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# DEFORMATION COHOMOLOGY FOR CYCLIC GROUPS ACTING ON POLYNOMIAL RINGS

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Abstract. We examine the Hochschild cohomology governing graded deformations for finite cyclic groups acting on polynomial rings. We classify the infinitesimal graded deformations of the skew group algebra $S \rtimes G$ for a cyclic group $G$ acting on a polynomial ring $S$. This gives all graded deformations of the first order. We are particularly interested in the case when the characteristic of the underlying field divides the order of the acting group, which complicates the determination of cohomology.

## 1. Introduction

Hochschild cohomology governs deformations of algebras: Every deformation arises from a Hochschild 2-cocycle, but the converse is false, and obstructions to lifting a 2-cocycle to a deformation are witnessed by the Gerstenhaber bracket (a Lie bracket) on Hochschild cohomology. The 2-cocycles are often called *infinitesimal deformations* or *deformations of the first order*. When $A$ is a graded algebra, the Hochschild cohomology $\mathrm{HH}^{\bullet}(A)$ inherits the grading, and the graded deformations of $A$ all arise from infinitesimal deformations of graded degree $-1$ (see [9]). Thus a first step in determining the graded deformations of a given algebra $A$ centers on describing the space $\mathrm{HH}^2_{-1}(A)$ of *infinitesimal graded deformations*, i.e., the space of *first-order graded deformations* of $A$. The groups $\mathrm{HH}^j_i(A)$ are more generally called *graded Hochschild cohomology groups*, see [10].

We consider a finite group $G \subset \mathrm{GL}(V)$ acting on a finite-dimensional vector space $V \cong \mathbb{F}^n$ over a field $\mathbb{F}$ and take the induced action on the polynomial ring $S(V)$, the symmetric algebra of $V$. Deformations of the natural semidirect product algebra $S(V) \rtimes G$ include Lusztig's graded Hecke algebras (see [14, 15]), rational Cherednik algebras and symplectic reflection algebras (see [10] and [7]), and Drinfeld Hecke algebras more generally (see [6]). The Hochschild cohomology of $A = S(V) \rtimes G$ has been described in the *nonmodular setting*, when $\operatorname{char}\mathbb{F}$ and the group order $|G|$ are coprime, for $\mathbb{F}$ algebraically closed (see Proposition 3.1 below). Much less is known in the *modular setting*, when $\operatorname{char}\mathbb{F}$ divides $|G|$, as the group ring $\mathbb{F}G$ is no longer semisimple and carries its own nontrivial cohomology (e.g., see [5] and [33]). Here we investigate the case when the acting group $G$ is cyclic and we assume $\operatorname{char}\mathbb{F} \neq 2$.

**Theorem 1.1.** *Let $G \subset \mathrm{GL}(V)$ be a finite cyclic group acting on $V \cong \mathbb{F}^n$. The space of infinitesimal graded deformations of $A = S(V) \rtimes G$ is isomorphic as an $\mathbb{F}$-vector space to*

$$\mathrm{HH}^2_{-1}(A) \;\cong\; (V^G/\operatorname{Im} T)^* \oplus \big(V \otimes \textstyle\bigwedge^2 V^*\big)^G \oplus \displaystyle\bigoplus_{\substack{h\in G\\ \operatorname{codim} V^h=1}} \big(\mathbb{F} \oplus \big(V/V_h \otimes (V^h)^*\big)\big)^{\chi_h} \oplus \bigoplus_{\substack{h\in G\\ \operatorname{codim} V^h=2}} (V/V_h)^{\chi_h}\,.$$

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Here, $\chi_h$ is an analogue of the Hochschild character, see Eq. (2.2), and $T : V \to V$ is the *transfer map* $T = \sum_{h\in G} h$ on $V$.

We recover the description of $\mathrm{HH}^2_{-1}(A)$ in the nonmodular setting for $\mathbb{F}$ algebraically closed, see [25] and Proposition 3.1 below. In that setting, *bireflections* (with fixed point spaces of codimension 2) along with the identity $1_G$ contribute to this space of infinitesimals, but *reflections* in the group do not. Over $\mathbb{R}$ or $\mathbb{C}$, graded deformations like the symplectic reflection algebras and rational Cherednik algebras (see [10] and [7]) have parameters only supported on bireflections and $1_G$ for this reason.

In the modular setting, we find reflections in the group $G$ also contributing to the space of infinitesimal graded deformations. We expect this given examples (see [12] and [29]) of graded deformations like Drinfeld Hecke algebras with parameters supported on reflections in the group, not just bireflections and $1_G$. In these graded deformations of $A = S(V) \rtimes G$, the reflections record some deforming of the semidirect product structure whereas the bireflections record some deforming of the commutativity of $S(V)$. The key complication in the modular setting centers on the frequent lack of a decomposition $V = V^h \oplus (V^h)^\perp$ preserved by the centralizer $Z(h)$ of $h$ in $G$, even for $G$ abelian (see Section 9). We obtain more information in the modular setting when a Jordan canonical form is available, see Corollary 8.9.

Marcos, Martínez-Villa, and Martins [17, 16] (see also Cibils and Redondo [4]) examine the Hochschild cohomology of semidirect products $S \rtimes G$ for an $\mathbb{F}$-algebra $S$ upon which a finite group $G$ acts by automorphisms and describe actions under which

$$\mathrm{HH}^i(S \rtimes G) \cong \Big( \coprod_{(g,h)\in G\times G} \mathrm{Ext}^i_{S^e}(Sg, Sh)\Big)^{G\times G}.$$

To describe the cohomology explicitly, we take a different approach here. For a cyclic group $G$, we directly construct a twisted product resolution (see [30]) of $A = S(V) \rtimes G$ that combines a convenient periodic resolution of $\mathbb{F}G$ with the Koszul resolution of $S(V)$. This approach allows us to give a concrete description of the cohomology, which can then be used to determine deformations that simultaneously generalize graded Hecke algebras and universal enveloping algebras, see Section 9, [21], and [26]. Also see Negron [20] and Briggs and Witherspoon [2] for related ideas.

**Outline of paper.** We recall some basic facts about skew group algebras and Hochschild cohomology in Section 2 and review a formulation of the cohomology $\mathrm{HH}^\bullet(S(V) \rtimes G)$ in the nonmodular setting in Section 3. In Section 4, we use a periodic resolution for a cyclic group $G$ to construct a twisted product resolution for $A = S(V) \rtimes G$ using the Koszul resolution for $S(V)$. We decompose the space $\mathrm{HH}^2_{-1}(A)$ of infinitesimal graded deformations into contributions from each group element in Section 5. We give cocycle conditions in Section 6 in terms of the dimension of fixed point spaces. Unique cocycle representatives giving cohomology are identified in Section 7, and we use these representatives in Section 8 to give the cohomology explicitly as a vector space. Lastly, in Section 9, we demonstrate how these results may be used to find deformations by considering the transvection groups acting on 2-dimensional vector spaces.

## 2. Hochschild cohomology and skew group algebras

We take a finite group $G \subset \mathrm{GL}(V)$ acting on $V \cong \mathbb{F}^n$ and consider the induced action of $G$ on the polynomial ring $S(V)$. We take all tensor products over the field $\mathbb{F}$, $\otimes = \otimes_{\mathbb{F}}$, and assume $\mathrm{char}\,\mathbb{F} \neq 2$ throughout. We assume all algebras are associative $\mathbb{F}$-algebras.

**Group actions.** For any $\mathbb{F}G$-module $M$, we write ${}^g m$ for the image of $m$ in $M$ under the action of $g$ in $G$ to distinguish from the product of $g$ and $m$ in any algebra containing both. We take the usual induced action on functions $f : M \to M'$ defined by $({}^g f)(m) = {}^g(f({}^{g^{-1}} m))$ for $g$ in $G$ and $m$ in $M$, for any $\mathbb{F}G$-module $M'$, and we always take the trivial $G$ action on $\mathbb{F}$.

**Transfer map.** We use the classical *transfer map* in modular invariant theory (see [3]) for a group $G \subset \mathrm{GL}(V)$ restricted to the vector space $V$: Define

$$T : V \longrightarrow V, \quad v \longmapsto \sum_{h \in G} {}^h v\,. \tag{2.1}$$

**Invariant subspaces and characters.** We denote the $G$-invariants in any $\mathbb{F}G$-module $M$ by $M^G = \{m \in M : {}^g m = m \text{ for all } g \in G\}$ and, more generally, denote the $\chi$-invariants in $M$ by

$$M^\chi = \{m \in M : {}^g m = \chi(g)\, m \text{ for all } g \in G\}$$

for any linear character $\chi : G \to \mathbb{F}^\times$ of $G$. Specifically for $M = V$ and $h$ in $G$, we set

$$V^h = \mathrm{Ker}\,(1-h) = \{v \in V \,:\, {}^h v = v\} \qquad \text{and} \qquad V_h = \mathrm{Im}(1-h) = \{v - {}^h v \,:\, v \in V\}\,.$$

When $G$ is abelian, $G$ fixes set-wise both $V^h$ and $V_h$ for any $h$ in $G$ and we define a linear character giving the determinant of $G$ acting on $V/V^h$, an analogue of the *Hochschild character:*

$$\chi_h : G \longrightarrow \mathbb{F}^\times, \quad \chi_h(g) := \det \big[g\big]_{V/V^h}\,. \tag{2.2}$$

**Skew group algebras.** Recall that the *skew group algebra* $S(V) \rtimes G = S\#G$ is the natural semidirect product algebra: $S(V) \rtimes G = S(V) \otimes \mathbb{F}G$ as an $\mathbb{F}$-vector space with multiplication given by

$$(s \otimes g) \cdot (s' \otimes g') = s({}^g s') \otimes gg' \quad \text{for all } s, s' \in S(V) \text{ and } g, g' \in G\,.$$

Note that we identify $1_G$ and $1_{\mathbb{F}}$ in $\mathbb{F}G$ and identify $\mathbb{F}G$ with $1_{\mathbb{F}} \otimes \mathbb{F}G$ and $S(V)$ with $S(V) \otimes 1_G$, subspaces of $S(V) \otimes \mathbb{F}G$.

