QUANTUM DRINFELD HECKE ALGEBRAS

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Abstract. We consider finite groups acting on quantum (or skew) polynomial rings. Deformations of the semidirect product of the quantum polynomial ring with the acting group extend symplectic reflection algebras and graded Hecke algebras to the quantum setting over a field of arbitrary characteristic. We give necessary and sufficient conditions for such algebras to satisfy a Poincaré-Birkhoff-Witt property using the theory of noncommutative Gröbner bases. We include applications to the case of abelian groups and the case of groups acting on coordinate rings of quantum planes. In addition, we classify graded automorphisms of the coordinate ring of quantum 3-space. In characteristic zero, Hochschild cohomology gives an elegant description of the Poincaré-Birkhoff-Witt conditions.

1. Introduction

Drinfeld Hecke algebras arise in a variety of settings: for example, as symplectic reflection algebras, rational Cherednik algebras, and Lusztig's graded version of the affine Hecke algebra. These algebras (also known as *graded Hecke algebras*) are natural deformations of the skew group algebra (the semi-direct product algebra) formed by a finite group G acting on a polynomial ring over some vector space V. They reflect the geometry of orbifold theory by serving as a noncommutative substitute for the coordinate ring (the ring of invariant polynomials $S(V)^G$) of the orbifold V/G (see Etingof and Ginzburg [13]). These algebras were also used to prove a version of the $n!$ conjecture for Weyl groups (see Gordon [14]).

In this article, we explore analogous deformations of a finite group acting on a quantum polynomial algebra over a field of arbitrary characteristic. Let V be a finite dimensional vector space over a field $\mathbb{K}$. The *quantum polynomial algebra* $S_Q(V)$ of V (also called the *skew polynomial ring*, or the *coordinate ring of multiparameter quantum affine space*) is the associative $\mathbb{K}$-algebra generated by a $\mathbb{K}$-basis $\{v_1, \ldots, v_n\}$ of V subject to the relations

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$v_j v_i = q_{ij} v_i v_j$ for $i < j$ for some (quantum) parameters q_{ij} in $\mathbb{K}^*$:

$$S_Q(V) := \mathbb{K}\langle v_1, \ldots, v_n\rangle \,/\, \langle v_j v_i - q_{ij} v_i v_j : 1 \leq i < j \leq n\rangle \ .$$

We augment the quantum polynomial algebra by a finite group G acting linearly on the vector space V. We introduce relations on the natural semi-direct product algebra $T(V) \rtimes G$ (for $T(V)$ the tensor algebra of V) which set q-commutators of vectors in V to elements in the group algebra. We call the resulting $\mathbb{K}$-algebra a *quantum Drinfeld Hecke algebra* if it satisfies a Poincaré-Birkhoff-Witt (PBW) property (see Definition 2.1).

We appeal to the theory of noncommutative Gröbner bases to investigate PBW properties. Explorations of related algebras often use Bergman's Diamond Lemma [5], a cornerstone of noncommutative Gröbner bases theory. We use Gröbner bases theory here as a rigorous and elegant refinement of Bergman's ideas. This refinement is well-suited to investigating PBW-like properties in a variety of settings. Indeed, the constructive nature of Gröbner bases theory often verifies a PBW-like property by explicitly giving a PBW-like basis; the theory also illuminates the failure of such properties to hold by supplying natural substitutes for PBW-like bases. Note that Gröbner bases theory emphasizes a fixed total well ordering on monomials, providing an expedient approach to non-noetherian algebras. Indeed, a Gröbner basis for a generating set of relations defining an algebra may be finite for one choice of monomial ordering but infinite for another choice; see Example 6.5. Moreover, Gröbner bases theory is algorithmic with several available implementations, providing computational aid to algebraic questions (on, e.g., ideal membership, kernels of algebra and module homomorphisms, and free and projective resolutions).

Although quantum Drinfeld Hecke algebras extend symplectic reflection algebras and graded Hecke algebras to the setting of quantum polynomial rings, our analysis requires tools previously unused in investigating the non-quantum setting. Since we are working over a field of arbitrary characteristic, many methods from the traditional theory of graded Hecke algebras no longer apply. (Note that the original proof of the technique of Braverman and Gaitsgory [7] does not automatically apply in our setting, as the group algebra $\mathbb{K}G$ may fail to be semi-simple; see [32] for an adaptation of the ideas of Braverman and Gaitsgory for arbitrary group algebras, including the modular case when the characteristic of the field $\mathbb{K}$ divides the order of the acting group G.) The set of quantum parameters also prevents us from regarding the algebra parameters as linear functions giving a wide class of uniform relations (see Remark 2.6), and thus we demote traditional linear algebra in favor of the analysis using noncommutative Gröbner bases.

After giving definitions (and examples) in Section 2, we show that every quantum Drinfeld Hecke algebra defines a quantum polynomial algebra upon which the group acts by automorphisms in Section 3. Tools from the theory of noncommutative Gröbner bases theory are given in Sections 4 and 5. In Section 6, we recall how a Gröbner basis may be used to find a monomial $\mathbb{K}$-basis for any quotient of a free algebra by one of its ideals. We also discuss general quotient algebras and associated graded algebra. (Some elementary algebraic properties of quantum Drinfeld Hecke algebras are also observed in this section.) We apply this theory in Section 7 to prove necessary and sufficient conditions for a factor algebra to define a quantum Drinfeld Hecke algebra. In Section 8, we describe all quantum Drinfeld Hecke algebras arising from an abelian group (acting diagonally). We relate the Poincaré-Birkhoff-Witt condition for quantum Drinfeld Hecke algebras to results in Hochschild cohomology and deformation theory by Naidu and Witherspoon [29] in Section 9.

In Section 10, we discuss groups which act as automorphisms on the coordinate ring of a quantum plane and classify all quantum Drinfeld Hecke algebras in two dimensions. We describe the automorphism group of the coordinate ring of quantum 3-space in Section 11. (We discuss the cases when quantum parameters are roots-of-unity explicitly.) Lastly, in Section 12, we demonstrate how to determine the complete set of quantum Drinfeld Hecke algebras associated to one fixed (nonabelian) group with a robust example.

2. Quantum Drinfeld Hecke Algebras

Let $Q = (q_{ij} \mid 1 \le i, j \le n)$ be a collection of arbitrary nonzero scalars in $\mathbb{K}$ and consider a finite group $G \subset \mathbb{GL}(V)$. Let $\{t_g \mid g \in G\}$ be a basis of the group algebra $\mathbb{K}G$ and $\{v_1, \dots, v_n\}$ a $\mathbb{K}$-basis of V. Define an associative $\mathbb{K}$-algebra $\mathcal{H}_{Q,\kappa}$ generated by

$$\{v_1, \dots, v_n\} \cup \{t_g \mid g \in G\}$$

subject to the following relations:

(a) $t_g t_h = t_{gh}$ for all g, h in G,
(b) $t_g v = g(v) t_g$ for all g in G and v in V,
(c) $v_j v_i = q_{ij}\ v_i v_j + \kappa(v_i, v_j)$ for $1 \le i, j \le n$,

where each parameter $\kappa(v_i, v_j)$ lies in $\mathbb{K}G$. Write

$$\kappa(v_i, v_j) = \sum_{g \in G} \kappa_g(v_i, v_j) t_g$$

for $\kappa_g(v_i, v_j)$ in $\mathbb{K}$. We identify the identity e of G and t_e of $\mathbb{K}G$ with 1 in $\mathbb{K}$ throughout this article and we set $G^* := G \setminus \{1\}$. We assume 0 lies in $\mathbb{N}$ and take all tensor products over $\mathbb{K}$.

Definition 2.1. We call $\mathcal{H}_{Q,\kappa}$ a *quantum Drinfeld Hecke algebra* if

$$B = \{v_1^{\alpha_1} \dots v_n^{\alpha_n} t_g \mid \alpha_i \in \mathbb{N},\ g \in G\}$$

is a $\mathbb{K}$-basis for $\mathcal{H}_{Q,\kappa}$. We call B the *standard PBW basis* in this case and its elements *quasi-standard monomials.*

One might alternatively call such algebras "quantum graded Hecke algebras" or "skew Drinfeld Hecke algebras". We use the phrase "PBW basis" in analogy with a Poincaré-Birkhoff-Witt basis for universal enveloping algebras of Lie algebras. Note that $\mathcal{H}_{Q,\kappa}$ is a quantum Drinfeld Hecke algebra if and only if its associated graded algebra is isomorphic to a skew group algebra $S_Q(V)\#G$ (see Sections 4 and 6).

The braided Cherednik algebras of Bazlov and Berenstein [4] are special cases of quantum Drinfeld Hecke algebras. If we set each $q_{ij} = 1$ in the above construction of $\mathcal{H}_{Q,\kappa}$ (and work over a field $\mathbb{K}$ of characteristic zero), we recover the classical (non-quantum) theory of *graded Hecke algebras*, also called *Drinfeld Hecke algebras* (see [18], for example), which include symplectic reflection algebras and rational Cherednik algebras. These algebras were first defined by Drinfeld [12] for arbitrary finite groups G. They were independently discovered and explored by Lusztig around the same time (see [25, 26]) as graded versions of the affine Hecke algebra in the special case that G is a Weyl group. (See [31] for basic properties of these algebras and an argument that Lusztig's algebras can be realized using Drinfeld's construction.) Etingof and Ginzburg [13] later rediscovered these algebras (from a viewpoint of symplectic geometry and orbifold theory) for G acting symplectically. We give some other examples with fixed quantum system of parameters.

Definition 2.2. A matrix $Q = (q_{ij} \mid 1 \le i, j \le n)$ with entries in $\mathbb{K}^*$ is a *quantum system of parameters* if $q_{ij} = q_{ji}^{-1}$ and $q_{ii} = 1$ for any i, j.

Example 2.3. Set $\kappa \equiv 0$, $G = 1$, and let Q be a quantum system of parameters. Then the factor algebra $\mathcal{H}_{Q,\kappa}$ is just the quantum polynomial algebra $S_Q(V)$.

Example 2.4. Again, let $\kappa \equiv 0$, $G = 1$, and let Q be a quantum system of parameters. Assume that char $\mathbb{K} \neq 2$ and set $-Q = (-q_{ij} \mid 1 \le i, j \le n)$. Then the factor algebra $\mathcal{H}_{-Q,\kappa}$ coincides with the *quantum exterior algebra* $\bigwedge_Q(V)$ of quantum affine space (corresponding to the quantum polynomial algebra $S_Q(V)$) generated over $\mathbb{K}$ by all products $v_{i_1} \wedge \cdots \wedge v_{i_k}$ (for $1 \le k \le n$) with multiplication

$$v_i \wedge v_j = -q_{ji}\ v_j \wedge v_i\ .$$

Although $S_Q(V)$ has the standard PBW basis (e.g., see [9, Example 5.1] or Proposition 6.4), the algebra $\bigwedge_Q(V)$ does not (as each $v_i \wedge v_i = 0$). (In fact, it is easy to see that any quantum Drinfeld Hecke algebra with $\kappa \equiv 0$ is a quantum polynomial algebra, see Proposition 3.5.)

Example 2.5. Let q, ω be roots of unity in $\mathbb{K}$ and let G be the subgroup of $\mathbb{GL}_4(\mathbb{K})$ generated by the diagonal matrix

$$h := \operatorname{diag}(q^2, \omega,\, \omega^{-1}, q^{-2})\,.$$

The $\mathbb{K}$-algebra $\mathcal{H}$ generated by v_1, v_2, v_3, v_4 and t_h with relations

$$\begin{aligned} t_g v_i &= g(v_i) t_g && \text{for } 1 \le i \le n \text{ and } g \text{ in } G, \\ v_i v_j &= q\, v_j v_i && \text{for } (i,j) \ne (2,3), \text{ and} \\ v_2 v_3 &= q\, v_3 v_2 + t_h \end{aligned}$$

is a quantum Drinfeld Hecke algebra.

In Section 8, we describe all quantum Drinfeld Hecke algebras arising from abelian groups acting diagonally. We classify all 2-dimensional quantum Drinfeld Hecke algebras in Section 10. Section 12 gives examples of quantum Drinfeld Hecke algebras arising from a nondiagonal group action.

