains both x^2 and x and hence can not be a basis for A. We set $P' = \{xy - x, yx - y, x^2 - x, y^2 - y\}$ to obtain a Gröbner basis of (P). Then $\mathcal{B}_{P'} = \{\text{monomials } m : m \text{ not divisible by } xy, yx, x^2, y^2\}$ is a k-basis for the algebra A.

Resolving ambiguities. Bergman focused on the problem of resolving ambiguities that arise when trying to rewrite elements of an algebra using different defining relations. Consider a k-algebra A defined by a finite set of generators and a finite set of relations

$$m_1 = f_1, \ m_2 = f_2, \ \ldots, \ m_k = f_k \,,$$

where the m_i are monomials (in the set of generators of A) and the f_i are linear combinations of monomials. Suppose we prefer the right side of our relations and try to eradicate the m_i whenever possible in writing the elements of A in terms of its generators. Can we define the notion of a canonical form for every element of A by deciding to replace each m_i by f_i whenever possible? We say an expression for an element of A is *reduced* if no m_i appears (as a subword anywhere), i.e., when no further replacements using the defining relations of A are possible. The idea of a *canonical form* for A then makes sense if the set of reduced expressions is a k-basis for A, i.e., if every element can be written uniquely in reduced form.

A natural ambiguity arises: If a monomial m contains both m_1 and m_2 as (overlapping) subwords, do we "reduce" first m_1 to f_1 or rather first m_2 to f_2 by replacing? (In the last example, the word xyx contains overlapping subwords xy and yx.) If the order of application of the two relations does not matter and we end up with the same reduced expression, then we say the *(overlap) ambiguity was resolvable*. The Composition-Diamond Lemma states that *knowing certain ambiguities resolve is enough to conclude that a canonical normal form exists for all elements in the algebra*.

Example 8.2. Again, let A be the k-algebra generated by symbols x and y and relations $xy = x$ and $yx = y$ (Example 4.1). We decide to eradicate xy and yx whenever possible in expressions for the elements of A using just the defining relations. On one hand, we may reduce xyx to x^2 (using the first relation); on the other hand, we may reduce xyx to xy (using the second relation) then to x (using the first relation). The words x and x^2 can not be reduced further using just the defining relations, so we consider them both "reduced". Yet they represent the same element xyx of A. Thus, a canonical "reduced" form does not make sense given this choice of defining relations for the algebra A.

The result of Bergman. One makes the above notions precise by introducing a monomial ordering and giving formal definitions for ambiguities, reduction, rewriting procedures, resolving, etc. We consider the quotient algebra $A = T/(P)$ where T (a tensor algebra) is the free k-algebra on the generators of A and P is a (say) finite set of generating relations. We single out a monomial m_i in each generating relation, writing

$$P = \{m_i - f_i : 1 \leq i \leq k\} ,$$

and choose a monomial ordering so that m_i is the leading monomial of each $m_i - f_i$ (assuming such an ordering exists). Then the reduced elements are exactly those spanned by $\mathcal{B}_P$. If all the ambiguities among elements of P are resolvable, we obtain a PBW-like basis, but Bokut' and Bergman give a condition that is easier to check. Instead of choosing to replace monomial m_1 by f_1 or monomial m_2 by f_2 when they both appear as subwords of a monomial m, we make *both* replacements separately and take the difference. If we can express this difference as a linear combination of elements p in the ideal (P) with $LM(p) < m$, then we say the ambiguity was resolvable *relative to the ordering*. (Bokut' used "closed under composition" to describe this condition along with minimality of P.) See [9, Theorem 1.2].

Diamond Lemma idea. *The following are equivalent:*

- *The set of reduced words is a k-basis of $T/(P)$.*
- *All ambiguities among elements of P are resolvable.*

- *All ambiguities among elements of P are resolvable relative to the ordering.*
- *Every element in (P) can be reduced to zero in $T/(P)$ by just using the relations in P.*

In essence, the lemma says that if the generating set of relations P is well-behaved with respect to some monomial ordering, then one can define a canonical form just by checking that nothing goes wrong with the set P instead of checking for problems implied by the whole ideal (P). Thus, resolving ambiguities is just another way of testing for a Gröbner basis (see [45]): The set P is a Gröbner basis for the ideal (P) if and only if all ambiguities among elements of P are resolvable.