**Identifications.** For any $\mathbb{F}G$-module $M$, we identify spaces under the $\mathbb{F}G$-module isomorphism

$$\mathrm{Hom}_{\mathbb{F}}(\textstyle\bigwedge^j V, M) \cong M \otimes \textstyle\bigwedge^j V^* \quad \text{for each } j \geq 0 \tag{2.3}$$

so that $\big(\mathrm{Hom}_{\mathbb{F}}(\bigwedge^j V, M)\big)^G = (M \otimes \bigwedge^j V^*)^G$. Any bimodule over an $\mathbb{F}$-algebra $A$ is a left $A^e$-module for $A^e = A \otimes A^{op}$ the enveloping algebra of $A$ with $A^{op}$ the opposite algebra of $A$, and we also identify the spaces $\mathrm{Hom}_{A^e}(A \otimes M \otimes A, A)$ and $\mathrm{Hom}_{\mathbb{F}}(M, A)$.

**Gradings.** The group $G$ acts on $S(V)$ by graded automorphisms when we take the natural grading on $S(V)$ by polynomial degree with generators forming a vector space basis of $V$ in degree 1. This grading induces a grading on $A = S(V) \rtimes G$ after we set the degree of each group element to zero.

**Hochschild cohomology.** For an $\mathbb{F}$-algebra $A$, the Hochschild cohomology of $A$ is

$$\mathrm{HH}^\bullet(A) := \mathrm{HH}^\bullet(A, A) = \mathrm{Ext}^\bullet_{A^e}(A, A)\,.$$

The cohomology $\mathrm{HH}^\bullet(A)$ may be computed as the homology of the complex that arises from applying $\mathrm{Hom}_{A^e}(—, A)$ to the bar resolution of $A$ with $m$-th term $A \otimes A^{\otimes m} \otimes A$ for $m \geq 0$, see [37] and [22] for example.

**Grading on cohomology.** If $A$ is a graded algebra, then $\mathrm{HH}^m(A)$ inherits the induced grading on the bar resolution [37], and we denote the homogeneous component of degree $i$ by $\mathrm{HH}^m_i(A)$. Specifically, we identify $\mathrm{Hom}_{A^e}(A \otimes A^m \otimes A, A)$ and $\mathrm{Hom}_{\mathbb{F}}(A^{\otimes m}, A)$ and take the usual grading on $A^{\otimes m}$ with $\deg(a_1 \otimes \cdots \otimes a_m) = \sum_j \deg(a_j)$ for $a_j$ homogeneous in $A$, so that $\gamma$ in $\mathrm{Hom}_{A^e}(A \otimes A^m \otimes A, A)$ has degree $i$ when $\deg(\gamma(a' \otimes a \otimes a'')) = i + \deg(a)$ for all $a \in A^{\otimes m}$ and $a', a'' \in A$.

**Graded deformations.** Recall that an $\mathbb{F}$-algebra $A_t$ is a graded deformation of a graded algebra $A$ if $A_t$ is a graded algebra over $\mathbb{F}[t]$ for $\deg t = 1$ with $A_t \cong A[t]$ as an $\mathbb{F}[t]$-vector space and $A_t/tA_t \cong A$ as an $\mathbb{F}$-algebra (see [9] and [1]). We may identify a graded deformation $A_t$ with $\mathbb{F}[t] \otimes_{\mathbb{F}} A$ and the multiplication is given by

$$a * b = ab + \mu_1(a \otimes b)t + \mu_2(a \otimes b)t^2 + \dots \qquad \text{for } a, b \in A$$

extended to be linear over $\mathbb{F}[t]$ for $\mathbb{F}$-linear maps $\mu_i : A \otimes A \to A$ homogeneous of degree $-i$. The first multiplication map $\mu_1$ is then necessarily a Hochschild 2-cocycle of $A$ of degree $-1$, called the *infinitesimal* of $A_t$. Thus to classify all graded deformations of $A$, we are interested in first determining the space

$$\mathrm{HH}^2_{-1}(A) = \{\text{infinitesimal graded deformations of } A\}.$$

If an infinitesimal $\mu$ in $\mathrm{HH}^2_{-1}(A)$ is the first multiplication map $\mu_1$ for a graded deformation of $A$, then we say $\mu$ *lifts (or integrates) to a graded deformation* of $A$.

**Koszul sign convention.** We use the *Koszul sign convention* throughout for the tensor product of maps: If $f : A \to B$ and $f' : A' \to B'$ are homogeneous maps of graded vector spaces, then $f \otimes f' : A \otimes A' \to B \otimes B'$ is the map satisfying, for homogeneous $a \in A$ and $a' \in A'$,

$$(f \otimes f')(a \otimes a') = (-1)^{\deg(a)\deg(f')} f(a) \otimes f'(a') \quad \text{(Koszul sign convention)}. \tag{2.4}$$

## 3. Cohomology in the nonmodular setting

We review a description of the Hochschild cohomology for $A = S(V) \rtimes G$ in the nonmodular setting before investigating the modular case. Consider an arbitrary finite group $G \subset \mathrm{GL}(V)$ with $\mathrm{char}\,\mathbb{F}$ and $|G|$ coprime and $\mathbb{F}$ algebraically closed for $V \cong \mathbb{F}^n$. Let $C$ be a set of representatives of the conjugacy classes of $G$ and let $Z(h)$ be the centralizer of $h$ in $G$. As $|G|$ is invertible in $\mathbb{F}$, there is a $G$-invariant inner product on $V$ (obtained by averaging any inner product over the group $G$). Then for any $h$ in $G$, $Z(h)$ preserves set-wise both $V^h$ and $(V^h)^\perp$, the orthogonal complement to $V^h$, and we may define the *Hochschild character*

$$\chi_h : Z(h) \to \mathbb{F}, \quad z \mapsto \det(z|_{(V^h)^\perp}),$$

recording the determinant of $Z(h)$ acting on $(V^h)^\perp$. Note that if $G$ is abelian, we may identify the Hochschild character $\chi_h$ with the linear character of Eq. (2.2) after identifying the $\mathbb{F}G$-modules $(V^h)^\perp$ and $V/V^h$ using the fact that $V$ decomposes as $V^h \oplus (V^h)^\perp$ as an $\mathbb{F}G$-module for $h$ in $G$.

**Proposition 3.1.** *Consider a finite group $G \subset \mathrm{GL}(V)$ acting on $V \cong \mathbb{F}^n$ with $|G|$ coprime to* $\operatorname{char}\mathbb{F}$ *and $\mathbb{F}$ algebraically closed. For $A = S(V) \rtimes G$,*

$$\mathrm{HH}^{\bullet}(A) \;\cong\; \bigoplus_{h\in C} \Big( S(V^h) \otimes \textstyle\bigwedge^{\bullet - \operatorname{codim} V^h} (V^h)^* \Big)^{\chi_h} \qquad \textit{and, in particular,}$$

$$\mathrm{HH}^2(A) \;\cong\; \big( S(V) \otimes \textstyle\bigwedge^2 V^* \big)^G \;\oplus \bigoplus_{\substack{h\in C \\ \operatorname{codim} V^h = 2 \\ \det h = 1}} \big( S(V^h) \big)^{\chi_h} .$$

*Proof.* The arguments in [38, Section 6] and [25, Theorem 3.1] (see also [37, Example 3.5.7]) using [36] give the first isomorphism (also see [8]). Note that each space of $\chi_h$-invariants is $\{0\}$ unless $\det(h) = \chi_h(h) = 1$ since $h$ itself acts trivially on $S(V^h) \otimes \bigwedge^{\bullet - \operatorname{codim} V^h} (V^h)^*$ (see [25, Equation (3.7)]). Since any reflection in $G$ is diagonalizable with determinant $\neq 1$, we thus need only sum over group elements whose fixed point spaces have codimension 0 or 2 to find $\mathrm{HH}^2(A)$. □

Under the induced grading on $\mathrm{HH}^{\bullet}(A)$, the graded component $\mathrm{HH}^m_i(A)$ is the subspace of $\mathrm{HH}^m(A)$ consisting of elements whose first tensor component has polynomial degree $i+m$:

**Proposition 3.2.** *Let $G \subset \mathrm{GL}(V)$ be a finite group acting on $V \cong \mathbb{F}^n$ with $|G|$ coprime to* $\operatorname{char}\mathbb{F}$ *and $\mathbb{F}$ algebraically closed. The space of infinitesimal graded deformations of $A = S(V) \rtimes G$ is isomorphic as an $\mathbb{F}$-vector space to*

$$\mathrm{HH}^2_{-1}(A) \;\cong\; (V \otimes \textstyle\bigwedge^2 V^*)^G \;\oplus \bigoplus_{\substack{h\in G \\ \operatorname{codim} V^h = 2 \\ \det h = 1}} \big( V^h \big)^{\chi_h} .$$

For $G$ cyclic, our main finding Theorem 8.1 recovers Proposition 3.2 in this nonmodular setting, see Remark 8.8. Note that the case where $S(V)$ is replaced by a quantum polynomial ring over fields of characteristic 0 was explored by Naidu, Shakalli, Shroff, and Witherspoon see [18, 23, 19, 31]. For deformations in that setting, see also [13, 32].

## 4. Periodic-twisted-Koszul Resolution for cyclic groups

We consider a finite cyclic group $G$ acting linearly on $V \cong \mathbb{F}^n$ for $\mathbb{F}$ a field of arbitrary characteristic. To compute the Hochschild cohomology of $A = S(V) \rtimes G$, we use a twisted product resolution for $A$: We twist together a periodic resolution for the group $G$ and the Koszul resolution for $S(V)$. See [27, 30] for the construction details (and requirements) of twisted product resolutions for general skew group algebras. We fix a generator $g$ of $G$.

**A periodic resolution for cyclic groups.** We may identify $\mathbb{F}G$ with $\mathbb{F}[x]/(x^{|G|} - 1)$ for some indeterminate $x$ and use a well-known resolution $P_\bullet$ of $\mathbb{F}G$ given by (see [11])

$$P_\bullet : \quad \cdots \xrightarrow{\ \gamma\ } \mathbb{F}G \otimes \mathbb{F}G \xrightarrow{\ \eta\ } \mathbb{F}G \otimes \mathbb{F}G \xrightarrow{\ \gamma\ } \mathbb{F}G \otimes \mathbb{F}G \xrightarrow{\ m\ } \mathbb{F}G \longrightarrow 0,$$

where $\gamma = g \otimes 1 - 1 \otimes g$, $\ \eta = g^{-1} \otimes 1 + g^{-2} \otimes g + \cdots + 1 \otimes g^{-1}$ and $m$ is multiplication.