Remark 2.6. (Bilinear Inextendability) We define parameters q, κ just on pairs of basis elements v_i, v_j, but we could (artificially) extend to functions $q : V \times V \to \mathbb{K}$ and $\kappa : V \times V \to \mathbb{K}G$. This approach is generally not useful for constructing factor algebras like those examined here (although it is helpful in translating results to the setting of cohomology, see Section 9).

For example, suppose we were to extend relation (3) defining the algebra $\mathcal{H}_{Q,\kappa}$ to all pairs v, w in V using a *bilinear* function κ and some function $q : V \times V \to \mathbb{K}$. Then in $\mathcal{H}_{Q,\kappa}$, for any distinct i, j, k,

$$(v_i + v_j)v_k = v_i v_k + v_j v_k = q(v_k, v_i)v_k v_i + q(v_k, v_j)v_k v_j + \kappa(v_k, v_i + v_j)$$

on one hand, while

$$(v_i + v_j)v_k = q(v_k, v_i + v_j)v_k(v_i + v_j) + \kappa(v_k, v_i + v_j)$$

on the other hand, forcing

$$q(v_k, v_i)v_k v_i + q(v_k, v_j)v_k v_j = q(v_k, v_i + v_j)v_k v_i + q(v_k, v_i + v_j)v_k v_j.$$

If $\mathcal{H}_{Q,\kappa}$ has the standard PBW basis, we may equate coefficients:

$$q(v_k, v_i) = q(v_k, v_i + v_j) = q(v_k, v_j).$$

This forces q constant on basis vectors, i.e., $q_{ij} = c$ for all i, j, for fixed c in $\mathbb{K}$. Note that q bilinear would generally imply that q is the zero function.

3. Quantum Polynomial Algebras, Quantum Determinants, and Skew Group Algebras

We show in this section that every quantum Drinfeld Hecke algebra defines a quantum polynomial algebra carrying an action of the group by automorphisms. We first give an easy lemma describing automorphisms of quantum polynomial algebras in terms of quantum minor determinants. Any automorphism h of the quantum exterior algebra $\bigwedge_Q(V)$ will act on the top degree piece $\mathbb{K}$-span$\{v_1 \wedge \cdots \wedge v_n\}$ by a scalar $\det_Q(h)$ which one might call the *quantum determinant* of h. We extend this idea: If a 2×2 matrix with entries a, b, c, d in K has determinant $ad - bc$, then we define its quantum determinant to be $ad - qbc$ where q is the quantum parameter of a 2-dimensional quantum polynomial ring. We define a quantum minor analogously.

Definition 3.1. For a linear transformation h acting on V via $h(v_j) = \sum_i h_i^j v_i$, we define the *quantum* (i, j, k, l)*-minor determinant* of h as

$$\det_{ijkl}(h) := h_k^i h_\ell^j - q_{ij} h_\ell^i h_k^j.$$

Lemma 3.2. *A transformation h in $\mathbb{GL}(V)$ acts as an automorphism on the quantum polynomial algebra (with quantum system of parameters Q)*

$$S_Q(V) := \mathbb{K}\langle v_1, \ldots, v_n\rangle \,/\, \langle v_j v_i = q_{ij} v_i v_j : 1 \leq i, j \leq n\rangle$$

if and only if

$$\det_{ijk\ell}(h) = -q_{\ell k} \det_{ij\ell k}(h) \qquad \textit{for all } 1 \leq i, j, k, \ell \leq n \ .$$

Proof. We write $h(v_j)h(v_i) - q_{ij} h(v_i)h(v_j) = \sum_{k,\ell} \det_{ij\ell k}(h)\, v_k v_\ell$ and express as a sum of standard monomials:

$$\sum_{k \leq \ell} \det_{ij\ell k}(h)\ v_k v_\ell + \sum_{k > \ell} \det_{ij\ell k}(h)\ v_k v_\ell \ = $$

$$\sum_{k<\ell} (\det_{ij\ell k}(h) + \det_{ijk\ell}(h)\ q_{k\ell})\ v_k v_\ell + \sum_k \det_{ijkk}(h)\ v_k v_k \, .$$

Since the set of standard monomials $\{v_1^{\alpha_1} \cdots v_n^{\alpha_n} : \alpha_i \in \mathbb{N}\}$ is a $\mathbb{K}$-basis of $S_Q(V)$ (see, e.g., [9, Example 5.1] or Corollary 6.4), the last expression vanishes in $S_Q(V)$ exactly when the coefficient of each $v_k v_\ell$ (for $k < \ell$) and of each v_k^2 is zero, yielding the result. $\square$

As an easy consequence (needed later), we observe the following corollary.

Corollary 3.3. A matrix h in $\mathbb{GL}(V)$ acts as an automorphism on $S_Q(V)$ if and only if its transpose acts as an automorphism on $S_Q(V)$.

Definition 3.4. We say that a parameter κ is a *quantum* 2*-form* if κ extends to an element of $\mathrm{Hom}_{\mathbb{K}}(\bigwedge_Q(V), \mathbb{K}G)$, i.e., each κ_g defines an element of

$$\left(\textstyle\bigwedge_Q(V)\right)^* \cong \textstyle\bigwedge_{Q^{-1}}(V^*)$$

where $Q^{-1} = (q_{ij}^{-1} : 1 \leq i, j \leq n)$. In other words, κ is a quantum 2-form exactly when $\kappa(v_i, v_i) = 0$ and $\kappa(v_j, v_i) = -q_{ij}^{-1} \kappa(v_i, v_j)$ for all i, j.

A PBW property on $\mathcal{H}_{Q,\kappa}$ implies an underlying quantum polynomial algebra.

Proposition 3.5. *Let $\mathcal{H}_{Q,\kappa}$ be a quantum Drinfeld Hecke algebra. Then*

- *the parameter κ is a quantum 2-form,*
- *the matrix Q is a quantum system of parameters, and*
- *the group G acts upon the quantum polynomial algebra $S_Q(V)$ by automorphisms.*

Proof. Since $\mathcal{H}_{Q,\kappa}$ exhibits the standard PBW basis, each $q_{ii} = 1$ and each $\kappa(v_i, v_i) = 0$ as $v_i^2 = q_{ii}v_i^2 + \kappa(v_i, v_i)$. In fact, for all i and j,

$$\begin{aligned} v_j v_i &= q_{ij} v_i v_j + \kappa(v_i, v_j) = q_{ij}\big(q_{ji} v_j v_i + \kappa(v_j, v_i)\big) + \kappa(v_i, v_j) \\ &= q_{ij} q_{ji} v_j v_i + q_{ij}\kappa(v_j, v_i) + \kappa(v_i, v_j) \ , \end{aligned}$$

and hence $q_{ij} \neq 0$, $q_{ij} = q_{ji}^{-1}$, and $\kappa(v_j, v_i) = -q_{ij}^{-1}\kappa(v_i, v_j)$. Thus κ is a quantum 2-form and Q defines a quantum system of parameters.

Additionally, for all h in G and $i \neq j$,

$$\begin{aligned} 0 &= (t_h v_j) v_i t_{h^{-1}} - t_h (v_j v_i) t_{h^{-1}} \\ &= h(v_j)(t_h v_i) t_{h^{-1}} - t_h \left(q_{ij} v_i v_j + \sum_{g \in G} \kappa_g(v_i, v_j) t_g \right) t_{h^{-1}} \\ &= h(v_j) h(v_i) - q_{ij} h(v_i) h(v_j) - \sum_{g \in G} \kappa_g(v_i, v_j) t_{hgh^{-1}} \\ &= \sum_{k,\ell} \det\nolimits_{ij\ell k}(h)\ v_k v_\ell - \sum_{g \in G} \kappa_{h^{-1}gh}(v_i, v_j) t_g \ . \end{aligned} \tag{1}$$

We separate the sum of $\det_{ij\ell k}(h)\ v_k v_\ell$ over $k > \ell$ and exchange v_ℓ and v_k to express Equation 1 using only quasi-standard monomials:

$$\begin{aligned} 0 = \quad & \sum_{k<\ell} \Big(q_{k\ell} \det\nolimits_{ijk\ell}(h) + \det\nolimits_{ij\ell k}(h) \Big)\ v_k v_\ell + \sum_k \det\nolimits_{ijkk}(h)\ v_k^2 \\ & \qquad - \sum_{g \in G} \left(\sum_{k<\ell} \det\nolimits_{ijk\ell}(h)\ \kappa_g(v_k, v_\ell) - \kappa_{h^{-1}gh}(v_i, v_j) \right) t_g \ . \end{aligned} \tag{2}$$

Since $\mathcal{H}_{Q,\kappa}$ has the standard PBW basis, the coefficient of each monomial $v_k v_\ell$ and v_k^2 in the above sum must be zero. Lemma 3.2 then implies that the action of G on V extends to an action of G on $S_Q(V)$ by automorphisms. $\square$

Recall that a matrix in $\mathbb{GL}_n(\mathbb{K})$ is *monomial* if each column and each row has exactly one nonzero entry. A subgroup $G \leq \mathbb{GL}(V)$ is called monomial with respect to a fixed basis of V if it acts by monomial matrices.

Corollary 3.6. Suppose $\mathcal{H}_{Q,\kappa}$ is a quantum Drinfeld Hecke algebra. If each $q_{ij} \neq 1$ with $i \neq j$, then G is a monomial group.

Proof. Fix h in G and write $h(v_a) = \sum_b h^a_b v_b$ for each $1 \leq a \leq n$. The previous proposition and lemma imply that $0 = \det_{ijkk}(h) = (1 - q_{ij}) h^i_k h^j_k$ and hence $h^i_k h^j_k = 0$ for all $i < j$ and all k. □

For any $\mathbb{K}$-algebra A upon which G acts via automorphisms, the *skew group algebra* (sometimes called the *crossed product algebra* or *smash product algebra*) $A\#G$ is the $\mathbb{K}$-vector space $A \otimes \mathbb{K}G$ with multiplication given by

$$(a \otimes g)(b \otimes h) = ag(b) \otimes gh$$

for all a, b in A and g, h in G. We write at_g for $a \otimes g$ so that the relation in $A\#G$ (or in $\mathcal{H}_{Q,\kappa}$) is simply $(at_g)(bt_h) = a\, g(b)\, t_{gh}$.

We may extend the action of G on V to a diagonal action on the tensor algebra $T(V)$ (so that G acts as automorphisms). Then the algebra $\mathcal{H}_{Q,\kappa}$ is just the factor algebra

$$\mathcal{H}_{Q,\kappa} = T(V)\#G / \langle v_j v_i - q_{ij} v_i v_j - \kappa(v_i, v_j) : 1 \leq i, j \leq n \rangle \ ,$$

where we write ab for the product $a \otimes b$ in $T(V)$. Hence, relation (2) defining $\mathcal{H}_{Q,\kappa}$ extends to all of $T(V)$.

If $\mathcal{H}_{Q,\kappa}$ is a quantum Drinfeld Hecke algebra, then Proposition 3.5 implies that G acts as automorphisms on the underlying quantum polynomial algebra $S_Q(V)$, and thus one may form a skew group algebra $S_Q(V)\#G$. The existence of the standard PBW basis here implies that the graded algebra associated to $\mathcal{H}_{Q,\kappa}$ is isomorphic to $S_Q(V)\#G$.

4. Noncommutative Gröbner Bases Theory

In this section, we recall the use of Gröbner bases in the theory of free associative algebras. Definitions and formulations used in noncommutative Gröbner bases theory often vary. Unfortunately, they differ widely among authors whose work we wish to combine, thus we give a concise, self-contained account in this section (and the next) of just those facts necessary for our main results. Standard references include [15, 28, 35].

Let $\langle X \rangle = \langle x_1, \ldots, x_n \rangle$ be the free monoid in symbols x_i. Its elements are the neutral element (empty word) and nonempty words in the alphabet $x_1, \ldots, x_n$ called *monomials*. Let $\mathbb{K}\langle X \rangle = \mathbb{K}\langle x_1, \ldots, x_n \rangle$ be the corresponding monoid algebra over the field $\mathbb{K}$ (i.e., the free associative algebra over $\mathbb{K}$). We call its elements *polynomials*. Identify the empty word in $\langle X \rangle$ with 1 in $\mathbb{K}$ so that $\mathbb{K}\langle X \rangle$ is spanned by monomials as a $\mathbb{K}$-vector space. Elements of the form $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ with α_i in $\mathbb{N}$ are called *standard monomials*.