Applications. Although the idea of the Composition-Diamond lemma can be phrased in many ways, the hypothesis to be checked in the various versions of the lemma requires very similar computations in application. One finds precursors of the ideas underlying the Composition-Diamond Lemma in the original proofs given by Poincaré, Birkhoff, and Witt of the PBW Theorem for universal enveloping algebras of Lie algebras. These techniques and computations have been used in a number of other settings. For example, explicit PBW conditions are given for Drinfeld Hecke algebras (which include symplectic reflection algebras) by Ram and Shepler [80]; see Section 10. In [86], we studied the general case of algebras defined by relations which set commutators to lower order terms using both a homological approach and the Composition-Diamond Lemma (as it holds in arbitrary characteristic). These algebras, called *Drinfeld orbifold algebras*, include Sridharan enveloping algebras, Drinfeld Hecke algebras, enveloping algebras of Lie algebras, Weyl algebras, and twists of these algebras with group actions. Gröbner bases were used explicitly by Levandovskky and Shepler [63] in replacing a commutative polynomial algebra by a skew (or quantum) polynomial algebra in the theory of Drinfeld Hecke algebras. Bergman's Diamond Lemma was used by Khare [61] to generalize the Drinfeld Hecke algebras of Section 10 from the setting of group actions to that of algebra actions.

Of course the Composition-Diamond Lemma and Gröbner-Shirshov bases have been used to explore many different kinds of algebras (and in particular to find PBW-like bases) that we will not discuss here. See Bokut' and Kukin [13] and Bokut' and Chen [12] for many such examples.

Note that some authors prove PBW theorems by creating a space upon which the algebra in question acts (see, e.g., Humphreys [55] or Griffeth [46, first version]). Showing that the given space is actually a module for the algebra requires checking certain relations that are similar to the conditions that one must check before invoking the Composition-Diamond Lemma.

9. Drinfeld-Jimbo quantum groups and related Hopf algebras

Quantized enveloping algebras (that is, Drinfeld-Jimbo quantum groups [30, 58]) are deformations of universal enveloping algebras of Lie algebras. (Technically, they are bialgebra deformations rather than algebra deformations.) Many mathematicians discovered PBW bases for these algebras, in particular Lusztig [68, 69, 70] in a very general setting and DeConcini and Kac [27] by defining a corresponding algebra filtration. Although there are many incarnations of these algebras, we restrict ourselves to the simply-laced case and to algebras over the complex numbers for ease of notation. We state a PBW theorem in this context and refer the reader to the literature for more general statements (see, e.g., [70]).

Quantum groups. Let $\mathfrak{g}$ be a finite dimensional semisimple complex Lie algebra of rank n with symmetric Cartan matrix (a_{ij}). Let q be a nonzero complex number, $q \neq \pm 1$. (Often q is taken to be an indeterminate instead.) The *quantum group* $U_q(\mathfrak{g})$ is the associative $\mathbb{C}$-algebra defined by generators

$$E_1, \ldots, E_n, F_1, \ldots, F_n, K_1^{\pm 1}, \ldots, K_n^{\pm 1}$$

and relations

$$\begin{aligned} K_i^{\pm 1} K_j^{\pm 1} = K_j^{\pm 1} K_i^{\pm 1}, &\qquad K_i K_i^{-1} = 1 = K_i^{-1} K_i, \\ K_i E_j = q^{a_{ij}} E_j K_i, &\qquad K_i F_j = q^{-a_{ij}} F_j K_i, \end{aligned}$$

$$\begin{array}{rcl} E_i F_j - F_j E_i &=& \delta_{ij} \dfrac{K_i - K_i^{-1}}{q - q^{-1}}, \\ E_i^2 E_j - (q+q^{-1}) E_i E_j E_i + E_j E_i^2 &=& 0 \quad \text{if } a_{ij} = -1, \quad E_i E_j = E_j E_i \quad \text{if } a_{ij} = 0, \\ F_i^2 F_j - (q+q^{-1}) F_i F_j F_i + F_j F_i^2 &=& 0 \quad \text{if } a_{ij} = -1, \quad F_i F_j = F_j F_i \quad \text{if } a_{ij} = 0. \end{array}$$

The last two sets of relations are called the quantum Serre relations.