To satisfy the compatibility requirements for constructing a twisted product resolution (see [30, Defintion 2.17]), we use the following $G$-grading on $\mathbb{F}G \otimes \mathbb{F}G$: For any $h$ in $G$ and $P_i = \mathbb{F}G \otimes \mathbb{F}G$ for $i \geq 0$, set

$$(P_i)_h = \begin{cases} \mathrm{Span}_{\mathbb{F}}\{a \otimes b \ : \ ab = h\} & \text{for } i \text{ even} \\ \mathrm{Span}_{\mathbb{F}}\{a \otimes b \ : \ ab = hg^{-1}\} & \text{for } i \text{ odd}. \end{cases}$$

**The Koszul complex.** Recall the bimodule Koszul complex for the symmetric algebra $S(V)$:

$$K_\bullet : \;\cdots \longrightarrow S(V)\otimes \textstyle\bigwedge^2 V\otimes S(V) \longrightarrow S(V)\otimes V\otimes S(V) \longrightarrow S(V)\otimes S(V) \longrightarrow S(V) \longrightarrow 0\,,$$

with differentials defined, for all $w_1\wedge\cdots\wedge w_j$ in $\bigwedge^j V$, by

$$\partial_K(1\otimes w_1\wedge\cdots\wedge w_j\otimes 1) = \sum_{\ell=1}^{j}(-1)^{\ell-1}(w_\ell\otimes w_1\wedge\cdots\wedge\hat{w}_\ell\wedge\cdots\wedge w_j\otimes 1 - 1\otimes w_1\wedge\cdots\wedge\hat{w}_\ell\wedge\cdots\wedge w_j\otimes w_\ell)\,.$$

This resolution satisfies the compatibility requirements of [30] for a twisted product resolution.

**Periodic-twisted-Koszul resolution.** The periodic-twisted-Koszul resolution $X_\bullet = P_\bullet \otimes^G K_\bullet$ of $A = S(V)\rtimes G$ is the total complex (see [27, 30])

$$X_m = \bigoplus_{i+j=m} X_{i,j} \quad\text{for}\quad X_{i,j} = P_i\otimes K_j = (\mathbb{F}G\otimes\mathbb{F}G)_i\otimes\big(S(V)\otimes\textstyle\bigwedge^j V\otimes S(V)\big) \quad\text{for } i,j\geq 0$$

with $A$-bimodule structure on each $P_i\otimes K_j$ given by

$$s(y_1\otimes y_2)a = y_1 a\otimes {}^{(ha)^{-1}}s\ {}^{a^{-1}}y_2 \qquad \text{for } y_1\in(P_i)_h,\ y_2\in K_j,\ a,h\in G,\ s\in S(V)$$

and differentials $d_m : X_m \to X_{m-1}$ given by $d_m = \sum_{i+j=m} d^{\text{hor}}_{i,j} + d^{\text{vert}}_{i,j}$ for horizontal and vertical maps

$$d^{\text{hor}}_{i,j} := \partial_P\otimes 1_K : X_{i,j}\to X_{i-1,j} \quad\text{and}\quad d^{\text{vert}}_{i,j} := 1_P\otimes\partial_K : X_{i,j}\to X_{i,j-1}\,.$$

Here, $\partial_P$ and $\partial_K$ are the differentials for $P_\bullet$ and $K_\bullet$, respectively. The complex $X_\bullet$ gives a free $A$-bimodule resolution of $A = S(V)\rtimes G$:

$$X_\bullet : \quad \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow A \longrightarrow 0\,.$$

We identify each $X_{i,j}$ with $A\otimes\bigwedge^j V\otimes A$ using the $A$-bimodule isomorphism

$$X_{i,j} \xrightarrow{\cong} A\otimes\textstyle\bigwedge^j V\otimes A$$

given by, for $a'\otimes a\in(P_i)_h$, $w_1\wedge\cdots\wedge w_j\in\bigwedge^j V$, and $s,r$ in $S(V)$,

$$(a'\otimes a)\otimes(s\otimes w_1\wedge\cdots\wedge w_j\otimes r) = ({}^h s\otimes a')\otimes {}^a(w_1\wedge\cdots\wedge w_j)\otimes({}^a r\otimes a)\,.$$

Note that after the identifications, the differential encodes the group action although the group algebra $\mathbb{F}G$ does not appear in $X_{i,j} = A\otimes\bigwedge^j V\otimes A$ as an overt tensor component.

The Hochschild cohomology $\mathrm{HH}^\bullet(A)$ is thus the homology of the complex

$$\cdots \longrightarrow \mathrm{Hom}_{A^e}(X_2,A) \longrightarrow \mathrm{Hom}_{A^e}(X_1,A) \longrightarrow \mathrm{Hom}_{A^e}(X_0,A) \longrightarrow 0\,.$$

The grading on each $\mathrm{HH}^m(A)$ induced by the grading on $A$ coincides with the grading on $\bigwedge V$ with $\bigwedge^j V$ in degree $j$ so that $\gamma\in\mathrm{Hom}_{A^e}(A\otimes\bigwedge^j V\otimes A, A)$ has degree $i$ when $\deg(\gamma(a\otimes v_1\wedge\cdots\wedge v_j\otimes a')) = i+j$ for $v_i\in V$ and $a,a'\in A$. We denote by $\mathrm{C}^m_i(A)$, $\mathrm{Z}^m_i(A)$, and $\mathrm{B}^m_i(A)$ the spaces of $m$-cochains, $m$-cocycles, and $m$-coboundaries, respectively, of graded degree $i$ so that

$$\mathrm{HH}^2_{-1}(A) \;=\; \mathrm{Z}^2_{-1}(A)/\mathrm{B}^2_{-1}(A)\,. \tag{4.1}$$

**Explicit differential.** We give the differential in the resolution $X_\bullet$ explicitly. We use the Koszul sign convention (Eq. (2.4)) with respect to homological degree noting that $\partial_P$ and $\partial_K$ each have degree $-1$ while the identity has degree 0: For $y_1 \in P_i = \mathbb{F}G \otimes \mathbb{F}G$ and $y_2$ in $S(V) \otimes \textstyle\bigwedge^j V \otimes S(V)$,

$$(1 \otimes \partial_K)(y_1 \otimes y_2) = (-1)^{(\deg y_1)(\deg \partial_K)}\big(y_1 \otimes \partial_K(y_2)\big) = (-1)^i\big(y_1 \otimes \partial_K(y_2)\big).$$

The horizontal differentials $\partial_P \otimes 1_K$ on $X_{i,j}$ are defined by

$$d^{\text{hor}}_{i,j}(\overline{w}) = \begin{cases} g \otimes w_1 \wedge \cdots \wedge w_j \otimes 1 - 1 \otimes {}^g(w_1 \wedge \cdots \wedge w_j) \otimes g & \text{if } i \text{ is odd} \\ \displaystyle\sum_{\ell=0}^{|G|-1} g^{-1-\ell} \otimes {}^{g^\ell}(w_1 \wedge \cdots \wedge w_j) \otimes g^\ell & \text{if } i \text{ is even} \end{cases}$$

and the vertical differentials $1_P \otimes \partial_K$ are defined by (correcting a misprint in [27, Example 4.6])

$$d^{\text{vert}}_{i,j}(\overline{w}) = \begin{cases} \displaystyle\sum_{\ell=1}^{j}(-1)^\ell({}^g w_\ell \otimes w_1 \wedge \cdots \wedge \hat{w}_\ell \wedge \cdots \wedge w_j \otimes 1 - 1 \otimes w_1 \wedge \cdots \wedge \hat{w}_\ell \wedge \cdots \wedge w_j \otimes w_\ell) & i \text{ odd} \\ \displaystyle\sum_{\ell=1}^{j}(-1)^{\ell-1}(w_\ell \otimes w_1 \wedge \cdots \wedge \hat{w}_\ell \wedge \cdots \wedge w_j \otimes 1 - 1 \otimes w_1 \wedge \cdots \wedge \hat{w}_\ell \wedge \cdots \wedge w_j \otimes w_\ell) & i \text{ even,} \end{cases}$$

for $\overline{w} = 1 \otimes w_1 \wedge w_2 \wedge \cdots \wedge w_j \otimes 1$ in $A \otimes \textstyle\bigwedge^j V \otimes A$.

## 5. Cocycle conditions and cohomology decomposition

We begin our analysis of the space of infinitesimal graded deformations with initial cocycle conditions. These conditions allow us to decompose the Hochschild cohomology into contributions from each group element. Again, we consider a finite cyclic group $G \subset \mathrm{GL}(V)$ acting on $V \cong \mathbb{F}^n$ and fix a generator $g$ of $G$ with which to construct the periodic-twisted-Koszul resolution $X_\bullet$ (see Section 4) of $A = S(V) \rtimes G$ giving the space $\mathrm{HH}^2_{-1}(A)$ of infinitesimal graded deformations of $A$. We examine the differential of the resolution $X = P_G \otimes^G K_S$ of $A$ to determine preliminary cocycle conditions.

**Decomposing cochains.** The space $\mathrm{C}^2_{-1}(A) = \big(\mathrm{Hom}_{A^e}(X_2, A)\big)_{-1}$ of 2-cochains of degree $-1$ decomposes into a space of maps on $V$ and a space of maps on $\textstyle\bigwedge^2 V$:

$$\mathrm{C}^2_{-1}(A) = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F}G) \oplus \mathrm{Hom}_{\mathbb{F}}(\textstyle\bigwedge^2 V,\, V \otimes \mathbb{F}G) = (V^* \otimes \mathbb{F}G) \oplus (V \otimes \textstyle\bigwedge^2 V^* \otimes \mathbb{F}G)\,, \tag{5.1}$$

since $X_2 = (A \otimes A) \oplus (A \otimes V \otimes A) \oplus (A \otimes \textstyle\bigwedge^2 V \otimes A)$ and $\mathbb{F}G$ is the degree 0 component of $A$, using the usual identifications (see Eq. (2.3)). Note here that the only $A^e$-homomorphism $A \otimes A \to A$ of degree $-1$ is the zero map.