A *monomial ordering* on $K\langle X \rangle$ is a total ordering $\succ$ on $\langle X \rangle$ compatible with monomial multiplication ($wu \succ wv$ and $uw \succ vw$ whenever $u \succ v$ for all u, v, w in $\langle X \rangle$) that is a well-ordering. We use the standard definition of *leading monomial* $\mathrm{lm}(f)$ and *leading coefficient* $\mathrm{lc}(f)$ of a polynomial f in $\mathbb{K}\langle X \rangle$. We say that a monomial v *divides* a monomial w if v is a proper subword of w, i.e., if there exist monomials m_1, m_2 in $\langle X \rangle$ such that $w = m_1 v\, m_2$. For a subset $S \subset \mathbb{K}\langle X \rangle$, the *leading ideal* of S is the two-sided ideal

$L(S) = \langle\, \mathrm{lm}(s) \mid s \in S \setminus \{0\}\rangle$ in $\mathbb{K}\langle X\rangle$. Recall that a subset $S \subset I$ is a *(two-sided) Gröbner basis* of the ideal I with respect to $\succ$ if $L(S) = L(I)$. In other words, for any nonzero f in I, there exists s in S with $\mathrm{lm}(s)$ dividing $\mathrm{lm}(f)$.

We are interested in *reduced* Gröbner bases. We say that f in $\mathbb{K}\langle X\rangle$ is *reduced with respect to* $S \subset A$ if no monomial of f is contained in $L(S)$. A subset $S \subset \mathbb{K}\langle X\rangle$ is called *reduced* if for any s in S, $\mathrm{lm}(s)$ does not divide any monomial of any polynomial from S except s itself.

In Lemma 4.2, we see that a monic reduced Gröbner basis is unique. We first define a normal form:

Definition 4.1. Let $\mathcal{S}$ be the set of all ordered subsets of $\mathbb{K}\langle X\rangle$ and let $\succ$ be a monomial ordering on $\mathbb{K}\langle X\rangle$. A map $\mathrm{NF} : \mathbb{K}\langle X\rangle \times \mathcal{S} \to \mathbb{K}\langle X\rangle$, $(p, S) \mapsto \mathrm{NF}(p, S)$ is called a *normal form* on $\mathbb{K}\langle X\rangle$ (with respect to $\succ$) if for all f in $\mathbb{K}\langle X\rangle$ and S in $\mathcal{S}$,

(i) $\mathrm{NF}(0, S) = 0$,
(ii) $\mathrm{NF}(f, S) \neq 0$ implies that $\mathrm{lm}\big(\mathrm{NF}(f, S)\big) \notin L(S)$, and
(iii) $f - \mathrm{NF}(f, S) \in \langle S\rangle$.

A normal form NF is called a *reduced normal form* if $\mathrm{NF}(f, S)$ is reduced with respect to S for all f. A reduced normal form always exists.

Lemma 4.2. *Let $I \subset \mathbb{K}\langle X\rangle$ be an ideal, $\succ$ a monomial ordering, $S \subset I$ a Gröbner basis of I with respect to $\succ$, and $\mathrm{NF}(\cdot, S)$ a normal form on $\mathbb{K}\langle X\rangle$ with respect to S and $\succ$.*

(i) *A polynomial f in $\mathbb{K}\langle X\rangle$ lies in I if and only if $\mathrm{NF}(f, S) = 0$.*
(ii) *If $J \subset \mathbb{K}\langle X\rangle$ is an ideal with $I \subset J$, then $L(I) = L(J)$ implies $I = J$. In particular, S generates I as an ideal of $\mathbb{K}\langle X\rangle$.*
(iii) *If $\mathrm{NF}(\cdot, S)$ is a reduced normal form, then it is unique up to a nonzero constant multiple.*

5. Computation of Gröbner Bases

We now explain how a Gröbner basis arises from an explicit construction of a reduced normal form and illustrate with group algebras. Fix an arbitrary monomial ordering $\succ$ on $\mathbb{K}\langle X\rangle$ throughout this section.

Definition 5.1. We say that f_1 and f_2 in $\mathbb{K}\langle X\rangle$ *overlap* if there exist monomials m_1, m_2 in X such that

(1) $\mathrm{lm}(f_1)m_2 = m_1 \,\mathrm{lm}(f_2)$,
(2) $\mathrm{lm}(f_1)$ does not divide m_1 and $\mathrm{lm}(f_2)$ does not divide m_2.

In this case, the *overlap relation* of f_1, f_2 by m_1, m_2 is the polynomial

$$o(f_1, f_2, m_1, m_2) = \mathrm{lc}(f_2)f_1m_2 - \mathrm{lc}(f_1)m_1f_2.$$

The overlap relation is a generalization of the s-polynomial from the theory of commutative Gröbner bases (see, e.g., [17]). Note that by construction,

$$\mathrm{lm}(o(f_1, f_2, m_1, m_2)) \prec \mathrm{lm}(f_2)m_2 = m_1\,\mathrm{lm}(f_2)\,.$$

Moreover, there are only finitely many overlaps between a fixed f_1 and f_2. Note also that a polynomial f can overlap itself.

We define *reduction* (also called "inclusion overlap" or "spoly", see [28, 15]) and the *reduction algorithm*, which provides a desirable coset representative of a polynomial modulo an ideal.

Definition 5.2. For any nonzero f, u in $\mathbb{K}\langle X\rangle$ with $\mathrm{lm}(u)$ dividing $\mathrm{lm}(f)$, define

$$\mathrm{NF}(f,u) := f - \mathrm{lc}(f)\,\mathrm{lc}(u)^{-1}\cdot m_1\,u\,m_2$$

where $\mathrm{lm}(f) = m_1\ \mathrm{lm}(u)\,m_2$ for monomials m_1, m_2 in X.

By construction, $\mathrm{lm}(\mathrm{NF}(f,u)) \prec \mathrm{lm}(f)$.

Definition 5.3. Let S be a subset of $\mathbb{K}\langle X\rangle$ and fix f in $\mathbb{K}\langle X\rangle$. Define *complete reduction of f with respect to S* to be the output $\mathrm{NF}(f,S)$ of the following procedure NF applied to f in $\mathbb{K}\langle X\rangle$:

(a) If $f = 0$, return f and stop.
(b) If the set $S' := \{u \in S : \mathrm{lm}(u) \text{ divides } \mathrm{lm}(f)\}$ is empty, return
$$\mathrm{lc}(f)\,\mathrm{lm}(f) + \mathrm{NF}(f - \mathrm{lc}(f)\,\mathrm{lm}(f), S).$$
(c) Otherwise, choose some u in S', replace f by $\mathrm{NF}(f,u)$, and go back to step (a).

The next two lemmas show that complete reduction defines an algorithm and that this algorithm is essentially independent of choices: $f \mapsto \mathrm{NF}(f,S)$ is a well-defined function up to a nonzero constant.

Lemma 5.4. *The procedure* NF *terminates in a finite number of steps.*

Proof. The procedure NF applied to a nonzero polynomial f produces a (non-unique) sequence of nonzero polynomials $f = f_0, f_1, f_2, \ldots$ with strictly decreasing leading monomials: $\mathrm{lm}(f) \succ \mathrm{lm}(f_1) \succ \mathrm{lm}(f_2) \succ \ldots$. (Indeed, we either apply Step (b) and set $f_{i+1} = f_i - \mathrm{lc}(f_i)\,\mathrm{lm}(f_i)$ (in order to recursively call NF) or we apply Step (c) and set $f_{i+1} = \mathrm{NF}(f_i, u)$ for some monomial u. In either case, $\mathrm{lm}(f_{i-1}) \succ \mathrm{lm}(f_i)$.) But $\succ$ is a well-ordering and thus the sequence $\mathrm{lm}(f), \mathrm{lm}(f_1), \mathrm{lm}(f_2), \ldots$ is finite. The procedure thus terminates. $\square$

Lemma 5.5. *Let $S \subset I$ be a Gröbner basis of an ideal $I \subset \mathbb{K}\langle X\rangle$. Then $\mathrm{NF}(\cdot, S)$ is a reduced normal form on $\mathbb{K}\langle X\rangle$.*

Proof. Recall that $\mathrm{NF}(0, S) = 0$ (see Definition 5.2). Suppose $h = \mathrm{NF}(f, S)$ for some nonzero polynomial f. Then the reduction algorithm gives

$$f = \sum_{u \in S} \mathrm{lc}(f)\,\mathrm{lc}(u)^{-1} \cdot a_u\, u\, b_u + h$$

where a_u, b_u are monomials in $\langle X \rangle$ for each u in S. Note that $f - h$ lies in $\langle S \rangle$ by construction and the claim holds for $h = 0$. Now suppose that $h \neq 0$. Then no monomial in h is divisible by $\mathrm{lm}(u)$ for any u in S. Hence, $\mathrm{lm}(h) \notin L(S)$ and h is reduced with respect to S as required. Moreover, $\mathrm{lm}(f) = \max_{<}(a_u \,\mathrm{lm}(u) b_u, \mathrm{lm}(h))$. Thus $\mathrm{lm}(h) = \mathrm{lm}(f)$ if and only if $f = c \cdot h + g$ for $c \in \mathbb{K} \setminus \{0\}$ and $g \in \langle S \rangle$ with $\mathrm{lm}(g) \prec \mathrm{lm}(f)$. Otherwise $\mathrm{lm}(h) \prec \mathrm{lm}(f)$. □

The following theorem provides the foundation for the generalized Buchberger's algorithm for the computation of Gröbner bases. Note that the corresponding algorithm belongs to the family of so-called "critical pair and completion" algorithms (see [8]).

Theorem 5.6 (e.g., [16])**.** *Let S be a subset of $\mathbb{K}\langle X \rangle$. Then S is a Gröbner basis of the ideal $\langle S \rangle$ if and only if for any nonzero f_1, f_2 in S and any overlap relation o of f_1, f_2 with some monomials m_1, m_2 in X,*

$$\mathrm{NF}(\ o(f_1, f_2, m_1, m_2),\ S) = 0.$$

We will apply this theorem to determine necessary and sufficient conditions for the set of relations defining $\mathcal{H}_{Q,\kappa}$ to be a Gröbner basis of the ideal it generates in the appropriate free algebra. In the meantime, we illustrate a computation of a Gröbner basis on the group algebra of a finite group.

Proposition 5.7. *Let G be a finite group. Then*

$$\begin{aligned} \mathbb{K}G &\cong \mathbb{K}\langle x_g : g \in G \rangle / \langle x_e - 1, x_g x_h - x_{gh} : g, h \in G \rangle \\ &\cong \mathbb{K}\langle x_g : g \in G^* \rangle / \langle S \rangle \end{aligned}$$

for $S = \{x_g x_h - x_{gh}, x_f x_{f^{-1}} - 1 : f, g, h \in G^*,\ gh \neq e\}$. Let $\succ$ be any monomial ordering on $\mathbb{K}\langle x_g : g \in G^* \rangle$ with $x_g x_h \succ x_{gh}$ for all $g, h \in G^*$. Then S is a reduced Gröbner basis with respect to $\succ$ of the ideal $\langle S \rangle$.

Proof. We apply Theorem 5.6. Consider the polynomial $p = x_g x_h - x_{gh}$ for fixed $g, h \in G^*$. Then $\mathrm{lm}(p) = x_g x_h$ has overlaps with leading monomials of the following four types of polynomials from S:

(a) $x_h x_f - x_{hf}$ for any $f \in G^*$; the overlap relation

$$o = (x_g x_h - x_{gh}) x_f - x_g (x_h x_f - x_{hf}) = x_g x_{hf} - x_{gh} x_f$$

reduces to $\mathrm{NF}(o) = x_{ghf} - x_{ghf} = 0$.

(b) $x_f x_g - x_{fg}$ for any $f \in G^*$; the overlap relation

$$o = x_f (x_g x_h - x_{gh}) - (x_f x_g - x_{fg}) x_h = x_{fg} x_h - x_f x_{gh}$$

reduces to $\mathrm{NF}(o) = x_{fgh} - x_{fgh} = 0$.