Let W be the Weyl group of $\mathfrak{g}$. Fix a reduced expression of the longest element w_0 of W. This choice yields a total order on the set Φ^+ of positive roots, $\beta_1, \ldots, \beta_m$. To each $\alpha \in \Phi^+$, Lusztig [68, 69, 70] assigned an element E_α (respectively, F_α) in $U_q(\mathfrak{g})$ determined by this ordering that is an iterated braided commutator of the generators $E_1, \ldots, E_n$ (respectively, $F_1, \ldots, F_n$). These "root vectors" then appear in a PBW basis:

PBW Theorem for Quantum Groups. *There is a basis of $U_q(\mathfrak{g})$ given by*

$$\{E_{\beta_1}^{a_1} \cdots E_{\beta_m}^{a_m} K_1^{b_1} \cdots K_n^{b_n} F_{\beta_1}^{c_1} \cdots F_{\beta_m}^{c_m} : a_i, c_i \geq 0,\ b_i \in \mathbb{Z}\}.$$

Moreover, there is a filtration on the subalgebra $U_q^{>0}(\mathfrak{g})$ (respectively, $U_q^{<0}(\mathfrak{g})$) generated by $E_1, \ldots, E_n$ (respectively, $F_1, \ldots, F_n$) for which the associated graded algebra is isomorphic to a skew polynomial ring.

The skew polynomial ring to which the theorem refers is generated by the images of the E_α (respectively, F_α), with relations $E_\alpha E_\beta = q_{\alpha\beta} E_\beta E_\alpha$ (respectively, $F_\alpha F_\beta = q_{\alpha\beta} F_\beta F_\alpha$) where each $q_{\alpha\beta}$ is a scalar determined by q and by α, β in Φ^+.

Example 9.1. The algebra $U_q^{>0}(\mathfrak{sl}_3)$ is generated by E_1, E_2. Let

$$E_{12} := E_1E_2 - qE_2E_1.$$

Then, as a consequence of the quantum Serre relations, $E_1E_{12} = q^{-1}E_{12}E_1$ and $E_{12}E_2 = q^{-1}E_2E_{12}$, and, by definition of E_{12}, we also have $E_1E_2 = qE_2E_1 + E_{12}$. In the associated graded algebra, this last relation becomes $E_1E_2 = qE_2E_1$. The algebras $U_q^{>0}(\mathfrak{sl}_n)$ are similar, however in general the filtration on $U_q^{>0}(\mathfrak{g})$ stated in the theorem is more complicated.

Proofs and related results. There are several proofs in the literature of the first statement of the above theorem and related results, beginning with Khoroshkin and Tolstoy [62], Lusztig [68, 69, 70], Takeuchi [93], and Yamane [98]. These generally involve explicit computations facilitated by representation theory. Specifically, one obtains representations of $U_q(\mathfrak{g})$ from those of the corresponding Lie algebra $\mathfrak{g}$ by deformation, and one then uses what is known in the classical setting to obtain information about $U_q(\mathfrak{g})$. Ringel [81] gave a different approach via Hall algebras. The filtration and structure of the associated graded algebra of $U^{>0}(\mathfrak{g})$ was first given by DeConcini and Kac [27]. For a general "quantum PBW Theorem" that applies to some of these algebras, see Berger [7].

In case q is a root of unity (of order ℓ), there are finite dimensional versions of Drinfeld-Jimbo quantum groups. The *small quantum group* $u_q(\mathfrak{g})$ may be defined as the quotient of $U_q(\mathfrak{g})$ by the ideal generated by all $E_\alpha^\ell, F_\alpha^\ell, K_\alpha^\ell - 1$. This finite dimensional algebra has k-basis given by elements in the PBW basis of the above theorem for which $0 \le a_i, b_i, c_i < \ell$.

The existence of PBW bases for $U_q(\mathfrak{g})$ and $u_q(\mathfrak{g})$ plays a crucial role in their representation theory, just as it does in the classical setting of Lie algebras. Bases of finite dimensional simple modules and other modules are defined from weight vectors and PBW bases [69]. R-matrices may be expressed in terms of PBW basis elements [30, 58, 82]. Computations of cohomology take advantage of the structure provided by the PBW basis and filtration (e.g., see [39], based on techniques developed for restricted Lie algebras [35]).