**Cochains decomposed by group contribution.** We decompose the vector space of cochains according to group elements with an extra *shift* by the generator $g$ of $G$. From Eq. (5.1),

$$\mathrm{C}^2_{-1}(A) = \bigoplus_{h \in G} (V^* \otimes \mathbb{F}h) \oplus \big(V \otimes \textstyle\bigwedge^2 V^* \otimes \mathbb{F}h\big) = \displaystyle\bigoplus_{h \in G} (V^* \otimes \mathbb{F}hg) \oplus \bigoplus_{h \in G} \big(V \otimes \textstyle\bigwedge^2 V^* \otimes \mathbb{F}h\big)\,.$$

For each $h \in G$, we set

$$\mathrm{C}^2_{-1}(h) := \underbrace{\big(V^* \otimes \mathbb{F}hg\big)}_{\lambda\text{-part}} \oplus \underbrace{\big(V \otimes \textstyle\bigwedge^2 V^* \otimes \mathbb{F}h\big)}_{\alpha\text{-part}} \qquad \text{so that} \qquad \mathrm{C}^2_{-1}(A) = \displaystyle\bigoplus_{h \in G} \mathrm{C}^2_{-1}(h)\,.$$

Furthermore, we define the coboundaries and cocycles for $h \in G$ by

$$\mathrm{B}^2_{-1}(h) := \mathrm{C}^2_{-1}(h) \cap \mathrm{B}^2_{-1}(A) \qquad \text{and} \qquad \mathrm{Z}^2_{-1}(h) := \mathrm{C}^2_{-1}(h) \cap \mathrm{Z}^2_{-1}(A)$$

and define the cohomology for $h$ by

$$\mathrm{HH}^2_{-1}(h) \;\; := \;\; \mathrm{Z}^2_{-1}(h)/\mathrm{B}^2_{-1}(h)\,. \tag{5.2}$$

We justify this terminology (and notation) in Proposition 5.5 below by exhibiting $\mathrm{HH}^2_{-1}(A)$ as the direct sum of the $\mathrm{HH}^2_{-1}(h)$. First, we establish two lemmas that will be useful here and later. We consider cocycle conditions in the first lemma using the transfer map $T$ (see Eq. (2.1)) and consider coboundary conditions in the second. We write any cochain $\gamma$ in $\mathrm{C}^2_{-1}(h)$ as $\gamma = (\lambda\otimes hg)\oplus(\alpha\otimes g)$ for $\lambda$ in $\mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F})$ and $\alpha$ in $\mathrm{Hom}_{\mathbb{F}}(\bigwedge^2 V, V)$.

**Lemma 5.3** (Cocycles)**.** *For any $h$ in $G$, a cochain $(\lambda\otimes hg)\oplus(\alpha\otimes h)$ in $\mathrm{C}^2_{-1}(h)$ is a cocycle in $\mathrm{Z}^2_{-1}(h)$ if and only if*

(1) $0=\lambda(\mathrm{Im}(T))$ in $\mathbb{F}$,

(2) $0=(\alpha-{}^{g^{-1}}\alpha)(u\wedge v)-\lambda(v)(u-{}^{h}u)+\lambda(u)(v-{}^{h}v)$ in $V$ *for all $u,v\in V$, and*

(3) $0=\alpha(u\wedge v)(w-{}^{h}w)+\alpha(v\wedge w)(u-{}^{h}u)+\alpha(w\wedge u)(v-{}^{h}v)$ in $S(V)$ *for all $u,v,w\in V$.*

*Proof.* We first consider a cochain $\gamma=\lambda\oplus\alpha$ in $\mathrm{C}^2_{-1}(A)$ using Eq. (5.1) for $\lambda$ in $\mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F}G)$ and $\alpha$ in $\mathrm{Hom}_{\mathbb{F}}(\bigwedge^2 V, V\otimes\mathbb{F}G)$. We examine $d\gamma$ for the differential $d$ on the resolution $X_\bullet$ (see Section 4) and conclude that $\gamma$ lies in $\mathrm{Z}^2_{-1}(A)$ if and only if

- $0=\lambda(\mathrm{Im}\,T)$,
- $0=g\alpha(u\wedge v)-\alpha({}^{g}u\wedge{}^{g}v)g-{}^{g}u\,\lambda(v)+\lambda(v)u+{}^{g}v\,\lambda(u)-\lambda(u)v$ for all $u,v\in V$, and
- $0=[u,\alpha(v\wedge w)]+[v,\alpha(w\wedge u)]+[w,\alpha(u\wedge v)]$ for all $u,v,w\in V$,

where the bracket is the commutator in $A$. Now decompose $\alpha$ and $\lambda$ according to group elements,

$$\lambda(v)=\sum_{h\in G}\lambda_h(v)h \qquad\text{and}\qquad \alpha(v\wedge w)=\sum_{h\in G}\alpha_h(v\wedge w)\otimes h \qquad\text{for all } v,w \text{ in } V$$

where $\lambda_h: V\to\mathbb{F}$ and $\alpha_h:\bigwedge^2V\to V$, so that $\gamma=\sum_{h\in G}\gamma_h$ for $\gamma_h=(\lambda_{hg}\otimes hg)\oplus(\alpha_h\otimes h)$. We compare coefficients to see that $\gamma$ satisfies the above three conditions if and only if each $\gamma_h$ does if and only if each $\gamma_h$ satisfies the conditions in the lemma. Hence $\gamma$ lies in $\mathrm{Z}^2_{-1}(A)$ if and only if each $\gamma_h$ lies in $\mathrm{Z}^2_{-1}(A)$ and thus in $\mathrm{Z}^2_{-1}(h)$. ☐

**Lemma 5.4** (Coboundaries)**.** *For any $h$ in $G$, a cochain $(\lambda\otimes hg)\oplus(\alpha\otimes h)$ in $\mathrm{C}^2_{-1}(h)$ is a coboundary in $\mathrm{B}^2_{-1}(h)$ if and only if there is some map $f: V\to\mathbb{F}$ with*

$$\lambda(u)=f(u-{}^{g}u) \qquad\textit{and}\qquad \alpha(u\wedge v)=f(v)(u-{}^{h}u)-f(u)(v-{}^{h}v) \qquad\textit{for all } u,v\in V.$$

*Proof.* We first write out conditions for a generic coboundary. Say $f'$ is a 1-cochain in $\mathrm{C}^1_{-1}(A)=V^*\otimes\mathbb{F}G$ and write $f'=\sum_{h\in G}f_h\otimes h$ with each $f_h\otimes h\in V^*\otimes\mathbb{F}h\subset \mathrm{C}^1_{-1}(A)$. Then again using the differential $d$ on the resolution $X_\bullet$ (see Section 4), we observe that

$$df'=\sum_{h\in G}d(f_h\otimes h) \qquad\text{with}\qquad d(f_h\otimes h)=(\lambda_{hg}\otimes hg)\oplus(\alpha_h\otimes h)\,,$$

for

$$\lambda_{gh}(u)=f_h(u-{}^{g}u) \qquad\text{and}\qquad \alpha_h(u\wedge v)=f_h(v)(u-{}^{h}u)-f_h(u)(v-{}^{h}v) \text{ for all } u,v\in V\,.$$

Then each $(\lambda_{hg}\otimes hg)\oplus(\alpha_h\otimes h)$ is a coboundary in $\mathrm{B}^2_{-1}(h)=\mathrm{C}^2_{-1}(h)\cap\mathrm{B}^2_{-1}(A)$ and satisfies the condition in the statement of the lemma. Thus for $h$ in $G$, if a cochain $(\lambda\otimes hg)\oplus(\alpha\otimes h)$ in $\mathrm{C}^2_{-1}(h)$ is $df'$ for some 1-cochain $f'$ in $\mathrm{C}^1_{-1}(A)$, then $f'=f\otimes h$ for some $f\in V^*$ satisfying the conclusion of the lemma. Conversely, if there is a function $f$ as in the statement of the lemma, then $f \otimes h$ is a cochain in $C^1_{-1}(A)$ with $d(f \otimes h) = (\lambda \otimes hg) \oplus (\alpha \otimes h)$. □

**Cohomology decomposed by group contribution.** Now we may decompose cohomology group element by group element:

**Proposition 5.5.** *Let $G \subset \mathrm{GL}(V)$ be a finite cyclic group acting on $V \cong \mathbb{F}^n$. Then*

$$\mathrm{B}^2_{-1}(A) \;=\; \bigoplus_{h \in G} \mathrm{B}^2_{-1}(h) \quad \textit{and} \quad \mathrm{Z}^2_{-1}(A) \;=\; \bigoplus_{h \in G} \mathrm{Z}^2_{-1}(h)\,.$$

*Thus the space of infinitesimal graded deformations of $A = S(V) \rtimes G$ is*

$$\mathrm{HH}^2_{-1}(A) \;\cong\; \bigoplus_{h \in G} \mathrm{HH}^2_{-1}(h)\,.$$

*Proof.* To verify that $\mathrm{B}^2_{-1}(A) \subset \bigoplus_{h\in G} \mathrm{B}^2_{-1}(h)$, consider $df$ in $\mathrm{B}^2_{-1}(A)$ with $f$ a 1-cochain in $\mathrm{C}^1_{-1}(A) = V^* \otimes \mathbb{F}G$. We saw in the proof of Lemma 5.4 that $df = \sum_{h\in G} d(f_h \otimes h)$ where $f = \sum_{h \in G} f_h \otimes h$ with each summand $f_h \otimes h$ in $V^* \otimes \mathbb{F}h \subset \mathrm{C}^1_{-1}(A)$ and $d(f_h \otimes \mathbb{F}h)$ in $\mathrm{C}^2_{-1}(h) \cap \mathrm{B}^2_{-1}(A) = \mathrm{B}^2_{-1}(h)$. Thus $df$ lies in $\bigoplus_{h\in G} \mathrm{B}^2_{-1}(h)$. The reverse inclusion is clear.