(c) $x_h x_{h^{-1}} - 1$ for any $h \in G^*$; the overlap relation reduces to zero as in part (a).
(d) $x_{g^{-1}} x_g - 1$ for any $g \in G^*$; the overlap relation reduces to zero as in part (b).

□

Note that there are several modern computer algebra systems implementing the theory of noncommutative Gröbner bases over free algebras: Bergman [3], MAGMA [6], GBNP [10] (a package for Gap 4), NCGB [19] (a package for Mathematica, partially written in C) and also Singular:Letterplace [21, 22].

6. Poincaré-Birkhoff-Witt Bases

A natural question arises when working with factor algebras: What properties must a set of relations exhibit to guarantee a PBW basis? In this section, we recall how one may establish a PBW property using Gröbner bases and construct a basis for the associated graded algebra. We encourage the reader to compare with Huishi Li's interesting and well-written text [24] on noncommutative Gröbner bases and associated graded algebras (which appeared in print after this article was completed); some of the ideas are similar although we are working in a different context (free algebras over group algebras).

Let I be an arbitrary ideal in the free algebra $\mathbb{K}\langle X\rangle$. We say that a set M of monomials in $\langle X\rangle$ is a *monomial* $\mathbb{K}$-*basis* of a factor algebra $\mathbb{K}\langle X\rangle/I$ if the cosets $m+I$ for m in M form a $\mathbb{K}$-vector space basis of $\mathbb{K}\langle X\rangle/I$. We begin by constructing a monomial $\mathbb{K}$-basis.

Definition 6.1. Let I be a two-sided ideal of $\mathbb{K}\langle X\rangle$ and $\succ$ any monomial ordering on $\mathbb{K}\langle X\rangle$. Define $B_\succ$ as the complement of the leading ideal $L(I)$:

$$B_{(\succ)} := \{\text{monomials } m \in \langle X\rangle : m \notin L(I)\} \ .$$

We call $B_{(\succ)}$ the *Gröbner coset basis* of $\mathbb{K}\langle X\rangle/I$.

The term *Gröbner coset basis* is justified by the following (folklore) proposition and the fact that $B_{(\succ)}$ is explicitly constructed from a Gröbner basis.

Proposition 6.2. *Let I be a two-sided ideal of $\mathbb{K}\langle X\rangle$ and let $\succ$ be any monomial ordering on $\mathbb{K}\langle X\rangle$. Then $B_{(\succ)}$ is a monomial $\mathbb{K}$-basis of $\mathbb{K}\langle X\rangle/I$.*

Proof. Let $B \subset \langle X\rangle$ be any set of monomials. Since $L(I)$ is a monomial ideal, $B + L(I)$ is a $\mathbb{K}$-basis of $\mathbb{K}\langle X\rangle/L(I)$ if and only if $B = B_{(\succ)}$. Any a in $\mathbb{K}\langle X\rangle$ is equivalent to the normal form $\mathrm{NF}(a,S)$ modulo I, where S is a reduced Gröbner basis of I. But since S and NF are reduced, every $\mathrm{NF}(a,S)$ lies in $\mathrm{Span}_{\mathbb{K}} B_{(\succ)}$ by definition. Hence, $B_{(\succ)}$ spans $\mathbb{K}\langle X\rangle/I$ as a $\mathbb{K}$-vector space. The set $B_{(\succ)}$ is also $\mathbb{K}$-independent modulo I: If any finite linear combination of monomials in $B_{(\succ)}$ would lie in I, then its leading monomial would lie in $L(I) \cap B_{(\succ)} = \emptyset$. □

Gröbner technology allows one to describe the explicit shape of relations lending themselves to a $\mathbb{K}$-basis of standard monomials.

Proposition 6.3. *Let I be a two-sided ideal of $\mathbb{K}\langle X\rangle$. Suppose there exists a monomial ordering $\succ$ with respect to which I has reduced Gröbner basis S of the form*

$$S = \{x_j x_i - p_{ij} : 1 \leq i < j \leq n\}$$

for some p_{ij} in $\mathbb{K}\langle X\rangle$ with $x_j x_i \succ \mathrm{lm}(p_{ij})$ for each $i < j$. Then the factor algebra $\mathbb{K}\langle X\rangle/I$ has monomial basis $\{x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}\}$.

Proof. Let S be a reduced Gröbner basis of I with respect to any monomial ordering $\succ$. Then the leading ideal $L(S)$ consists of all non-standard monomials if and only if $L(S)$ is generated by $x_j x_i$ for $1 \leq i < j \leq n$ (since $L(S)$ is a monomial ideal). As S is reduced, this is equivalent to $S = \{x_j x_i -\ p_{ij} :\ x_j x_i \succ \mathrm{lm}(p_{ij})\}$. Thus, $B_{\succ}$ is the set of standard monomials if and only if S has the given form. The result then follows from Proposition 6.2. □

The last proposition gives an immediate proof of the well-known fact that quantum polynomial algebras satisfy a PBW property.

Corollary 6.4. Let $S = \{v_j v_i - q_{ij} v_i v_j : 1 \leq i < j \leq n\} \subset \mathbb{K}\langle v_1, \dots, v_n\rangle$. Then for any monomial ordering $\succ$ on $\mathbb{K}\langle v_1, \dots, v_n\rangle$, S is a Gröbner basis of $\langle S\rangle$. Hence $\{v_1^{\alpha_1} \dots v_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}\}$ is a monomial $\mathbb{K}$-basis of $S_Q(V) = \mathbb{K}\langle v_1, \dots, v_n\rangle/\langle S\rangle$.

Example 6.5. Consider $A = \mathbb{K}\langle x, y\rangle/\langle xy - y^2\rangle$. Suppose $\succ$ is any monomial ordering of $\mathbb{K}\langle x, y\rangle$ with $x \succ y$. Then $xy \succ y^2$ and $\{xy - y^2\}$ is a Gröbner basis of the ideal I it generates with respect to $\succ$. The Gröbner coset basis, $B_{(\succ)} = \{y^a x^b : a, b \in \mathbb{N}\}$, is a monomial $\mathbb{K}$-basis of $\mathbb{K}\langle x, y\rangle/I$ as Proposition 6.3 implies. On the other hand, the set of standard monomials $\{x^a y^b : a, b \in \mathbb{N}\}$ does not form a monomial $\mathbb{K}$-basis of $\mathbb{K}\langle x, y\rangle/I$ since, e.g., $xy + \langle xy - y^2\rangle = -y^2 + \langle xy - y^2\rangle$.

Now consider instead a monomial ordering $>$ on $\mathbb{K}\langle x, y\rangle$ with $y > x$. Then $y^2 > xy$ and the Gröbner basis S of $\langle -y^2 + xy\rangle$ with respect to $>$ is an infinite set, $S = \{yx^n y - x^{n+1} y : n \in \mathbb{N}\}$ (see [35]). Notice that x^2, xy, yx all lie in the Gröbner coset basis $B_{(>)}$, as they do not lie in the ideal of leading monomials of S. By Proposition 6.2, the Gröbner coset basis $B_{(>)}$ is a monomial $\mathbb{K}$-basis of $\mathbb{K}\langle x, y\rangle/I$, yet it is not a Poincaré-Birkhoff-Witt basis (as it contains x^2, xy, yx). Note that $\{xy - y^2\}$ is not a Gröbner basis of the ideal it generates with respect to $>$.

The Poincaré-Birkhoff-Witt theorem for universal enveloping algebras of Lie algebras has several possible analogs in the setting of finitely presented associative $\mathbb{K}$-algebras. Applying a fixed permutation to the indices in the set of standard monomials $x_1^{\alpha_1} \dots x_m^{\alpha_m}$ may yield a monomial $\mathbb{K}$-basis for $\mathbb{K}\langle X\rangle/I$ for some permutations but not others, as we saw in the last example. We appeal to the associated graded algebra. Let $A = \mathbb{K}\langle X\rangle/I$ be an arbitrary factor algebra (with I a two-sided ideal in $\mathbb{K}\langle X\rangle$). Let $\mathcal{A} = \{A_i : i \geq -1\}$ be an ascending $\mathbb{N}$-filtration of A. Note that any $\mathbb{N}$-filtration on $\mathbb{K}\langle X\rangle$ (for example, by degree) induces an $\mathbb{N}$-filtration on a factor algebra of $\mathbb{K}\langle X\rangle$. Recall that the associated graded algebra $\mathrm{Gr}^{\mathcal{A}}(A)$ of A with respect to the filtration $\mathcal{A}$ is

$$\mathrm{Gr}(A) = \mathrm{Gr}^{\mathcal{A}}(A) = \bigoplus_{i\in\mathbb{N}} A_i \,/\, A_{i-1} \ .$$

One may choose any $\mathbb{K}$-vector space (direct sum) complement to A_{i-1} in A_i to obtain a vector space isomorphism, $A \cong \mathrm{Gr}(A)$.

We say that any f in $\mathbb{K}\langle X\rangle$ has $\mathcal{A}$-*degree* $d \geq 0$ in $\mathbb{N}$ whenever $f + I \in A_d$ but $f + I \notin A_{d-1}$ and we write $\deg_{\mathcal{A}}(f) = d$ in this case. Set $\deg_{\mathcal{A}}(f) = -\infty$ for any f in I. We call a monomial ordering $\succ$ on $\mathbb{K}\langle X\rangle$ *compatible* with the filtration $\mathcal{A}$ if

$$\deg_{\mathcal{A}}(f) > \deg_{\mathcal{A}}(f') \quad \text{implies} \quad \mathrm{lm}(f) \succ \mathrm{lm}(f')$$

for all f, f' in $\mathbb{K}\langle X\rangle$. Note that many compatible monomial orderings exist for a fixed $\mathbb{N}$-filtration on A. We say a set M of monomials in $\mathbb{K}\langle X\rangle$ is a *monomial* $\mathbb{K}$-*basis* of the associated graded algebra $\mathrm{Gr}(A)$ if the elements $m + I + A_{\deg_{\mathcal{A}}(m)-1}$ for m in M form a $\mathbb{K}$-basis of $\mathrm{Gr}(A)$, and we record a straightforward observation.

Proposition 6.6. *Let $\mathbb{K}\langle X\rangle/I$ be an $\mathbb{N}$-filtered algebra. Then*

(1) *Any monomial $\mathbb{K}$-basis of $\mathrm{Gr}(\mathbb{K}\langle X\rangle/I)$ is also a monomial $\mathbb{K}$-basis of $\mathbb{K}\langle X\rangle/I$.*
(2) *The set $B_{(\succ)}$ is a monomial $\mathbb{K}$-basis for both $\mathbb{K}\langle X\rangle/I$ and $\mathrm{Gr}(\mathbb{K}\langle X\rangle/I)$, for any monomial ordering $\succ$ compatible with the $\mathbb{N}$-filtration.*

Proof. One may check directly that the set of $m + I$ for m in a monomial $\mathbb{K}$-basis of $\mathrm{Gr}(A)$ spans $A = \mathbb{K}\langle X\rangle/I$ and is linearly independent. Now suppose some nonzero, finite $\mathbb{K}$-linear combination of monomials m_i in $B_{(\succ)}$ has degree d with respect to the filtration. Then the compatibility of $\succ$ and the filtration force each $\deg(m_i) \leq d$. By Proposition 6.2, $B_{(\succ)}$ is a monomial $\mathbb{K}$-basis of A. Hence for each d, the set $\{m+I : m \in B_{(\succ)},\ \deg(m) \leq d\}$ spans A_d and $\{m + I + A_d : m \in B_{(\succ)},\ \deg(m) = d\}$ spans A_d/A_{d-1} over $\mathbb{K}$. This set is also $\mathbb{K}$-linearly independent: If any nonzero, finite linear combination of monomials in $B_{(<)}$ of $\deg d$ defined the zero class in A_d/A_{d-1}, the degrees of all the monomials in the combination would be $d - 1$ instead of d. Thus $B_{(\succ)}$ is also a $\mathbb{K}$-monomial basis for $\mathrm{Gr}(A)$. $\square$

In the special case that our factor algebra is $\mathcal{H} = \mathcal{H}_{Q,\kappa}$, we may relate the PBW property of the original algebra to that of the quantum polynomial algebra. Formally, we filter the free associative algebra

$$\mathcal{F} = \mathbb{K}\langle v_1, \ldots, v_n, t_g : g \in G^* \rangle = \bigoplus_{i=0}^{\infty} \mathcal{F}_i$$

by assigning degree 0 to all t_g (for g in G) and degree 1 to all v in V and consider the associated graded algebra

$$\mathrm{Gr}\ \mathcal{H} := \oplus_{i=0}^{n}\ \mathcal{H}_i/\mathcal{H}_{i-1}\ ,$$

where $\mathcal{H}_i$ is the image of $\mathcal{F}_i$ under the projection $\mathcal{F} \to \mathcal{H}$. Assuming that Q is a quantum system of parameters, the graded algebra $\mathrm{Gr}\,\mathcal{H}$ is isomorphic to a quotient of the quantum polynomial ring $S_Q(V)\#G$, and $\mathcal{H}$ has the standard PBW basis if and only if $\mathrm{Gr}\,\mathcal{H}$ and $S_Q(V)\#G$ are isomorphic (as graded algebras). In fact, Naidu and Witherspoon [29] observe that every quantum Drinfeld Hecke algebra is isomorphic to a formal deformation of $S_Q(V)\#G$.