More generally, PBW bases and some Lie-theoretic structure appear in a much larger class of Hopf algebras. Efforts to understand finite dimensional Hopf algebras of various types led in particular to a study of those arising from underlying Nichols algebras. Consequently, a classification of some types of pointed Hopf algebras was completed by Andruskiewitsch and Schneider [1], Heckenberger [51], and Rosso [83]. A Nichols algebra is a "braided" graded Hopf algebra that is connected, generated by its degree 1 elements, and whose subspace of primitive elements is precisely its degree 1 component. The simplest Nichols algebras are those of "diagonal type," and these underlie the Drinfeld-Jimbo quantum groups and the Hopf algebras in the above-mentioned classification. These algebras have

PBW bases just as does $U_q^{>0}(\mathfrak{g})$ or $u_q^{>0}(\mathfrak{g})$; a proof given by Kharchenko [60] uses a combinatorial approach such as in Section 8.

10. Symplectic reflection algebras, rational Cherednik algebras, and graded (Drinfeld) Hecke algebras

Drinfeld [29] and Lusztig [66, 67] originally defined the algebras now variously called symplectic reflection algebras, rational Cherednik algebras, and graded (Drinfeld) Hecke algebras, depending on context. These are PBW deformations of group extensions of polynomial rings (skew group algebras) defined by relations that set commutators of vectors to elements of a group algebra. Lusztig explored the representation theory of these algebras when the acting group is a Weyl group. Crawley-Boevey and Holland [24] considered subgroups of $\mathrm{SL}_2(\mathbb{C})$ and studied subalgebras of these algebras in relation to corresponding orbifolds. Initial work on these types of PBW deformations for arbitrary groups began with Etingof and Ginzburg [33] and Ram and Shepler [80]. Gordon [40] used the rational Cherednik algebra to prove a version of the $n!$-conjecture for Weyl groups and the representation theory of these algebras remains an active area. (See [16], [41], [42], and [84].) We briefly recall and compare these algebras. (See also [25] for a survey of symplectic reflection algebras and rational Cherednik algebras in the context of Hecke algebras and representation theory.)

Let G be a group acting by automorphisms on a k-algebra S. The *skew group algebra* $S\#G$ (also written as a semidirect y product $S \rtimes G$) is the k-vector space $S \otimes kG$ together with multiplication given by $(r \otimes g)(s \otimes h) = r({}^g s) \otimes gh$ for all r, s in S and g, h in G, where ${}^g s$ is the image of s under the automorphism g.

Drinfeld's "Hecke algebra". Suppose G is a finite group acting linearly on a finite dimensional vector space V over $k = \mathbb{C}$ with symmetric algebra $S(V)$. Consider the quotient algebra

$$\mathcal{H}_\kappa = T(V)\#G \big/ (v_1 \otimes v_2 - v_2 \otimes v_1 - \kappa(v_1, v_2) : v_1, v_2 \in V)$$

defined by a bilinear parameter function $\kappa : V \times V \to \mathbb{C}G$. We view $\mathcal{H}_\kappa$ as a filtered algebra by assigning degree one to vectors in V and degree zero to group elements in G. Then the algebra $\mathcal{H}_\kappa$ is a PBW deformation of $S(V)\#G$ if its associated graded algebra is isomorphic to $S(V)\#G$. Drinfeld [29] originally defined these algebras for arbitrary groups, and he also counted the dimension of the parameter space of such PBW deformations for Coxeter groups. For more information and a complete characterization of parameters κ yeilding the PBW property for arbitrary groups, see [33], [80], [85], [86], and [87].