To verify that $\mathrm{Z}^2_{-1}(A) = \bigoplus_{h\in G} \mathrm{Z}^2_{-1}(h)$, we refer to the proof of Lemma 5.3: For any $\gamma = \sum_{h\in G} \gamma_h$ in $\mathrm{C}^2_{-1}(A)$ with each $\gamma_h$ in $\mathrm{C}^2_{-1}(h)$, $\gamma$ lies in $\mathrm{Z}^2_{-1}(A)$ if and only if each $\gamma_h$ lies in $\mathrm{Z}^2_{-1}(h)$. □

## 6. Cocycle conditions in terms of codimension

In this section, we detangle the cocycle conditions for the space $\mathrm{HH}^2_{-1}(A)$ of infinitesimal graded deformations of $A = S(V) \rtimes G$ for $G \subset \mathrm{GL}(V)$ a finite cyclic group acting on $V \cong \mathbb{F}^n$. We again fix a generator $g$ of $G$ to define the resolution $X_\bullet$, see Section 4.

**Vector space complements and projections.** We describe $\mathrm{HH}^2_{-1}(A)$ using a choice of cohomology representatives depending on projection maps. Recall that a $G$-invariant inner product on $V$ may not exist, but we use the notation of an orthogonal complement in any case in analogy with the nonmodular setting. We choose a vector space complement $(V^h)^\perp$ to $V^h$ for each $h$ in $G$ so $V = V^h \oplus (V^h)^\perp$. Note that $V^g = V^G$ and so this gives a decomposition $V = V^G \oplus (V^G)^\perp$. We also choose a vector space complement $(V_h)^\perp$ to $V_h$ with projection map $\pi_h : V \to V_h$:

$$V = \mathrm{Im}(1-h) \oplus \mathrm{Im}(1-h)^\perp = V_h \oplus (V_h)^\perp\,. \tag{6.1}$$

**Cocycle condition (1).** We interpret the first cocycle condition of Lemma 5.3. Recall that $h$ in $G$ is a *reflection* when the fixed-point space $V^h$ is a hyperplane, i.e., $\operatorname{codim} V^h = 1$. Note that if $h$ is a reflection, then either $h$ is diagonalizable with order $|h|$ coprime to $\operatorname{char}\mathbb{F}$ or $h$ is nondiagonalizable with order $|h| = \operatorname{char}\mathbb{F}$ (see [35]). Also note that the following lemma fails when $\operatorname{char}\mathbb{F} = 2 = n = |G|$, but we have excluded $\operatorname{char}\mathbb{F} = 2$ from consideration throughout.

**Lemma 6.2.** *If $G$ contains a nondiagonalizable reflection, then* $\mathrm{Im}\, T = \{0\}$.

*Proof.* Let $h$ be a nondiagonalizable reflection with $H = \langle h \rangle \subset G$ and let $p = \operatorname{char}\mathbb{F} = |h|$. We write $T^G = \sum_{a\in G} a$ and $T^H = \sum_{a \in H} a$ as transformations on $V$ and note that for a transversal $g_1, g_2, \ldots, g_m$ of $H$ in $G$ (coset representatives for $G/H$) with $m = [G : H]$ (see [34, Theorem 4.1] and [24]),

$$T^G(v) = \sum_{a\in G} {}^a v = \sum_{i=1}^m \sum_{a \in H} {}^{g_i a} v = \sum_{i=1}^m {}^{g_i}\Big( \sum_{a\in H} {}^a v\Big) = \sum_{i=1}^m {}^{g_i}\big(T^H(v)\big) \text{ for } v \in V\,.$$

We argue that $\operatorname{Im} T^G$ is zero by showing $\operatorname{Im} T^H$ is zero. There is a basis $v_1, \dots, v_n$ of $V$ with

$$ {}^h v_i = v_i \quad \text{for } i < n \qquad \text{and} \qquad {}^h v_n = v_1 + v_n \,. $$

Then $T^H(v_i) = |H|\, v_i = 0$ for $i < n$ and

$$ T^H(v_n) = \sum_{i=0}^{p-1} {}^{h^i} v_n = \sum_{i=0}^{p-1} (i v_1 + v_n) = \tfrac{p(p-1)}{2}\, v_1 = 0 \qquad \text{as well.} $$

□

**Example 6.3.** Note that $\operatorname{Im} T \neq 0$ for $G \subset \mathrm{GL}_4(\mathbb{F}_3)$ generated by $\left(\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{smallmatrix}\right)$, but $\operatorname{Im} T = 0$ for $G \subset \mathrm{GL}_3(\mathbb{F}_3)$ generated by $g = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$ by Lemma 6.2 since $g^4$ is a nondiagonalizable reflection.

**Cocycle condition (3).** We also interpret the third cocycle condition of Lemma 5.3 for cochains. Recall that we write any $\gamma$ in $C^2_{-1}(h) = (V^* \otimes \mathbb{F}hg) \oplus (V \otimes \bigwedge^2 V^* \otimes \mathbb{F}h)$ as $\gamma = (\lambda \otimes hg) \oplus (\alpha \otimes g)$ for $\lambda$ in $\operatorname{Hom}_{\mathbb{F}}(V, \mathbb{F})$ and $\alpha$ in $\operatorname{Hom}_{\mathbb{F}}(\bigwedge^2 V, V)$.

**Lemma 6.4.** *For any $h$ in $G$, if a cochain $\gamma = (\lambda \otimes hg) \oplus (\alpha \otimes h)$ in $\mathrm{C}^2_{-1}(h)$ satisfies Lemma 5.3(3), then either*

(a) *$h$ is the identity element of $G$, or*
(b) *$\operatorname{codim} V^h = 1$ and $\alpha(u \wedge v) = 0$ for all $u, v$ in $V^h$, or*
(c) *$\operatorname{codim} V^h = 2$ and $\alpha(u \wedge v)$ lies in $\mathbb{F}$-span$\{v - {}^h v\}$ for $u \in V^h$ and $v \in V$, or*
(d) *$\operatorname{codim} V^h > 2$ and $\alpha(u \wedge v)$ lies in $\mathbb{F}$-span$\{u - {}^h u,\ v - {}^h v\}$ for all $u, v \in V$.*

*Proof.* For brevity, write $\hat{u} = u - {}^h u$ for all $u \in V$ so that Lemma 5.3(3) implies that

$$ 0 = \alpha(u \wedge v)\hat{w} + \alpha(v \wedge w)\hat{u} + \alpha(w \wedge u)\hat{v} \qquad \text{in } S(V) \text{ for all } u, v, w \in V\,. \tag{6.5} $$

If $u, v \in V^h$ and $\operatorname{codim} V^h \geq 1$, then $\alpha(u \wedge v) = 0$ as we may choose $w \notin V^h$. If $u \in V^h$ and $v \in (V^h)^\perp$ and $\operatorname{codim} V^h \geq 2$, then $\alpha(u \wedge v) \in \mathbb{F}\hat{v}$ as we may choose $w$ with $\hat{v}$ and $\hat{w}$ independent. Lastly, if $u, v \in (V^h)^\perp$ with $\operatorname{codim} V^h > 2$, then $\alpha(u \wedge v) \in \mathbb{F}\hat{u} + \mathbb{F}\hat{v}$ as we may choose $w$ with $\hat{w} \notin \mathbb{F}\hat{u} + \mathbb{F}\hat{v}$. □

**Lemma 6.6** (A partial converse to Lemma 6.4)**.** *For any $h$ in $G$, suppose $\gamma = (\lambda \otimes hg) \oplus (\alpha \otimes h)$ is a cochain in $\mathrm{C}^2_{-1}(h)$ with either*

(a) *$h$ is the identity element of $G$, or*
(b) *$\operatorname{codim} V^h = 1$ and $\alpha(u \wedge v) = 0$ for all $u, v$ in $V^h$, or*
(c) *$\operatorname{codim} V^h = 2$ and $\alpha(u \wedge v) = 0$ for $u \in V^h$ and $v \in V$, or*
(d) *$\operatorname{codim} V^h > 2$ and $\alpha(u \wedge v) = 0$ for all $u, v \in V$.*

*Then $\gamma$ satisfies Lemma 5.3(3).*

*Proof.* We again write $\hat{v} = v - {}^h v$ for all $v \in V$ and verify that

$$ 0 = \alpha(u \wedge v)\hat{w} + \alpha(v \wedge w)\hat{u} + \alpha(w \wedge u)\hat{v} \qquad \text{for all } u, v, w \in V\,. \tag{6.7} $$

We assume $\operatorname{codim} V^h$ is 1 or 2 else the statement is trivial and consider $V = V^h \oplus (V^h)^\perp$. Notice that the right hand side of Eq. (6.7) vanishes automatically for $u, v, w \in (V^h)^\perp$ since it defines an alternating linear function and $\bigwedge^3 (V^h)^\perp = \{0\}$. If $\operatorname{codim} V^h = 1$, then the right hand side also vanishes for $u, v \in V^h$, $w \in V$ by (b) and for $u \in V^h$, $v, w \in (V^h)^\perp$ since it is alternating in $v, w$ for fixed $u$ and $\bigwedge^2 (V^h)^\perp = \{0\}$. Lastly, if $\operatorname{codim} V^h = 2$, then the right side of Eq. (6.7) vanishes for $u \in V^h$, $v, w \in V$ as (c) implies that $\alpha(u \wedge v) = 0 = \alpha(u \wedge w)$. □

## 7. Unique cohomology representatives

In this section, we identify unique representatives for the cohomology classes in the space $\mathrm{HH}^2_{-1}(A)$ of infinitesimal graded deformations of $A = S(V) \rtimes G$ for a finite cyclic group $G \subset \mathrm{GL}(V)$ generated by $g$ acting on $V \cong \mathbb{F}^n$. By Proposition 5.5,

$$\mathrm{HH}^2_{-1}(A) \;=\; \bigoplus_{h\in G} \mathrm{HH}^2_{-1}(h) \qquad \text{with} \qquad \mathrm{HH}^2_{-1}(h) \;=\; \mathrm{Z}^2_{-1}(h)/\mathrm{B}^2_{-1}(h)\,,$$

and we describe coset representatives for each $\mathrm{HH}^2_{-1}(h)$ using a choice of vector space complement $(V_h)^\perp$ to $V_h$ and complement $(V^h)^\perp$ to $V^h$ (see Eq. (6.1)) with projection map $\pi_h : V \to V_h$. Recall that the space of cochains for each $h$ in $G$ is $\mathrm{C}^2_{-1}(h) = (V^* \otimes \mathbb{F}gh) \oplus (V \otimes \textstyle\bigwedge^2 V^* \otimes \mathbb{F}h)$.