We end this section by recording a few other facts about quantum Drinfeld Hecke algebras.

Theorem 6.7. *If $\mathcal{H}_{Q,\kappa}$ is a quantum Drinfeld Hecke algebra, then*

(i) *$\mathcal{H}_{Q,\kappa}$ is Noetherian;*
(ii) *$\mathcal{H}_{Q,\kappa}$ is an integral domain if and only if G is trivial;*
(iii) *the Gel'fand-Kirillov dimension of $\mathcal{H}_{Q,\kappa}$ is*

$$\mathrm{GKdim}\,\mathcal{H}_{Q,\kappa} = n + \mathrm{GKdim}\,\mathbb{K}\ ;$$

(iv) *if $|G|$ is not divisible by* $\mathrm{char}\,\mathbb{K}$*, then the global homological dimension of $\mathcal{H}_{Q,\kappa}$ is at most n.*

Proof.

(i) Since $S_Q(V)$ is Noetherian (e.g., see [9]), so is $S_Q(V)\#G$ (see [30, Proposition 1.6]). Then as $\mathrm{Gr}\,\mathcal{H}_{Q,\kappa} \cong S_Q(V)\#G$, the filtered algebra $\mathcal{H}_{Q,\kappa}$ is as well (see, e.g., [27]).
(ii) In $\mathbb{K}G$, $0 = 1 - (t_g)^d = (1 - t_g)(1 + t_g + \ldots + t_{g^{d-1}})$ for any $g \in G$ of order $d > 1$.
(iii) Consider the filtration $\{\mathcal{H}_k : k \geq -1\}$ of $\mathcal{H} = \mathcal{H}_{Q,\kappa}$. Let $d = |G|$. Then $\limsup_{k\to\infty} \log_k(k^n \frac{d}{n!} + \ldots) = n$ as the PBW property implies that

$$\dim_{\mathbb{K}} \mathcal{H}_k = \binom{k+n}{n} \cdot d = k^n \frac{d}{n!} + \text{ (lower order terms)}.$$

(iv) In the non-modular case (see [27, Theorem 7.5.6]),

$$\mathrm{gl.\,dim}(S_Q(V)\#G) = \mathrm{gl.\,dim}(S_Q(V)) = n\,.$$

Then $\mathrm{gl.\,dim}(\mathcal{H}_{Q,\kappa}) \leq \mathrm{gl.\,dim}(\mathrm{Gr}\,\mathcal{H}_{Q,\kappa}) = n$ (by [27, Theorem 7.6.18]) since $\mathrm{Gr}\,\mathcal{H}_{Q,\kappa} \cong S_Q(V)\#G$. $\square$

Remark 6.8. One might ask about the possibility of grading $\mathcal{H}_{Q,\kappa}$ directly. The group algebra $\mathbb{K}G$ is graded if and only if the graded degree (weight) of each t_g is 0. The relation $t_g v_k = g(v_k) t_g$ in $\mathcal{H}_{Q,\kappa}$ is graded if all v_i have the same weight, say 1. But the relation $v_j v_i = q_{ij} v_i v_j + \kappa(v_i, v_j)$ is graded only in two cases: Either the weight of every v_k is zero or each $\kappa(v_i, v_j)$ is zero, since otherwise the graded degree of $\kappa(v_i, v_j)$ is zero while the graded degree of $v_j v_i - q_{ij} v_i v_j$ is 2. In the first case, $\mathcal{H}_{Q,\kappa}$ is trivially graded (all weights zero). In the second case, $\mathcal{H}_{Q,\kappa} = S_Q(V) \# G$.

7. Conditions on Parameters

In this section, we deploy the theory of Gröbner bases to rigorously establish necessary and sufficient conditions for $\mathcal{H}_{Q,\kappa}$ to define a quantum Drinfeld Hecke algebra. We write the factor algebra $\mathcal{H}_{Q,\kappa}$ as $\mathcal{F} / \langle R' \rangle$, where $\mathcal{F}$ is the free associative $\mathbb{K}$-algebra

$$\mathcal{F} = \mathbb{K}\langle v_1, \dots, v_n, t_g : g \in G^* \rangle \tag{3}$$

and $\langle R' \rangle$ is the ideal in $\mathcal{F}$ generated by relations defining $\mathcal{H}_{Q,\kappa}$,

$$\begin{aligned} R' = \{ t_g t_h - t_{gh},\ t_g v_i - g(v_i) t_g,\ v_j v_i - q_{ij} v_i v_j - \kappa(v_i, v_j) : \\ \text{for all } g, h \in G^*, 1 \le i, j \le n \}. \end{aligned} \tag{4}$$

Moreover, let us define the smaller set of relations

(5)

$$\begin{aligned} R = \{ t_g t_h - t_{gh},\ t_g v_i - g(v_i) t_g,\ v_j v_i - q_{ij} v_i v_j - \kappa(v_i, v_j) : \\ \text{for all } g, h \in G^*, 1 \le i < j \le n \}. \end{aligned}$$

Before expressing the PBW property of $\mathcal{H}_{Q,\kappa}$ in terms of a Gröbner basis, we must ensure that the given monomial ordering is compatible.

Definition 7.1. Consider a monomial ordering $\succ$ on the free algebra $\mathcal{F}$ which satisfies $v_1 \succ \ldots \succ v_n \succ t_g$ for all $g \in G^*$. We say that $\succ$ *preserves the rewriting procedure* of relations of $\mathcal{H}_{Q,\kappa}$ if

- $t_g t_h \succ t_{gh}$ for all $g, h \in G^*$,
- $v_j v_i \succ t_g$ for all i, j and $g \in G^*$,
- $v_j v_i \succ v_i v_j$ for all $i < j$ ("first misordering preference"), and
- $t_g v_i \succ v_j t_g$ for all i, j and $g \in G^*$ ("second misordering preference").

Remark 7.2. A monomial ordering which preserves the rewriting procedure always exists. One example can be constructed as follows. We assign degree 1 to each t_g for g in G^* and to each v_i for $1 \le i \le n$. Two monomials in $\mathcal{F}$ are first compared by their total degree. In the case of equal degrees, misordering preferences are applied. If two monomials are of the same total degree and no misordering preference can be applied, we compare monomials further with left lexicographical ordering.

Proposition 7.3. *Suppose $\succ$ is any monomial ordering on $\mathcal{F}$ with $v_1 \succ \ldots \succ v_n \succ t_g$ which preserves the rewriting procedure of $\mathcal{H}_{Q,\kappa}$. If R is a Gröbner basis of $\langle R\rangle$ with respect to $\succ$, then $\mathcal{F}/\langle R\rangle$ has monomial $\mathbb{K}$-basis*

$$B = \{v_1^{\alpha_1} \cdots v_n^{\alpha_n} t_g : \alpha \in \mathbb{N}^n, g \in G\} \ .$$

Proof. The set of leading monomials of R in $\mathcal{F}$ is

$$L := \{v_j v_i,\ t_g v_i,\ t_g t_h \ \mid g, h \in G^*, 1 \le i < j \le n\},$$

and $B_{(\succ)} = \langle X\rangle \setminus (\langle X\rangle \cap \langle L\rangle) = B$. Thus if R is a Gröbner basis of $\langle R\rangle$, then $\langle L\rangle = L(R) = L(\langle R\rangle)$ and B is a monomial $\mathbb{K}$-basis of $\mathcal{F}/\langle R\rangle$ by Proposition 6.2. ☐

We now give conditions for R to be a Gröbner basis.

Theorem 7.4. *Let $\succ$ be any monomial ordering on $\mathcal{F}$ with $v_1 \succ \ldots \succ v_n \succ t_g$ that preserves the rewriting procedure of $\mathcal{H}_{Q,\kappa}$. Then R is a Gröbner basis of $\langle R\rangle$ with respect to $\succ$ if and only if for all g, h in G and $1 \le i < j < k \le n$,*

$$\text{(i)} \qquad \begin{aligned} 0 \ = \ &\big(q_{ik}q_{jk}\, hv_k - v_k\big)\ \kappa_h(v_i, v_j) + \big(q_{jk}v_j - q_{ij}\, hv_j\big)\ \kappa_h(v_i, v_k) \\ &+ \big(hv_i - q_{ij}q_{ik}v_i\big)\ \kappa_h(v_j, v_k)\,, \end{aligned}$$

$$\text{(ii)} \qquad g(v_j)g(v_i) = q_{ij}g(v_i)g(v_j)\ , \quad \textit{and}$$

$$\text{(iii)} \qquad \kappa_{h^{-1}gh}(v_i, v_j) = \sum_{k<\ell} \det\nolimits_{ijkl}(h)\, \kappa_g(v_k, v_\ell)\,.$$

Moreover, if R is a Gröbner basis, it is reduced.

Proof. We derive necessary and sufficient conditions under which R is a Gröbner basis of the ideal it generates in the free associative algebra $\mathcal{F}$ using Theorem 5.6: We examine all overlap polynomials $o = o(f_1, f_2, m_1, m_2)$ with f_1, f_2 in R and m_1, m_2 monomials in $\mathcal{F}$. Setting the complete reduction $\mathrm{NF}(o, R)$ of each overlap to zero in $\mathcal{F}$ gives a set of necessary and sufficient conditions for R to be a Gröbner basis of the ideal $\langle R\rangle$ in $\mathcal{F}$.

By Lemmas 5.5 and 4.2, the algorithm NF produces a reduced normal form and hence its output is unique up to a nonzero constant. Thus the algorithm NF gives a result independent (up to a nonzero scalar) of any choices in the algorithm. We forgo the explicit computations and just record the results here.

Since the set of relations of the group algebra $\mathbb{K}G$ forms a Gröbner basis of the ideal it generates in the free algebra $\mathbb{K}\langle t_g \mid g \in G^*\rangle$ (by Proposition 5.7, for example), we are left with only three kinds of possibly nonzero overlaps between elements from R:

(a) There is an overlap relation between $t_h t_g - t_{hg}$ and $t_g v - g(v)t_g$ for any $v = v_i$ and g, h in G, namely,

$$o = (t_h t_g - t_{hg})v - t_h(t_g v - g(v)t_g) \ .$$

The complete reduction algorithm applied to $o = -t_{hg}v + t_h g(v)t_g$ yields zero: $\mathrm{NF}(o, R) = 0$ for this type of overlap.

(b) There is an overlap relation between elements $v_k v_j - q_{jk} v_j v_k - \kappa(v_j, v_k)$ and $v_j v_i - q_{ij} v_i v_j - \kappa(v_i, v_j)$ for distinct $1 \le i, j, k \le n$ obtained by multiplying the first on the right by v_i and the second on the left by v_k. Applying the complete reduction algorithm gives $\mathrm{NF}(o, R)$ as the *non-degeneracy expression* (see [23])

$$\sum_g (q_{ik} q_{jk} g(v_k) - v_k) a_g^{ij} t_g + (q_{jk} v_j - q_{ij} g(v_j)) a_g^{ik} t_g + (g(v_i) - q_{ij} q_{ik} v_i) a_g^{jk} t_g$$

(where we abbreviate $a_g^{ij} := \kappa_g(v_i, v_j)$), which is zero in $\mathcal{F}$ if and only if

$$\begin{aligned} 0 \;=\; & (q_{ik} q_{jk} g v_k - v_k)\ \kappa_g(v_i, v_j) + \ (q_{jk} v_j - q_{ij} g v_j)\ \kappa_g(v_i, v_k) \\ & \qquad + (g v_i - q(i,j) q_{ik} v_i)\ \kappa_g(v_j, v_k) \end{aligned}$$

for each g in G. This is precisely condition (i) of the theorem.