Example 10.1. Let V be a vector space of dimension 3 with basis v_1, v_2, v_3, and let G be the symmetric group S_3 acting on V by permuting the chosen basis elements. The following is a PBW deformation of $S(V)\#G$, where $(i\,j\,k)$ denotes a 3-cycle in S_3:

$$\mathcal{H}_\kappa = {}^{T(V)\#S_3}\big/\big(v_i \otimes v_j - v_j \otimes v_i - (i\,j\,k) + (j\,i\,k) : \{i,j,k\} = \{1,2,3\}\big)\,.$$

Lusztig's graded affine Hecke algebra. While exploring the representation theory of groups of Lie type, Lusztig [66, 67] defined a variant of the affine Hecke algebra for Weyl groups which he called "graded" (as it was obtained from a particular filtration of the affine Hecke algebra). He gave a presentation for this algebra $\mathbb{H}_\lambda$ using the same generators as those for Drinfeld's Hecke algebra $\mathcal{H}_\kappa$, but he gave relations preserving the structure of the polynomial ring and altering the skew group algebra relation. (Drinfeld's relations do the reverse.) The *graded affine Hecke algebra* $\mathbb{H}_\lambda$ (or simply the "graded Hecke algebra") for a finite Coxeter group G acting on a finite dimensional complex vector space V (in its natural reflection representation) is the $\mathbb{C}$-algebra generated by the polynomial algebra $S(V)$ together with the group algebra $\mathbb{C}G$ with relations

$$gv = {}^g v g + \lambda_g(v) g$$

for all v in V and g in a set $\mathcal{S}$ of simple reflections (generating G) where λ_g in V^* defines the reflecting hyperplane ($\ker \lambda_g \subseteq V$) of g and $\lambda_g = \lambda_{hgh^{-1}}$ for all h in G. (Recall that a *reflection* on a finite dimensional vector space is just a nonidentity linear transformation that fixes a hyperplane pointwise.)

Note that for g representing a fixed conjugacy class of reflections, the linear form λ_g is only well-defined up to a nonzero scalar. Thus one often fixes once and for all a choice of linear forms $\lambda = \{\lambda_g\}$ defining the orbits of reflecting hyperplanes (usually expressed using Demazure/BGG operators) and then introduces a formal parameter by which to rescale. This highlights the degree of freedom arising from each orbit; for example, one might replace

$$\lambda_g(v) \quad \text{by} \quad c_g\, \langle v, \alpha_g^\vee \rangle = c_g\, \left(\frac{v - {}^g v}{\alpha_g}\right)$$

for some conjugation invariant formal parameter c_g after fixing a G-invariant inner product and root system $\{\alpha_g : g \in \mathcal{S}\} \subset V$ with coroot vectors $\alpha_g^\vee$. (Note that for any reflection g, the vector $(v - {}^g v)$ is a nonzero scalar multiple of α_g and so the quotient of $v - {}^g v$ by α_g is a scalar.) Each graded affine Hecke algebra $\mathbb{H}_\lambda$ is filtered with vectors in degree one and group elements in degree zero and defines a PBW deformation of $S(V)\#G$.

Finite Group	Any $G \leq \mathrm{GL}(V)$	Coxeter $G \leq \mathrm{GL}(V)$
Algebra	$\mathcal{H}_\kappa$ (Drinfeld)	$\mathbb{H}_\lambda$ (Lusztig)
generated by	V and $\mathbb{C}G$	V and $\mathbb{C}G$
with relations	$gv = {}^g v g$, $vw = wv + \kappa(v, w)$ $(\forall v, w \in V,\ \forall g \in G)$	$gv = {}^g v g + \lambda_g(v) g$, $vw = wv$ $(\forall v, w \in V,\ \forall g \in \mathcal{S})$

Comparing algebras. Ram and Shepler [80] showed that Lusztig's graded affine Hecke algebras are a special case of Drinfeld's construction: For each parameter λ, there is a parameter κ so that the filtered algebras $\mathbb{H}_\lambda$ and $\mathcal{H}_\kappa$ are isomorphic (see [80]). Etingof and Ginzburg [33] rediscovered Drinfeld's algebras with focus on groups G acting symplectically (in the context of orbifold theory). They called algebras $\mathcal{H}_\kappa$ satisfying the PBW property *symplectic reflection algebras*, giving necessary and sufficient conditions on κ. They used the theory of Beilinson, Ginzburg, and Soergel [6] of Koszul rings to generalize Braverman and Gaitsgory's conditions to the setting where the ground field is replaced by the semisimple group ring $\mathbb{C}G$. (The skew group algebra $S(V)\#G$ is Koszul as a ring over the semisimple subring $\mathbb{C}G$.) Ram and Shepler [80] independently gave necessary and sufficient PBW conditions on κ (for arbitrary groups acting linearly over $\mathbb{C}$) and classified all such quotient algebras for complex reflection groups. Their proof relies on the Composition-Diamond Lemma. (See Sections 7 and 8 for a comparison of these two techniques for showing PBW properties.) Both approaches depend on the fact that the underlying field $k = \mathbb{C}$ has characteristic zero (or, more generally, has characteristic that does not divide the order of the group G). See Section 11 for a discussion of PBW theorems in the modular setting when $\mathbb{C}$ is replaced by a field whose characteristic divides $|G|$.