**Proposition 7.1.** *Fix $h \in G$. Each coset in $\mathrm{HH}^2_{-1}(h)$ has a unique representative $\gamma = (\lambda \otimes hg) \oplus (\alpha \otimes h)$ in $\mathrm{Z}^2_{-1}(h)$ for $\lambda \in V^*$ and $\alpha \in \mathrm{Hom}_{\mathbb{F}}(\bigwedge^2 V, V)$ with $\pi_h\alpha \equiv 0$ and*

(a) *when $\operatorname{codim} V^h = 0$ (so $h = 1_G$), $\lambda \equiv 0$ on $(V^G)^\perp$,*
(b) *when $\operatorname{codim} V^h = 1$, $\alpha(u \wedge v) = 0$ for all $u, v \in V^h$ and $\chi_h$ nontrivial implies $\lambda \equiv 0$ on $(V^h)^\perp$,*
(c) *when $\operatorname{codim} V^h = 2$, $\alpha(u \wedge v) = 0$ for all $u \in V^h$, $v \in V$ and $\lambda \equiv 0$ on $V^h$, and*
(d) *when $\operatorname{codim} V^h > 2$, $\gamma = 0$.*

*Proof.* Fix $\gamma = (\lambda \otimes hg) \oplus (\alpha \otimes h)$ in $\mathrm{Z}^2_{-1}(h)$. We show $\gamma$ is in the same coset as a cocycle $\gamma - df$ satisfying the given conditions and then show this cocycle is unique. We construct the linear function $f : V \to \mathbb{F}$ explicitly and write $\gamma - df = \gamma' = \lambda' \oplus \alpha'$. Recall that we identify $\lambda$ with a function $\lambda : V \to \mathbb{F}$ and $\alpha$ with a function $\alpha : \bigwedge^2 V \to V$. Set $\hat{v} = v - {}^h v$ for any $v$ in $V$.

**Existence of representatives.** Assume $\operatorname{codim} V^h = 0$, i.e., $h = 1_G$. For $v \in (V^G)^\perp$, define $f(v - {}^g v) = \lambda(v)$ and extend to a linear function $f : V \to \mathbb{F}$. Note that $f$ is well-defined: If $v - {}^g v = w - {}^g w$ then $v - w \in V^G$, so $v = w$ for $v, w$ in $(V^G)^\perp$. Then $df \in \mathrm{B}^2_{-1}(1_G)$ and $\gamma' = \gamma - df$ satisfies the conditions in the statement by Lemma 5.4 as $\lambda'(v) = \lambda(v) - f(v - {}^g v) = 0$ for $v \in (V^G)^\perp$.

Now assume $\operatorname{codim} V^h = m \geq 2$ with $v_1, \dots, v_m$ a basis of $(V^h)^\perp$ and observe that $\hat{v}_1, \dots, \hat{v}_m$ form a basis of $V_h$. Define a map $f : (V^h)^\perp \to \mathbb{F}$ by setting $f(v_i)$ and $f(v_j)$ to be the unique constants such that

$$\pi_h\alpha(v_i \wedge v_j) = f(v_j)\hat{v}_i - f(v_i)\hat{v}_j \qquad \text{for } 1 \leq i \neq j \leq m$$

using Lemma 5.3 and Lemma 6.4(d) when $\operatorname{codim} V^h > 2$ and using the fact that $V_h = \mathbb{F}\text{-span}\{\hat{v}_1, \hat{v}_2\}$ when $\operatorname{codim} V^h = 2$. Notice that $f$ is well-defined: If $\pi_h\alpha(v_i \wedge v_j) = a\hat{v}_i + b\hat{v}_j$ and $\pi_h\alpha(v_i \wedge v_k) = c\hat{v}_i + d\hat{v}_k$ for $a, b, c, d$ in $\mathbb{F}$ and $i, j, k$ distinct, then $(a-c)\hat{v}_i + b\hat{v}_j - d\hat{v}_k = \pi_h\alpha(v_i \wedge (v_j - v_k))$ is a linear combination of $\hat{v}_i$ and $\hat{v}_j - \hat{v}_k$ by Lemma 6.4(d) so $b = d$. By Lemma 6.4, we may extend $f$ to a map $f : V \to \mathbb{F}$ satisfying

$$\alpha(v \wedge u) = f(u)\hat{v} \qquad \text{for } u \in V^h,\ v \in V\,.$$

Again, this is well-defined as $\alpha((v - w) \wedge u)$ lies in the span of $\hat{v} - \hat{w}$ for all $u \in V^h$, $v, w$ in $V$.

We claim that $\gamma - df = \gamma' = \lambda' \oplus \alpha'$ satisfies the conditions in the statement. By Lemma 5.4,

$$\alpha'(v \wedge u) = \alpha(v \wedge u) - f(u)\hat{v} + f(v)\hat{u} \qquad \text{for } u, v \in V\,. \tag{7.2}$$

Thus $\alpha'(v \wedge u)$ vanishes for $u, v \in V^h$ as $\alpha(v \wedge u) = 0$ by Lemmas 5.3 and 6.4 and also vanishes for $u \in V^h$, $v \in (V^h)^\perp$ as $\alpha(v \wedge u) = f(u)\hat{v}$ by construction of $f$. For $u, v \in (V^h)^\perp$,

$$\pi_h\alpha'(v \wedge u) = \pi_h\alpha(v \wedge u) - f(u)\hat{v} + f(v)\hat{u} = \pi_h\alpha(v \wedge u) - \pi_h\alpha(v \wedge u) = 0$$

as all functions involved are linear: For $v = \sum_i a_i v_i$ and $u = \sum_j b_j v_i$ with $a_i, b_j \in \mathbb{F}$

$$\alpha(v \wedge u) = \sum_{i,j} a_i b_j (f(v_j)\hat{v}_i - f(v_i)\hat{v}_j) = f(u)\hat{v} - f(v)\hat{u}\,.$$

Hence $\pi_h \alpha' \equiv 0$. Note in particular that this implies that $\alpha' \equiv 0$ when $\operatorname{codim} V^h > 2$ by Lemma 6.4(d).

Since $\gamma - d\!f = \gamma'$ is a cocycle, Lemma 5.3(2) implies that

$$(\alpha' - {}^{g^{-1}}\alpha')(u \wedge v) = \lambda'(v)\hat{u} - \lambda'(u)\hat{v} \qquad \text{for } u, v \in V\,. \tag{7.3}$$

Observe in particular that this implies $\lambda' \equiv 0$ when $\operatorname{codim} V^h > 2$: In that case, $\alpha' \equiv 0$ so

$$0 = \lambda'(v)\hat{u} - \lambda'(u)\hat{v} \quad \text{for all } u, v \in V\,;$$

to see that $\lambda'(u) = 0$, choose $v \notin V^h$ when $u \in V^h$ and choose $v \notin V^h$ with $\hat{u}$ and $\hat{v}$ linearly independent when $u \notin V^h$. Thus $\gamma - d\!f = \gamma' \equiv 0$ when $\operatorname{codim} V^h > 2$.

Now we argue that $\lambda' \equiv 0$ on $V^h$ when $\operatorname{codim} V^h = 2$. Fix $u \in V^h$. We saw above that Eq. (7.2) implies that $\alpha'(u \wedge v) = 0$ for all $v \in V$ and so $\alpha'({}^g u \wedge {}^g v) = 0$ for all $v$ as well since $G$ preserves $V^h$ set-wise. Thus by Eq. (7.3) above, $\lambda'(u)\hat{v} = 0$ for all $v$ and $\lambda'(u) = 0$. Thus $\lambda' \equiv 0$ on $V^h$.

Lastly, suppose $\operatorname{codim} V^h = 1$. Fix nonzero $x \in (V^h)^\perp$ so $\hat{x}$ spans $V_h \supset \operatorname{Im} \pi_h \alpha$ and set $w = x - {}^g x$. Let $f : V \to \mathbb{F}$ be the linear function with

$$\begin{aligned}
f(u)\,\hat{x} &= \pi_h \alpha(x \wedge u) && \text{for } u \in V^h,\\
f(x)\,\hat{x} &= \big(1 - \chi_h(g)\big)^{-1}\big(\lambda(x)\,\hat{x} - \pi_h\alpha(x \wedge w)\big) && \text{for } \chi_h \not\equiv 1,\\
f(x) \quad &= 0 && \text{for } \chi_h \equiv 1\,.
\end{aligned}$$

We verify that $\gamma - d\!f = \gamma' = \lambda' \oplus \alpha'$ satisfies the conditions in the statement. By Lemma 5.4,

$$\alpha'(u \wedge v) = \alpha(u \wedge v) - f(v)\hat{u} + f(u)\hat{v} \qquad \text{for } u, v \in V\,.$$

Thus $\alpha'(u \wedge v)$ is zero for $u, v \in V^h$ as $\alpha(u \wedge v) = 0$ by Lemmas 5.3 and 6.4. It also vanishes for $u, v \in (V^h)^\perp$ as it defines an alternating function in $u$ and $v$ and $\bigwedge^2 (V^h)^\perp = \{0\}$ as $\operatorname{codim} V^h = 1$. And for $u \in V^h$,

$$\alpha'(x \wedge u) = \alpha(x \wedge u) - f(u)\hat{x} = \alpha(x \wedge u) - \pi_h \alpha(x \wedge u)\,.$$

Hence $\pi_h \alpha' \equiv 0$.