(c) For all h in G and $i \neq j$, there is an overlap relation o between $t_h v_j - h(v_j) t_h$ and $v_j v_i - q_{ij} v_i v_j - \kappa(v_i, v_j)$ obtained by multiplying the first on the right by v_i and the second on the left by t_h:

$$\begin{aligned} o =\; & t_h (v_j v_i - q_{ij} v_i v_j - \kappa(v_i, v_j)) - (t_h v_j - h(v_j) t_h) v_i \\ =\; & -q_{ij} t_h v_i v_j - t_h \kappa(v_i, v_j) + h(v_j) t_h v_i\ . \end{aligned}$$

The complete reduction algorithm reduces o to

(6)
$$\begin{aligned} \mathrm{NF}(o, R) \;=\; & \sum_{k<\ell} \Big(q_{k\ell} \det_{ijk\ell}(h) + \det_{ij\ell k}(h) \Big)\ v_k v_\ell\, t_h + \sum_k \det_{ijkk}\ v_k^2\, t_h \\ & + \sum_{g \in G} \left(\sum_{k<\ell} \det_{ijk\ell}(h)\ \kappa_g(v_k, v_\ell)\ - \kappa_{h^{-1}gh}(v_i, v_j) \right) t_{gh}\,. \end{aligned}$$

But $\mathrm{NF}(o, R)$ vanishes in $\mathcal{F}$ exactly when the coefficient of each monomial in Equation 6 vanishes. Lemma 3.2 implies that the coefficient of each $v_k v_\ell t_{gh}$ and each $v_m^2 t_{gh}$ vanish in Equation 6 if and only if condition (ii) of the theorem holds. The coefficient of each t_{gh} in Equation 6 vanishes for all $i \neq j$ exactly when condition (iii) of the theorem holds. □

Remark 7.5. Observe that if $\mathcal{H}_{Q,\kappa}$ has the standard PBW basis, then conditions (i), (ii), (iii) for $i < j < k$ in the above theorem are equivalent to conditions (i), (ii), (iii) with *arbitrary* indices i, j, k. Indeed, from the definition of quantum minor:

$$\det_{jikl}(h) = -q_{ji} \det_{ijkl}(h)$$

for all h in G. If $\mathcal{H}_{Q,\kappa}$ has the standard PBW basis, then Proposition 3.5 implies in addition that $\kappa(v_j, v_i) = q_{ij}^{-1} \kappa(v_i, v_j)$. These two facts allow us to replace increasing by arbitrary indices in the conditions of the theorem when it might be helpful.

We now use the last theorem and the connection between Gröbner bases and standard bases in the last section to show that $\mathcal{H}_{Q,\kappa} = \mathcal{F}/\langle R' \rangle$ has the standard PBW basis if and only if the conditions of the last theorem hold. (We will see in Section 9 that these conditions have a natural interpretation in terms of Hochschild cocycles.)

Theorem 7.6. *The factor algebra $\mathcal{H}_{Q,\kappa}$ is a quantum Drinfeld Hecke algebra if and only if the following four conditions hold:*

(i) *The matrix Q is a quantum system of parameters and G acts on the quantum polynomial algebra $S_Q(V)$ as automorphisms,*

(ii) *The parameter κ defines a quantum 2-form:*

$$\kappa(v_i, v_j) = -q_{ji}^{-1}\kappa(v_j, v_i) \text{ for distinct } i, j \ ,$$

(iii) *For all h in G and $1 \leq i < j < k \leq n$,*

$$\begin{aligned} 0 \ = \ \ & \big(q_{ik}q_{jk}\, hv_k - v_k\big)\ \kappa_h(v_i, v_j) \\ & + \big(q_{jk}v_j - q_{ij}\, hv_j\big)\ \kappa_h(v_i, v_k) \\ & + \big(hv_i - q_{ij}q_{ik}v_i\big)\ \kappa_h(v_j, v_k)\ , \end{aligned}$$

(iv) *For all g, h in G and all $1 \leq i < j \leq n$,*

$$\kappa_{h^{-1}gh}(v_i, v_j) = \sum_{k<\ell} \det_{ijkl}(h)\, \kappa_g(v_k, v_\ell)\ .$$

Proof. Fix any monomial ordering $\succ$ on $\mathcal{F}$ which satisfies the rewriting procedure with $v_1 \succ v_2 \succ \ldots v_n \succ t_g$ (see Remark 7.2 for an explicit choice). By Theorem 7.4, the conditions of the theorem imply that R is a Gröbner basis of the ideal it generates and that $\mathcal{H}_{Q,\kappa} = \mathcal{F}/\langle R' \rangle = \mathcal{F}/\langle R \rangle$. Thus $\mathcal{H}_{Q,\kappa}$ has the standard PBW basis by Proposition 7.3.

Conversely, assume that $\mathcal{H}_{Q,\kappa}$ has the standard PBW basis. Proposition 3.5 implies conditions (i) and (ii) and thus $\mathcal{H}_{Q,\kappa} = \mathcal{F}/\langle R' \rangle = \mathcal{F}/\langle R \rangle$. We saw in the proof Theorem 7.4 that the overlap polynomial o of any elements in R has normal form $\mathrm{NF}(o)$ lying in $\mathrm{span}_{\mathbb{K}}(B)$. But each $\mathrm{NF}(o)$ lies in $\langle R \rangle$ as well (since each overlap o does). Thus each $\mathrm{NF}(o)$ gives a linear dependence modulo $\langle R \rangle$ among elements of B. As B is a standard PBW basis, each $\mathrm{NF}(o)$ must then be zero in the free algebra $\mathcal{F}$. Thus R is a Gröbner basis of the ideal it generates by Theorem 5.6. The result then follows from Theorem 7.4. $\square$

The last theorem immediately implies (set $\kappa \equiv 0$) the following corollary.

Corollary 7.7. Suppose G acts as automorphisms on a quantum polynomial algebra $S_Q(V)$. Then $B = \{v_1^{\alpha_1} \ldots v_n^{\alpha_n} t_g \mid \alpha_i \in \mathbb{N},\ g \in G\}$ is a monomial $\mathbb{K}$-basis for $S_Q(V)\#G$.

Remark 7.8. Fix some q in $\mathbb{K}$ and suppose $q_{ij} = q$ for all $1 \leq i < j \leq n$. Then for all i, j, k, the first part of condition (iii) of the last theorem is equivalent to

$$0 \ = \ (q^2 g v_k - v_k)\kappa_g(v_i, v_j) + q(v_j - g v_j)\kappa_g(v_i, v_k) + (g v_i - q^2 v_i)\kappa_g(v_j, v_k) \ .$$

Remark 7.9. Condition (iii) from the last theorem can be written explicitly in terms of the entries of any matrix h in G. Again, fix scalars h^a_b in $\mathbb{K}$ with $h(v_a) = \sum_b h^a_b v_b$ and abbreviate a^{ij}_g for $\kappa_g(v_i, v_j)$. Then condition (iii) holds exactly when

- $0 = (q_{ik}q_{jk}h^k_k - 1)a^{ij}_g - q_{ij}h^j_k a^{ik}_g + h^i_k a^{jk}_g$,
- $0 = q_{ik}q_{jk}h^k_j a^{ij}_g + (q_{jk} - q_{ij}h^j_j)a^{ik}_g + h^i_j a^{jk}_g$,
- $0 = q_{ik}q_{jk}h^k_i a^{ij}_g - q_{ij}h^j_i a^{ik}_g + (h^i_i - q_{ij}q_{ik})a^{jk}_g$, and
- $0 = q_{ik}q_{jk}h^k_\ell a^{ij}_g - q_{ij}h^j_\ell a^{ik}_g + h^i_\ell a^{jk}_g$

for all g in G, all $i < j < k$, and any ℓ not in $\{i, j, k\}$.

8. Abelian Groups

In this section, we assume G in $\mathbb{GL}_n(\mathbb{K})$ is abelian acting diagonally on $v_1, \ldots, v_n$. Let $\chi_i : G \to \mathbb{K}^*$ be the linear character recording the i-th diagonal entry, i.e., $g v_i = \chi_i(g) v_i$ for all g in G and $1 \leq i \leq n$. We deform the skew group algebra $S_Q(V)\#G$ by setting each q-commutator $v_i v_j - q_{ji} v_j v_i$ to a group element g whose i-th and j-th entries are inverse and whose k-th entry is the scalar that arises upon interchanging $v_i v_j$ and v_k in the quantum algebra $S_Q(V)\#G$:

$$(v_i v_j)v_k = (q_{ki}q_{kj}) \ v_k(v_i v_j).$$

In fact, we will take linear combinations of such group elements g and also insist that *every* element in G has inverse i, j entries: $\chi_i = \chi_j^{-1}$. Indeed, we apply Theorem 7.6 carefully for diagonal actions to deduce the following corollary.

Corollary 8.1. Suppose G is abelian acting diagonally. Then $\mathcal{H}_{Q,\kappa}$ is a quantum Drinfeld Hecke algebra if and only if the following hold:

- Q is a quantum system of parameters,
- κ is a quantum 2-form,
- for all g in G and $i \neq j$, $\kappa_g(v_i, v_j) \neq 0$ implies that $\chi_i = \chi_j^{-1}$ and $\chi_k(g) = q_{ki}\, q_{kj}$ for all $k \neq i, j$.

The next proposition gives a complete description of quantum Drinfeld Hecke algebra in the abelian setting.

Proposition 8.2. *Suppose G is an abelian group acting diagonally on the basis $v_1, \ldots, v_n$. Then the set of quantum Drinfeld Hecke algebras comprises all factor algebras of the form*

$$\mathbb{K}\langle v_1, \ldots, v_n\rangle \# G / \Big\langle v_j v_i - q_{ij} v_i v_j - \sum_{g \in G^{ij}} c_g^{ij}\, g : \ 1 \le i < j \le n \text{ with } \chi_i = \chi_j^{-1} \Big\rangle$$

where $G^{ij} = \{g \in G : \chi_k(g) = q_{ki} q_{kj}$ for all $k \neq i, j\}$ and the c_g^{ij} are arbitrary scalars in $\mathbb{K}$.

Remark 8.3. Suppose $\mathcal{H}_{Q,\kappa}$ is a quantum Drinfeld Hecke algebra with G acting diagonally on $v_1, \ldots, v_n$. Fix q in $\mathbb{K}$ and suppose $q_{ij} = q$ for all $i < j$. Then if $c_g^{ji} \neq 0$ in the last corollary, g is a diagonal matrix

$$g = \operatorname{diag}(q^2, \ldots, q^2, c, 1, \ldots, 1, c^{-1}, q^{-2}, \ldots, q^{-2})$$

with entries c and c^{-1} at i-th and j-th locations, respectively, for some $c \in \mathbb{K}$.

Example 8.4. Let q be a primitive odd k-th root-of-unity in $\mathbb{K}$ and let G be the group of order k generated by the diagonal matrix

$$g = \operatorname{diag}(q^2, 1, q^{-2})\,.$$

Then the $\mathbb{K}$-algebra $\mathcal{H}$ generated by symbols v_1, v_2, v_3, t_g with relations

$$t_g^k = 1,\ t_g v_1 = q^2\, v_1 t_g,\ t_g v_2 = v_2 t_g,\ t_g v_3 = q^{-2}\, v_3 t_g,$$
$$v_2 v_1 = q\, v_1 v_2,\ v_3 v_2 = q\, v_2 v_3,\ v_3 v_1 = q\, v_1 v_3 + \sum_{i=1}^{k} c_i t_g^i\,,$$

for arbitrary constants c_i in $\mathbb{K}$, is a quantum Drinfeld Hecke algebra.

See Naidu and Witherspoon [29] for an explicit description of the related Hochschild cohomology (and the cocycles defining these algebras) for groups acting diagonally in characteristic zero.