Rational Cherednik algebras. The rational Cherednik algebra is a special case of a quotient algebra $\mathcal{H}_\kappa$ satisfying the PBW property (in fact, a special case of a symplectic reflection algebra) for reflection groups acting diagonally on two copies of their reflection representations ("doubled up"). These algebras are regarded as "doubly degenerate" versions of the double affine Hecke algebra introduced by Cherednik [23] to solve the Macdonald (constant term) conjectures in combinatorics. We simply recall the definition here in terms of reflections and hyperplane arrangements.

Suppose G is a finite group generated by reflections on a finite dimensional complex vector space V. (If G is a Coxeter group, then extend the action to one over the complex numbers.) Then the induced diagonal action of G on $V \oplus V^*$ is generated by *bireflections* (linear transformations that each fix a subspace of codimension 2 pointwise), i.e., by *symplectic reflections* with respect to a natural symplectic form on $V \oplus V^*$.

Let $\mathcal{R}$ be the set of all reflections in G acting on V. For each reflection s in $\mathcal{R}$, let α_s in V and α_s^* in V^* be eigenvectors ("root vectors") each with nonindentity eigenvalue. We define an algebra generated by $\mathbb{C}G$, V, and V^* in which vectors in V commute with each other and vectors in V^* commute with each other, but passing a vector from V over one from V^* gives a linear combination of reflections (and the identity). As parameters, we take a scalar t and a set of scalars $\mathbf{c} = \{c_s : s \in \mathcal{R}\}$ with $c_s = c_{hsh^{-1}}$ for all h in G. The *rational Cherednik algebra* $\mathbf{H}_{t,\mathbf{c}}$ with parameters $t, \mathbf{c}$ is then the $\mathbb{C}$-algebra generated by the vectors in V and V^* together with the group algebra $\mathbb{C}G$ satisfying the relations

$$\begin{aligned} gu &= {}^g u g, \quad uu' = u'u, \\ vv^* &= v^*v + t\, v^*(v) - \sum_{s\in\mathcal{R}} c_s\ \alpha_s^*(v)\ v^*(\alpha_s)\ s \end{aligned}$$

for any g in G, v in V, v^* in V^*, and any u, u' both in V or both in V^*. Note that α_s and α_s^* are only well-defined up to a nonzero scalar, and we make some conjugation invariant choice of normalization in this definition, say, by assuming that $\alpha_s^*(\alpha_s) = 1$. One often replaces $\mathbb{C}$ by $\mathbb{C}[t, \mathbf{c}]$ to work in a formal parameter space.

The relations defining the rational Cherednik algebra are often given in terms of the arrangement of reflecting hyperplanes $\mathcal{A}$ for G acting on V. For each hyperplane H in $\mathcal{A}$, choose a linear form α_H^* in V^* defining H (so $H = \ker \alpha_H^*$) and let α_H be a nonzero vector in V perpendicular to H with respect to some fixed G-invariant inner product. Then the third defining relation of $\mathbf{H}_{t,\mathbf{c}}$ can be rewritten (without a choice of normalization) as

$$vv^* - v^*v = tv^*(v) - \sum_{H\in\mathcal{A}} \frac{\alpha_H^*(v)\ v^*(\alpha_H)}{\alpha_H^*(\alpha_H)} \big(c_{s_H} s_H + c_{s_H^2} s_H^2 + \ldots + c_{s_H^{a_H}} s_H^{a_H}\big)$$

where s_H is the reflection in G about the hyperplane H of maximal order $a_H + 1$. Again, this is usually expressed geometrically in terms of the inner product on V and induced product on V^*:

$$\frac{\alpha_H^*(v)\ v^*(\alpha_H)}{\alpha_H^*(\alpha_H)} = \frac{\langle v, \alpha_H^\vee\rangle \langle \alpha_H, v^*\rangle}{\langle \alpha_H, \alpha^\vee\rangle}\ .$$