Now assume $\chi_h$ is nontrivial so $\chi_h(g) \neq 1$. We argue that $\lambda' \equiv 0$ on $(V^h)^\perp$. First note that ${}^g x = u + \chi_h(g) x$ for some $u \in V^h$ so that $w = x - {}^g x = u + (1 - \chi_h(g))\,x$. Then as $\alpha(x \wedge u) = \alpha(x \wedge w)$,

$$f(w)\,\hat{x} = f(u)\,\hat{x} + \big(1 - \chi_h(g)\big)\,f(x)\,\hat{x} = \pi_h\alpha(x \wedge u) + \lambda(x)\,\hat{x} - \pi_h\alpha(x \wedge w) = \lambda(x)\,\hat{x}$$

and Lemma 5.4 implies that

$$\lambda'(x)\,\hat{x} = \big(\lambda(x) - f(x - {}^g x)\big)\,\hat{x} = \lambda(x)\,\hat{x} - f(w)\,\hat{x} = 0\,.$$

**Uniqueness of representatives.** We argue these coset representatives are unique. Suppose $\gamma = (\lambda \otimes hg) \oplus (\alpha \otimes h)$ and $\gamma' = (\lambda' \otimes hg) \oplus (\alpha' \otimes h)$ lie in the same coset of $\mathrm{HH}^2_{-1}(h)$ and both satisfy the conditions in the statement. Then $\gamma - \gamma' = d\!f$ for some $f : V \to \mathbb{F}$ and Lemma 5.4 implies that

$$(\alpha - \alpha')(u \wedge v) = f(v)\hat{u} - f(u)\hat{v} \quad \text{and} \quad (\lambda - \lambda')(u) = f(u - {}^g u) \qquad \text{for all } u, v \in V.$$

Then $\operatorname{Im}(\alpha - \alpha') \subset V_h$ but $\pi_h(\alpha - \alpha') \equiv 0$ by assumption, so $\alpha \equiv \alpha'$ and

$$0 = f(v)\hat{u} - f(u)\hat{v} \qquad \text{for all } u, v \in V. \tag{7.4}$$

To show that $\lambda \equiv \lambda'$, we argue that $f(u - {}^g u) = 0$ for all $u$ by considering the codimension of $V^h$.

Assume $\operatorname{codim} V^h \geq 2$ and consider some nonzero $w = u - {}^g u$ in $V$. By Eq. (7.4),

$$0 = f(v)\hat{w} - f(w)\hat{v} \quad \text{ for all } v \in V\,.$$

To see that $f(w) = 0$, choose any $v \notin V^h$ when $w \in V^h$ and choose $v$ with $\hat{v}$ and $\hat{w}$ independent in $V_h$ when $w \notin V^h$. Thus $(\lambda - \lambda')(u) = f(w) = 0$ and $\lambda \equiv \lambda'$.

Now assume that $\operatorname{codim} V^h = 1$ and let $x$ span $(V^h)^\perp$. First notice that $f$ is zero on $V^h$ since Eq. (7.4) implies that $0 = f(u)\hat{x}$ for all $u \in V^h$. Then $f(u - {}^g u) = 0$ for all $u$ in $V^h$ as $G$ fixes $V^h$ set-wise. We show $f(x - {}^g x) = 0$ as well. As above, $x - {}^g x = (1 - \chi_h(g))\, x$ modulo $V^h$ and thus $f(x - {}^g x) = (1 - \chi_h(g)) f(x)$. This is zero when $\chi_h$ is the trivial character of course, and when $\chi_h$ is nontrivial, then $\lambda$ and $\lambda'$ both vanish on $(V^h)^\perp$ by condition (b) so already $0 = (\lambda - \lambda')(x) = f(x - {}^g x)$. Hence $\lambda \equiv \lambda'$.

Finally, assume $h = 1_G$. By assumption, $\lambda$ and $\lambda'$ are zero on $(V^G)^\perp$ and $\lambda \equiv \lambda'$ on $V^G$ since $(\lambda - \lambda')(v) = f(v - {}^g v)$ for all $v$. Hence $\lambda \equiv \lambda'$. ☐

## 8. The main result: Graded deformation cohomology

We are now ready to describe the space of infinitesimal graded deformations for a finite cyclic group $G \subset \mathrm{GL}(V)$ with $V \cong \mathbb{F}^n$ acting on a polynomial ring $S(V)$ over an arbitrary field $\mathbb{F}$ with $\operatorname{char} \mathbb{F} \neq 2$. We describe the Hochschild cohomology $\mathrm{HH}^2_{-1}(A)$ for $A = S(V) \rtimes G$ in terms of the subspaces $V^h$ and $V_h = \operatorname{Im}(1 - h)$ of $V$ for $h$ in $G$ which are stabilized set-wise by $G$ (see Eq. (2.2)) and the linear character $\chi_h : G \to \mathbb{F}^\times$ defined by $\chi_h(g) := \det \big[g\big]_{V/V^h}$, for each $h$ in $G$. In addition, we take the trivial action of $G$ on $\mathbb{F}$ so $\mathbb{F}^{\chi_h} = \{0\}$ unless $\chi_h \equiv 1$, the trivial character. We describe the cohomology giving all graded deformations of first order in terms of these linear characters, compare with Proposition 3.1, and establish the theorem in the introduction. Again we use the transfer map on $V$ given by $T : V \to V$, $v \mapsto \sum_{h \in G} {}^h v$, which is zero when $G$ contains a diagonalizable reflection by Lemma 6.2.

**Theorem 8.1.** *Let $G \subset \mathrm{GL}(V)$ be a finite cyclic group acting on $V \cong \mathbb{F}^n$. The space of infinitesimal graded deformations of $A = S(V) \rtimes G$ is isomorphic as an $\mathbb{F}$-vector space to*

$$\mathrm{HH}^2_{-1}(A) \;\cong\; (V^G/\operatorname{Im} T)^* \oplus \big(V \otimes \textstyle\bigwedge^2 V^*\big)^G \;\oplus \displaystyle\bigoplus_{\substack{h \in G\\ \operatorname{codim} V^h = 1}} \big(\mathbb{F} \oplus \big(V/V_h \otimes (V^h)^*\big)\big)^{\chi_h} \;\oplus \bigoplus_{\substack{h \in G\\ \operatorname{codim} V^h = 2}} (V/V_h)^{\chi_h}\,.$$

*Proof.* We choose a generator $g$ of $G$ and construct the periodic-twisted-Koszul resolution $X_\bullet$ (see Section 4) of $A = S(V) \rtimes G$ to express $\mathrm{HH}^\bullet(A)$. We then use Proposition 5.5 to decompose $\mathrm{HH}^2_{-1}(A)$ according to the contribution of each group element:

$$\mathrm{HH}^2_{-1}(A) \;\cong\; \bigoplus_{h \in G} \mathrm{HH}^2_{-1}(h)\,.$$

By Proposition 7.1, $\mathrm{HH}^2_{-1}(h) = 0$ when $\operatorname{codim} V^h > 2$ and we analyze the remaining cases to show

$$\begin{aligned}
\mathrm{HH}^2_{-1}(h) \;&\cong\; (V^G/\operatorname{Im} T)^* \oplus \big(V \otimes \textstyle\bigwedge^2 V^*\big)^G && \text{when } \operatorname{codim} V^h = 0,\\
\mathrm{HH}^2_{-1}(h) \;&\cong\; \big(\mathbb{F} \oplus \big(V/V_h \otimes (V^h)^*\big)\big)^{\chi_h} && \text{when } \operatorname{codim} V^h = 1, \text{ and}\\
\mathrm{HH}^2_{-1}(h) \;&\cong\; (V/V_h)^{\chi_h} && \text{when } \operatorname{codim} V^h = 2\,.
\end{aligned}$$

In each case, we establish the isomorphism by defining a map $\Phi$ from a set of distinguished (cocycle) coset representatives

$$\gamma = (\lambda \otimes hg) \oplus (\alpha \otimes h) \quad \text{ in } \mathrm{Z}^2_{-1}(h) \qquad \text{ for } \lambda \in V^* \text{ and } \alpha \in \operatorname{Hom}_{\mathbb{F}}(\textstyle\bigwedge^2 V, V)$$

of $\mathrm{HH}^2_{-1}(h)$ given in Proposition 7.1 to the indicated vector space and then construct a map $\Phi'$ the opposite direction with $\Phi\Phi' = 1$. Note that this choice of representatives depends on a choice of vector space complement $(V^h)^\perp$ to $V^h$ and complement $(V_h)^\perp$ to $V_h$ for each $h$ in $G$, see Eq. (6.1).

**Contribution of the identity:** For $h = 1_G$, define

$$\Phi : \mathrm{HH}^2_{-1}(1_G) \longrightarrow (V^G/\operatorname{Im} T)^* \oplus (V \otimes \textstyle\bigwedge^2 V^*)^G \quad \text{by} \quad (\lambda \otimes g) \oplus (\alpha \otimes 1_G) + \mathrm{B}^2_{-1}(1_G) \mapsto \lambda' \oplus \alpha$$

for $\lambda' : V^G/\operatorname{Im} T \to \mathbb{F}$ the extension of $\lambda|_{V^G}$ to $V^G/\operatorname{Im} T$ where $(\lambda \otimes g) \oplus (\alpha \otimes 1_G)$ is a distinguished coset representative of $\mathrm{HH}^2_{-1}(1_G)$ as in Proposition 7.1. Note here that $\lambda|_{\operatorname{Im}(T)} \equiv 0$ by Lemma 5.3(1) so $\lambda'$ is well-defined. Also observe that $\Phi$ has the indicated codomain because $\alpha$ lies in $(V \otimes \bigwedge^2 V^*)^G$ by Lemma 5.3(2):

$$0 = {}^g(\alpha(u \wedge v)) - \alpha({}^g u \wedge {}^g v) \quad \text{for all } u, v \in V\,.$$

Now we construct an inverse map. Define

$$\Phi' : \mathrm{HH}^2_{-1}(1_G) \longleftarrow (V^G/\operatorname{Im} T)^* \oplus (V \otimes \textstyle\bigwedge^2 V^*)^G \quad \text{by} \quad \big((\lambda \otimes g) \oplus (\alpha \otimes 1_G)\big) + \mathrm{B}^2_{-1}(1_G) \leftarrow\!\shortmid \lambda' \oplus \alpha\,,$$

where $\lambda : V \to \mathbb{F}$ is defined by $\lambda(u) = \lambda'(u + \operatorname{Im} T)$ for $u$ in $V^G$ and $\lambda \equiv 0$ on $(V^G)^\perp$. We verify that $(\lambda \otimes g) \oplus (\alpha \otimes 1_G)$ lies in $\mathrm{Z}^2_{-1}(1_G)$ and thus $\Phi'$ is well-defined by checking the three cocycle conditions in Lemma 5.3: Condition (1) holds by construction of $\lambda$ and Conditions (2) and (3) hold since $h = 1_G$ and $\alpha$ is invariant. In fact, we observe that $\Phi'(\lambda' \oplus \alpha)$ is a distinguished coset representative as in Proposition 7.1 and $\Phi\Phi' = 1$.