9. Hochschild Cohomology

Using results of Naidu and Witherspoon [29], one may interpret the conditions of Theorem 7.6 in terms of the Hochschild cohomology of the associated skew group algebra, $\mathrm{HH}^\bullet(S_Q(V)\#G)$. We assume $\mathbb{K} = \mathbb{C}$ (the complex numbers) in this section and fix a quantum system of parameters $Q = (q_{ij} \mid 1 \le i, j \le n)$ defining a quantum polynomial algebra $S_Q(V)$. Recall that Hochschild cohomology is a generalization of group cohomology to a bimodule setting: For a $\mathbb{K}$-algebra C, $\mathrm{HH}^\bullet(C) = \mathrm{Ext}^\bullet_{C^e}(C, C)$ where C^e is the enveloping algebra $C \otimes C^{op}$.

We may regard the quantum exterior algebra $\bigwedge_Q(V)$ (see Example 2.4) as a factor algebra of a quantum polynomial algebra with respect to a nearly opposite set of scalars:

$$\bigwedge_Q(V) = S_{Q'}(V)/\langle v_1^2, \ldots, v_n^2\rangle$$

where $Q' = (q'_{ij} \,|\, 1 \leq i, j \leq n)$ is the quantum system of parameters with $q'_{ij} = -q_{ij}$ for $i \neq j$ and $q'_{ii} = 1$ for each i. Proposition 3.2 (together with Corollary 3.3) applied to $S_{Q'}(V)$ then easily implies the following two observations (where h^t is the transpose of h).

Corollary 9.1. The group G acts as automorphisms on $\bigwedge_Q(V)$ if and only if for all h in G,

(i) $\det_{ijk\ell}(h) = q_{\ell k} \det_{ij\ell k}(h)$ for all $1 \leq i, j, k, \ell \leq n$, and
(ii) $\det_{ijkk}(h^t) = 0$ for all k and $i < j$.

Corollary 9.2. G acts on as automorphisms on *both* $S_Q(V)$ and on $\bigwedge_Q[V]$ if and only if for all h in G,

(i) $\det_{ijkl}(h) = 0$ for all i, j, k, l, and
(ii) $\det_{ijkk}(h^t) = 0$ for all k and $i < j$.

As in Shepler and Witherspoon [33, 34] (in the nonquantum setting), Naidu and Witherspoon recommend associating a Hochschild cocycle to the parameters Q, κ defining a factor algebra $\mathcal{H}_{Q,\kappa}$. Any quantum 2-form κ (see Proposition 3.5) extends to an element of

$$\mathrm{Hom}_{\mathbb{K}}\Big(\bigwedge_Q^2 V,\ S_Q(V)\#G\Big) \cong \mathrm{Hom}_{S_Q(V)^e}\Big(S_Q(V)^e \otimes \bigwedge_Q^2 V,\ S_Q(V)\#G\Big),$$

and thus defines a 2-cochain in the theory of Hochschild cohomology

$$\mathrm{HH}^\bullet(S_Q(V),\ S_Q(V)\#G)$$

computed using a quantum Koszul resolution on $S_Q(V)$ (see [29]). But (see [29, Theorem 3.5])

$$\mathrm{HH}^\bullet(S_Q(V),\ S_Q(V)\#G)^G \cong \mathrm{HH}^\bullet(S_Q(V)\#G)\ .$$

Thus, one wonders: When does κ define a class in the Hochschild cohomology $\mathrm{HH}^\bullet(S_Q(V)\#G)$, the cohomology theory detecting all algebraic deformations of $S_Q(V)\#G$? Results of Naidu and Witherspoon [29] imply the following proposition.

Proposition 9.3. *Assume G acts as automorphisms on both $\bigwedge_Q(V)$ and $S_Q(V)$ and κ is a quantum 2-form. Then*

- *Condition (iii) of Theorem 7.6 holds if and only if κ is a cocycle.*
- *Condition (iv) of Theorem 7.6 holds if and only if κ is invariant.*

Theorem 7.6 and Proposition 9.3 together with Theorem 3.5 of Naidu and Witherspoon [29] therefore give another interpretation of the necessary and sufficient PBW conditions:

Theorem 9.4. *Assume G acts on both the quantum polynomial algebra $S_Q(V)$ and the quantum exterior algebra $\bigwedge_Q(V)$ as automorphisms. Let κ be a quantum 2-form. Then the factor algebra $\mathcal{H}_{Q,\kappa}$ is a quantum Drinfeld Hecke algebra if and only if κ induces a Hochschild cocycle for $S_Q(V)\#G$.*

Naidu and Witherspoon [29, Theorem 4.4] in fact show that every "constant" Hochschild 2-cocycle gives rise to a quantum Drinfeld Hecke algebra (extending a theorem from the nonquantum setting; see [33]).

10. Automorphisms of Coordinate Rings of Quantum Planes

In this section, we consider automorphisms of quantum polynomial algebras and quantum Drinfeld Hecke algebras over any 2-dimensional vector space V. Recall that every quantum Drinfeld Hecke algebra $\mathcal{H}_{\kappa,Q}$ arises from a group acting as automorphisms on some quantum polynomial algebra $S_Q(V)$ (by Proposition 3.5). Every graded $\mathbb{K}$-automorphism of a quantum polynomial algebra $S_Q(V)$ restricts to a linear map on V and thus defines an element of $\mathbb{GL}_n(\mathbb{K})$. Conversely, a transformation in $\mathbb{GL}_n(\mathbb{K})$ extends to a graded $\mathbb{K}$-automorphism of $S_Q(V)$ when it satisfies the condition of Lemma 3.2.

We write q for the parameter q_{12}. Recall that the monomial matrices in $\mathbb{GL}_2(\mathbb{K})$ are simply those which are either diagonal or anti-diagonal. For $n = 2$, it is not difficult to determine the group $\operatorname{Aut}_{\mathbb{K}} S_Q(V)$ of graded $\mathbb{K}$-automorphisms of $S_Q(V)$ explicitly (see, e.g., Alev-Chamarie [1]):

Proposition 10.1. *If $n = 2$, then $\operatorname{Aut}_{\mathbb{K}} S_Q(V)$ is*
- $\mathbb{GL}_2(\mathbb{K})$ *when $q = 1$,*
- $(\mathbb{K}^*)^2$ *(the torus) when $q \neq \pm 1$, and*
- *the subgroup of monomial matrices of* $\mathbb{GL}_2(\mathbb{K})$ *when $q = -1$.*

We describe the set of quantum Drinfeld Hecke algebra in each of the above three cases by applying Theorem 7.6.

Remark 10.2. Condition (iv) of Theorem 7.6 for $n = 2$ implies that for any commuting g and h in G, $\kappa_g(v_1, v_2) = \det_Q(h)\, \kappa_g(v_1, v_2)$, where $\det_Q$ is the *quantum determinant* defined by

$$\det_Q \begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - q\, bc \ .$$

Thus, for any quantum Drinfeld Hecke algebra $\mathcal{H}_{\kappa,Q}$ and for any g in G, the parameter κ_g is identically zero unless the centralizer subgroup $Z_G(g)$ of g in G lies in the set of quantum-determinant-one matrices,

$$\{M \in \mathbb{GL}_2(\mathbb{K}) : \det_Q(M) = 1\}\, .$$

In particular, every quantum Drinfeld Hecke algebra is supported on group elements of quantum determinant one.

10.1. **Coordinate Ring of Nonquantum Plane.** $(n = 2, q = 1)$

When $q = 1$, the set of quantum Drinfeld Hecke algebras comprises all quotients of the form

$$\mathbb{K}\langle x, y\rangle \# G / \Big\langle\, xy - yx - \sum_{\substack{g\in G \\ \det(g)=1}} c_g t_g \,\Big\rangle$$

where the scalars c_g in $\mathbb{K}$ are arbitrary for g in a set of determinant-one conjugacy class representatives of $G \leq \mathbb{GL}_2(\mathbb{K})$ and $c_{h^{-1}gh} = \det(h)\, c_g$ for all h in G. Note that the coefficient c_g is zero or the centralizer $Z_G(g)$ is a subgroup of $\mathbb{SL}_2(\mathbb{K})$. (In particular, the coefficient of the identity group element is zero unless $G \leq \mathbb{SL}_2(\mathbb{K})$.) These nonquantum algebras are called *graded Hecke algebras* (see [13] and [31], for example). (In fact, Remark 10.2 is an quantum analogue of an aspect of the characteristic zero theory of graded Hecke algebras.)

10.2. **Coordinate Ring of Transcendental Quantum Plane.** $(n = 2, q \neq \pm 1)$

Quantum Drinfeld Hecke algebras in 2 dimensions for $q \neq \pm 1$ (including the case of q transcendental over a subfield) all arise from an abelian group G acting diagonally and are described in Section 8. If each element of G has quantum determinant 1 ($\det_Q(g) = 1$ for all g in G), then the set of quantum Drinfeld Hecke algebras comprises all quotients of the form

$$\mathbb{K}\langle x, y\rangle \# G / \big\langle\, xy - q\, yx - \sum_{g\in G} c_g t_g \,\big\rangle$$

where the scalars c_g in $\mathbb{K}$ are arbitrary. If some element of G has non-unity quantum determinant, then κ is identically zero (by Remark 10.2), and $\mathcal{H}_{\kappa,Q}$ is just the quantum polynomial algebra $S_Q(V) = S_q(V)$ on two variables.

10.3. **Coordinate Ring of Skew Quantum Plane.** $(n = 2, q = -1)$

The set of quantum Drinfeld Hecke algebras in 2 dimensions when $q = -1$ comprises all quotients of the form

$$\mathbb{K}\langle x, y\rangle \# G / \big\langle\, xy + yx - \sum_{g\in G} c_g t_g \,\big\rangle$$

where the scalars c_g in $\mathbb{K}$ are arbitrary for g in a set of conjugacy class representatives of a monomial group $G \leq \mathbb{GL}_2(\mathbb{K})$ and $c_{h^{-1}gh} = \det_Q(h)\, c_g$ for all h in G. In particular, $c_g = 0$ if some element h of the centralizer $Z_G(g)$ has non-unity quantum determinant ($\det_Q(h) \neq 1$).

11. Automorphisms of the Coordinate Ring of Quantum 3-space

Various authors examine automorphisms and graded automorphisms of quantum polynomial algebras and their generalizations (for example, see Kirkman, Kuzmanovich, and Zhang [20], Alev and Chamarie [1], and Artamonov and Wisbauer [2]). The group $(\mathbb{K}^*)^n$ of diagonal matrices is always a subgroup of the group of graded automorphisms, $\mathrm{Aut}_{\mathbb{K}} S_Q(V)$, of $S_Q(V)$. When the parameters q_{ij} are independent over $\mathbb{K}^*$, $\mathrm{Aut}_{\mathbb{K}} S_Q(V)$ contains no other automorphisms. For arbitrary parameters, the situation is more complicated to describe. In this section, we give $\mathrm{Aut}_{\mathbb{K}} S_Q(V)$ for $n = 3$ explicitly. A careful analysis of Lemma 3.2 for $n = 3$ (with help from the computer algebra system Singular [11]) leads to the following theorem, whose proof we omit for the sake of brevity.