The PBW theorem then holds for the algebra $\mathbf{H}_{t,\mathbf{c}}$ (see [33]):

PBW Theorem for Rational Cherednik Algebras. *The rational Cherednik algebra $\mathbf{H}_{t,\mathbf{c}}$ is isomorphic to $S(V) \otimes S(V^*) \otimes \mathbb{C}G$ as a complex vector space for any choices of parameters t and c, and its associated graded algebra is isomorphic to $(S(V) \otimes S(V^*))\#G$.*

Connections between rational Cherednik algebras and other fields of mathematics are growing stronger. For example, Gordon and Griffeth [43] link the Fuss-Catalan numbers in combinatorics to the representation theory of rational Cherednik algebras. These investigations also bring insight to the classical theory of complex reflection groups, especially to the perplexing question of why some reflection groups acting on n-dimensional space can be generated by n reflections (called "well-generated" or "duality" groups) and others not. (See [26, 44, 89] for other recent applications.)

11. Positive characteristic and nonsemisimple ground rings

Algebras displaying PBW properties are quite common over ground fields of positive characteristic and nonsemisimple ground rings, but techniques for establishing PBW theorems are not all equally suited for work over arbitrary fields and rings. We briefly mention a few results of ongoing efforts to establish and apply PBW theorems in these settings.

The algebras of Section 10 make sense in the *modular setting*, that is, when the characteristic of k is a prime dividing the order of the finite group G. In this case, however, the group algebra kG is not semisimple, and one must take more care in proofs. PBW conditions on κ were examined by Griffeth [46] by construction of an explicit $\mathcal{H}_\kappa$-module, as is done in one standard proof of the PBW Theorem for universal enveloping algebras. (See also Bazlov and Berenstein [5] for a generalization.) The Composition-Diamond Lemma, being characteristic free, applies in the modular setting; see our paper [86] for a proof of the PBW property using this lemma that applies to graded (Drinfeld) Hecke algebras over fields of arbitrary characteristic. (Gröbner bases are explicitly used in Levandovskyy and Shepler [63].) Several authors consider representations of rational Cherednik algebras in the modular setting, for example, Balagovic and Chen [4], Griffeth [46], and Norton [75].

The theory of Beilinson, Ginzburg, and Soergel of Koszul rings over semisimple subrings, used in Braverman-Gaitsgory style proofs of PBW theorems, does not apply directly to the modular setting. However it may be adapted using a larger complex replacing the Koszul complex: In [88], we used this approach to generalize the Braverman-Gaitsgory argument to arbitrary Koszul algebras with finite group actions. This replacement complex has an advantage over the Composition-Diamond Lemma or Gröbner basis theory arguments in that it contains information about potentially new types of deformations that do not occur in the nonmodular setting.

Other constructions generalize the algebras of Section 10 to algebras over ground rings that are not necessarily semisimple. Etingof, Gan, and Ginzburg [32] considered deformations of algebras that are extensions of polynomial rings by acting algebraic groups or Lie algebras. They used a Braverman-Gaitsgory approach to obtain a Jacobi condition by realizing the acting algebras as inverse limits of finite dimensional semsimple algebras. Gan and Khare [37] investigated actions of $U_q(\mathfrak{sl}_2)$ on the quantum plane (a skew polynomial algebra), and Khare [61] looked at actions of arbitrary cocommutative algebras on polynomial rings. In both cases PBW theorems were proven using the Composition-Diamond Lemma. A general result for actions of (not necessarily semisimple) Hopf algebras on Koszul algebras is contained in Walton and Witherspoon [96] with a Braverman-Gaitsgory style proof. See also He, Van Oystaeyen, and Zhang [50] for a PBW theorem using a somewhat different complex in a general setting of Koszul rings over not necessarily semisimple ground rings. One expects yet further generalizations and applications of the ubiquitous and potent PBW Theorem.

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