Theorem 11.1. *Let k be a field and $\mathbb{K} = k(q_{12}, q_{13}, q_{23})$ an extension. Consider the coordinate ring of the quantum affine 3-space*

$$S_Q(V) = \mathbb{K}\langle v_1, v_2, v_3 \mid v_2v_1 = q_{12}v_1v_2,\ v_1v_3 = q_{31}v_3v_1,\ v_3v_2 = q_{23}v_2v_3\rangle.$$

Then $\mathrm{Aut}_{\mathbb{K}} S_Q(V)$ is exactly one of the following groups:

(i) *If all $q_{ij} = 1$, then $\mathrm{Aut}_{\mathbb{K}} S_Q(V) = \mathbb{GL}_3(k)$. (Here, $\mathbb{K} = k$ and $\mathrm{tr.deg}_k \mathbb{K} = 0$.)*

(ii) *If all $q_{ij} = -1$, then $\mathrm{Aut}_{\mathbb{K}} S_Q(V)$ is the subgroup of monomial matrices in $\mathbb{GL}_3(k)$. (See Corollary 3.6.) Also, $\mathbb{K} = k$ and $\mathrm{tr.deg}_k \mathbb{K} = 0$.*

(iii) $\mathrm{Aut}_{\mathbb{K}} S_Q(V) = (\mathbb{K}^*)^3$ *and* $\mathrm{tr.deg}_k \mathbb{K} \leq 3$ *unless*
- $q_{12} = q_{23} = q_{31}$, *or*
- $\{q_{12}, q_{23}, q_{31}\} = \{\pm 1, c, c^{-1}\}$ *for some c in $\mathbb{K}^*$.*

(iv) *If $q_{12} = q_{23} = q_{31} \neq \pm 1$, then $\mathrm{tr.deg}_k \mathbb{K} \leq 1$ and $\mathrm{Aut}_{\mathbb{K}} S_Q(V)$ is generated by*

$$\left\{ \begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix} \right\} \subset \mathbb{GL}_n(\mathbb{K}) .$$

(v) *If $\{q_{12}, q_{23}, q_{31}\} = \{1, c, c^{-1}\}$ for some $c \neq 1$, then $\mathrm{tr.deg}_k \mathbb{K} \leq 1$* [1] *and three cases arise:*

 (a) *If $q_{23} = 1$ and $q_{12} = q_{31}^{-1} \neq 1$, then*

$$\mathrm{Aut}_{\mathbb{K}} S_Q(V) = \left\{ \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \right\} \leq \mathbb{GL}_n(\mathbb{K}).$$

[1] We give upper bounds for tr. deg, allowing further evaluation of quantum parameters q_{ij} in addition to the given conditions on them.

(b) *If* $q_{31} = 1$ *and* $q_{12} = q_{23}^{-1} \neq 1$, *then*

$$\operatorname{Aut}_{\mathbb{K}} S_Q(V) = \left\{ \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix} \right\} \leq \mathbb{GL}_n(\mathbb{K}).$$

(c) *If* $q_{12} = 1$ *and* $q_{23} = q_{31}^{-1} \neq 1$, *then*

$$\operatorname{Aut}_{\mathbb{K}} S_Q(V) = \left\{ \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \right\} \leq \mathbb{GL}_n(\mathbb{K}).$$

(vi) *If* $\{q_{12}, q_{23}, q_{31}\} = \{-1, c, c^{-1}\}$ *for some* $c \neq -1$ *in* $\mathbb{K}^*$, *then* $\operatorname{tr.deg}_k \mathbb{K} \leq 1$ *and* $\operatorname{Aut}_{\mathbb{K}} S_Q(V)$ *is generated by* $(\mathbb{K}^*)^3$ *together with*

$$\left\{ \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix} \right\} \subset \mathbb{GL}_n(\mathbb{K}) \ \textit{if}\ q_{23} = -1, q_{12} = q_{31}^{-1}\ ,$$

$$\left\{ \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix} \right\} \subset \mathbb{GL}_n(\mathbb{K}) \ \textit{if}\ q_{31} = -1, q_{12} = q_{23}^{-1}\ ,$$

or

$$\left\{ \begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \right\} \subset \mathbb{GL}_n(\mathbb{K}) \ \textit{if}\ q_{12} = -1, q_{31}^{-1} = q_{23}\ .$$

In the next section, we will give an example using this theorem.

12. Example

In this section, we show how to use our results to work out the complete set of quantum Drinfeld Hecke algebras arising from a fixed group. We assume the characteristic of $\mathbb{K}$ is not two in this example.

Consider the subgroup G of $\mathbb{GL}_3(\mathbb{K})$ generated by the two matrices

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

and note that G is isomorphic to the dihedral group D_8 of order 8. Set $g_1 = e, g_2 = M, g_3 = N$,

$$g_4 = MN = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad g_5 = NM = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

$$g_6 = MNM = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g_7 = NMN = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \text{and}$$

$$g_8 = MNMN = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

We use Theorems 7.6 and 11.1 and the computer algebra system SINGULAR [11] to determine parameters q_{ij} and $\kappa(v_i, v_j)$ such that $\mathcal{H}_{Q,\kappa}$ is a quantum Drinfeld Hecke algebra. Condition (i) is satisfied when $q_{12}q_{23} = 1$ and $q_{13} = \pm 1$. Conditions (ii), (iii), and (iv) provide us with a linear system in terms of the $\kappa_g(v_i, v_j)$. We abbreviate notation and write $\kappa_k(i, j)$ for $\kappa_{g_k}(v_i, v_j)$. Computing minimal associated prime ideals from a primary decomposition in the affine space of parameters, we arrive at all possibilities yielding a factor algebra $\mathcal{H} := \mathcal{H}_{Q,\kappa}$ which satisfies the PBW property. The following relations $\mathcal{R}$ define all quantum Drinfeld Hecke algebras $\mathcal{H} \cong \mathbb{K}\langle v_1, v_2, v_3\rangle \# G/\langle \mathcal{R}\rangle$.

(I) For $q_{13} = 1, q_{12}q_{23} = 1$:

(a) If $q_{12} \neq q_{23}$, then the relations are

$$v_2v_1 = q_{12}v_1v_2,\ v_3v_2 = q_{12}^{-1}v_2v_3,\ v_3v_1 = v_1v_3\,.$$

(b) If $q_{12} = q_{23}$, then the parameter $\kappa_4(1, 3)$ can be chosen freely in $\mathbb{K}$ and the relations are

$$v_2v_1 = q_{12}v_1v_2,\ v_3v_2 = q_{12}v_2v_3,\ v_3v_1 = v_1v_3+\kappa_4(1,3)(t_{g_4}-t_{g_5})\,.$$

(II) For $q_{13} = -1, q_{12}q_{23} = 1$:

(c) If $q_{12}^2 = -1$ (giving a primitive fourth-root-of-unity), then $\kappa_2(1, 3)$ can be chosen freely in $\mathbb{K}$ and

$$v_2v_1 = q_{12}v_1v_2,\ v_3v_2 = -q_{12}v_2v_3,\ v_3v_1 = -v_1v_3+\kappa_2(1,3)(t_{g_2}-t_{g_7})\,.$$

(d) Otherwise, the relations are

$$v_2v_1 = q_{12}v_1v_2,\ v_3v_2 = q_{12}^{-1}v_2v_3,\ v_3v_1 = -v_1v_3\,.$$

Note that in the nonquantum setting, when $q_{13} = q_{12} = q_{23} = 1$, we recover a one-parameter family of classical Hecke Drinfeld algebras from Case (I)(b). In the quantum setting, we obtain several other one-parameter families of algebras.

References

1. J. Alev and M. Chamarie, *Dérivations et automorphismes de quelques algèbres quantiques*, Comm. Algebra **20** (1992), no. 6, 1787–1802.
2. V. Artamonov and R. Wisbauer, *Homological properties of quantum polynomials*, Algebr. Represent. Theory **4** (2001), no. 3, 219–247.
3. J. et al. Backelin, *The Gröbner basis calculator* BERGMAN, 2006, `http://servus.math.su.se/bergman`.
4. Y. Bazlov and A. Berenstein, *Noncommutative Dunkl operators and braided Cherednik algebras*, Selecta Math. **14** (2009), no. 3–4, 325–372.
5. G. Bergman, *The diamond lemma for ring theory*, Advances in Mathematics **29** (1978), no. 2, 178–218.
6. W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I: The user language*, J. Symbolic Computation **24** (1997), no. 3-4, 235–265.
7. A. Braverman and D. Gaitsgory, *Poincaré-Birkhoff-Witt theorem for quadratic algebras of Koszul type*, J. Algebra **181** (1996), no. 2, 315–328.

8. Bruno Buchberger, *Basic features and development of the critical-pair/completion procedure*, Rewriting Techniques and Applications (J.-P. Jouannaud, ed.), Lecture Notes in Computer Science, vol. 202, Springer, 1985, `http://dx.doi.org/10.1007/3-540-15976-2_1`, pp. 1–45.
9. J. Bueso, J. Gómez-Torrecillas, and A. Verschoren, *Algorithmic methods in non-commutative algebra. Applications to quantum groups*, Kluwer Academic Publishers, 2003.
10. A. M. Cohen and D. A. H. Gijsbers, GBNP*, a non–commutative Gröbner bases package for* GAP *4*, 2007, `http://www.win.tue.nl/~amc/pub/grobner`.
11. W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann, Singular *3-1-6 — A computer algebra system for polynomial computations*, (2012), `http://www.singular.uni-kl.de`.
12. V. G. Drinfeld, *Degenerate affine Hecke algebras and Yangians*, Funct. Anal. Appl. **20** (1986), 58–60.
13. P. Etingof and V. Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math. **147** (2002), no. 2, 243–348.
14. Iain Gordon, *On the quotient ring by diagonal invariants*, Invent. Math. **153** (2003), no. 3, 503–518.
15. Edward L. Green, *An introduction to noncommutative Gröbner bases*, Computational algebra (K. Fischer, ed.), vol. 151, Dekker. Lect. Notes Pure Appl. Math., 1994, pp. 167–190.
16. ———, *Noncommutative Groebner bases, and projective resolutions*, Computational methods for representations of groups and algebras (P. Dräxler, ed.), Birkhäuser, 1999, pp. 29–60.
17. G.-M. Greuel and G. Pfister, *A Singular introduction to commutative algebra*, 2nd ed., Springer, 2008.
18. S. Griffeth, *Towards a combinatorial representation theory for the rational Cherednik algebra of type $G(r,p,n)$*, Proc. Edinb. Math. Soc. (2) **53** (2010), no. 2, 419–445.
19. J. W. Helton and M. Stankus, NCGB *4.0, a noncommutative Gröbner basis package for* Mathematica, 2012, `http://www.math.ucsd.edu/~ncalg/`.
20. E. Kirkman, J. Kuzmanovich, and J. J. Zhang, *Shephard-Todd-Chevalley theorem for skew polynomial rings*, Algebr. Represent. Theory **13** (2010), no. 2, 127–158.
21. R. La Scala and V. Levandovskyy, *Letterplace ideals and non-commutative Gröbner bases*, J. Symbolic Computation **44** (2009), no. 10, 1374–1393.
22. ———, *Skew polynomial rings, Gröbner bases and the letterplace embedding of the free associative algebra*, J. Symbolic Computation **48** (2013), no. 1, 110–131, `http://dx.doi.org/10.1016/j.jsc.2012.05.003`.
23. V. Levandovskyy, *PBW bases, non-degeneracy conditions and applications*, Representation of algebras and related topics. (R.-O. Buchweitz and H. Lenzing, eds.), vol. 45, AMS. Fields Institute Communications, 2005, pp. 229–246.
24. Huishi Li, *Gröbner bases in ring theory*, World Scientific Publishing, 2012.
25. G. Lusztig, *Cuspidal local systems and graded Hecke algebras. I*, Inst. Haute Études Sci. Publ. Math. **67** (1988), 145–202.
26. ———, *Affine Hecke algebras and their graded version*, J. Amer. Math. Soc. **2** (1989), no. 3, 599–635.
27. J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, Graduate Studies in Mathematics, American Mathematical Society (AMS), 2001, With the cooperation of L. W. Small.
28. T. Mora, *An introduction to commutative and non-commutative Gröbner bases*, Theor. Comp. Sci. **134** (1994), 131–173.
29. D. Naidu and S. Witherspoon, *Hochschild cohohomology and quantum Drinfeld Hecke algebras*, (2011), Submitted. Preprint at `http://arxiv.org/abs/1111.5243v1`.

30. Donald S. Passman, *Infinite crossed products*, Pure and Applied Mathematics, 135. Boston, MA: Academic Press, 1989.
31. Arun Ram and Anne V. Shepler, *Classification of graded Hecke algebras for complex reflection groups*, Comment. Math. Helv. **78** (2003), no. 2, 308–334.
32. Anne V. Shepler and Sarah Witherspoon, *A Poincaré-Birkhoff-Witt theorem for quadratic algebras with group actions*, Trans. Amer. Math. Soc., To appear.
33. ———, *Hochschild cohomology and graded Hecke algebras*, Trans. Amer. Math. Soc. **360** (2008), no. 8, 3975–4005.
34. ———, *Group actions on algebras and the graded Lie structure of Hochschild cohomology*, Journal of Algebra **351** (2012), 350–381.
35. V. Ufnarovski, *Introduction to noncommutative Gröbner bases theory*, Gröbner bases and applications (B. Buchberger and F. Winkler, eds.), Cambridge University Press, 1998, pp. 259–280.

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