**Codimension one contributions:** Fix $h$ in $G$ with $\operatorname{codim} V^h = 1$ and consider $V = V^h \oplus (V^h)^\perp$ with fixed element $x$ spanning $(V^h)^\perp$. Define

$$\Phi : \mathrm{HH}^2_{-1}(h) \longrightarrow \big(\mathbb{F} \oplus (V/V_h \otimes (V^h)^*)\big)^{\chi_h} \quad \text{by} \quad (\lambda \otimes hg) \oplus (\alpha \otimes h) + \mathrm{B}^2_{-1}(h) \mapsto \lambda(x) \oplus \alpha'\,,$$

for $\alpha'$ in $V/V_h \otimes (V^h)^*$ the map $V^h \to V/V_h$ defined by

$$\alpha'(u) = \alpha(x \wedge u) + V_h \qquad \text{for } u \in V^h$$

where $(\lambda \otimes hg) \oplus (\alpha \otimes h)$ is a distinguished coset representative of $\mathrm{HH}^2_{-1}(h)$ as in Proposition 7.1.

We verify that each $\lambda(x) \oplus \alpha'$ is $\chi_h$-invariant and hence $\Phi$ has the indicated codomain. First notice that $\lambda(x) \in \mathbb{F}$ is $\chi_h$-invariant since $G$ acts trivially on $\mathbb{F}$ and $\lambda(x) = 0$ when $\chi_h$ is not the trivial character. Now we argue $\alpha'$ is $\chi_h$-invariant. Since $\alpha \equiv 0$ on $\bigwedge^2 V^h$ and ${}^g x = \chi_h(g)\, x$ modulo $V^h$ (as $\dim_{\mathbb{F}}(V/V^h) = 1$),

$$\chi_h(g)\, \alpha(x \wedge u) = \alpha\big(\chi_h(g) x \wedge u\big) = \alpha({}^g x \wedge u) \qquad \text{for all } u \in V^h\,, \tag{8.2}$$

and, as $G$ preserves $V_h$ and $V^h$ set-wise,

(8.3)
$$\begin{aligned}\big(\alpha' - \chi_h(g)\ {}^{g^{-1}}\alpha'\big)(u) &= \alpha'(u) - \chi_h(g)\ {}^{g^{-1}}\big(\alpha'({}^g u)\big) = (\alpha(x \wedge u) + V_h) - \chi_h(g)\ {}^{g^{-1}}\big(\alpha(x \wedge {}^g u) + V_h\big) \\ &= \big(\alpha(x \wedge u) - {}^{g^{-1}}(\alpha({}^g x \wedge {}^g u))\big) + V_h = (\alpha - {}^{g^{-1}}\alpha)(x \wedge u) + V_h \quad \text{for } u \in V^h\,.\end{aligned}$$

But Lemma 5.3(2) implies that $(\alpha - {}^{g^{-1}}\alpha)(u \wedge v)$ lies in $V_h$ for all $u, v$ in $V$ so this last expression is zero and thus $\alpha'$ is also $\chi_h$-invariant.

Now we construct an inverse map to $\Phi$,

$$\Phi' : \mathrm{HH}^2_{-1}(h) \longleftarrow \big(\mathbb{F} \oplus (V/V_h \otimes (V^h)^*)\big)^{\chi_h}\,, \qquad \big((\lambda \otimes hg) \oplus (\alpha \otimes h)\big) + \mathrm{B}^2_{-1}(h) \leftarrow\!\shortmid \lambda' \oplus \alpha'\,.$$

For a pair $\lambda'$ in $\mathbb{F}$ and $\alpha' : V^h \to V/V_h$, define $\alpha : \bigwedge V^2 \to V$ and $\lambda : V \to \mathbb{F}$ as follows. Let $\alpha(x \wedge u)$ be the unique coset representative in $(V_h)^\perp$ of $\alpha'(u)$ for $u \in V^h$ and extend to a linear function on $\bigwedge^2 V$ by setting $\alpha \equiv 0$ on $\bigwedge^2 V^h$. Then for $u \in V^h$, $\pi_h \alpha(x \wedge u) = 0$ (for projection map $\pi_h : V \mapsto V_h$) and $\alpha'(u) = \alpha(x \wedge u) + V_h$, and as $\alpha$ vanishes on $\bigwedge^2 V^h$, Eq. (8.2) and Eq. (8.3) hold and imply that

$$(\alpha - {}^{g^{-1}}\alpha)(x \wedge u) + V_h = (\alpha' - \chi_h(g)\ {}^{g^{-1}}\alpha')(u) = 0\,, \tag{8.4}$$

i.e., $(\alpha - {}^{g^{-1}}\alpha)(x \wedge u)$ lies in $V_h = \mathbb{F}\text{-span}\{x - {}^h x\}$. Thus we can define $\lambda : V \to \mathbb{F}$ as the linear function satisfying

$$\lambda(u)(x - {}^h x) = (\alpha - {}^{g^{-1}}\alpha)(x \wedge u) \quad \text{for } u \text{ in } V^h \qquad \text{and} \qquad \lambda(x) = \lambda'\,.$$

We argue that $(\lambda \otimes hg) \oplus (\alpha \otimes h)$ is a cocycle by checking the conditions in Lemma 5.3. For Lemma 5.3(1), we verify that $\lambda(\operatorname{Im} T) = 0$. Since $\operatorname{codim} V^h = 1$, $h$ is a reflection and either $|h| = p$ or $|h|$ and $p$ are coprime for $p = \operatorname{char} \mathbb{F}$ (see Section 6). If $|h| = p$, then $h$ is nondiagonalizable and $\lambda(\operatorname{Im} T) = 0$ by Lemma 6.2. Assume now that $p$ and $|h|$ are coprime. Then $h$ is diagonalizable with $V^h \cap V_h = \{0\}$, and we may assume $(V_h)^\perp$ is chosen as $V^h$ in the construction of coset representatives from Proposition 7.1. We argue that $(\alpha - {}^{g^{-1}}\alpha)(x \wedge u) = 0$ for all $u$ in $V^h$. On one hand, $(\alpha - {}^{g^{-1}}\alpha)(x \wedge u)$ lies in $V_h$ by Eq. (8.4). On the other hand, we claim that $(\alpha - {}^{g^{-1}}\alpha)(x \wedge u)$ lies in $V^h$. By construction, $\pi_h \alpha \equiv 0$ and so $V^h = (V_h)^\perp$ contains both $\alpha(x \wedge u)$ and $\alpha({}^g x \wedge {}^g u)$ and also $({}^{g^{-1}}\alpha)(x \wedge u) = {}^{g^{-1}}\big(\alpha({}^g x \wedge {}^g u)\big)$ as $G$ preserves $V^h$; hence the difference $(\alpha - {}^{g^{-1}}\alpha)(x \wedge u)$ lies in $V^h$. Then, as $V^h \cap V_h = \{0\}$, we must have $(\alpha - {}^{g^{-1}}\alpha)(x \wedge u) = 0$. Hence $\lambda(\operatorname{Im} T) = 0$ as $\operatorname{Im} T \subset V^G \subset V^h$ and Lemma 5.3(1) holds.

We now verify Lemma 5.3(2), i.e.,

$$(\alpha - {}^{g^{-1}}\alpha)(u \wedge v) = \lambda(v)(u - {}^h u) - \lambda(u)(v - {}^h v) \quad \text{ for all } u, v \in V\,.$$

The equality holds for $u = x$ and $v \in V^h$ by definition of $\lambda$. For $u, v \in V^h$, the right-hand-side is zero and the left side is zero as well since $\alpha$ vanishes on $\bigwedge^2 V^h$ by construction and $G$ fixes $V^h$ set-wise, so $\alpha({}^g u \wedge {}^g v) = 0$. For $u, v \in (V^h)^\perp$, both sides vanish as they are alternating in $u$ and $v$ and $(V^h)^\perp$ has dimension 1. Lastly, Lemma 6.6 implies Lemma 5.3(3) is satisfied. Therefore, $(\lambda \otimes hg) \oplus (\alpha \otimes h)$ is a cocycle by Lemma 5.3.

In fact, we observe that $(\lambda \otimes hg) \oplus (\alpha \otimes h)$ is a distinguished coset representative as in Proposition 7.1: $\pi_h \alpha \equiv 0$, $\alpha \equiv 0$ on $\bigwedge^2 V^h$, and, whenever $\chi_h$ is nontrivial, $\mathbb{F}^{\chi_h} = \{0\}$ so $\lambda(x) = \lambda' = 0$. It is straightforward to check that $\Phi\Phi' = 1$.

**Codimension two contributions:** Fix $h$ in $G$ with $\operatorname{codim} V^h = 2$ and write $V = V^h \oplus (V^h)^\perp$ with fixed basis elements $v_1$ and $v_2$ of $(V^h)^\perp$. Then $\hat{v}_1$ and $\hat{v}_2$ form a basis for $V_h$. Define

$$\Phi : \mathrm{HH}^2_{-1}(h) \longrightarrow (V/V_h)^{\chi_h} \qquad \text{by} \qquad (\lambda \otimes hg) \oplus (\alpha \otimes h) \mapsto \alpha(v_1 \wedge v_2) + V_h\,,$$

where $(\lambda \otimes hg) \oplus (\alpha \otimes h)$ is a distinguished coset representative for $\mathrm{HH}^2_{-1}(h)$ as in Proposition 7.1. We show that $\alpha(v_1 \wedge v_2) + V_h$ is $\chi_h$-invariant and hence $\Phi$ has the indicated codomain. First observe that $g$ acts on the 1-dimensional space $\bigwedge^2(V/V^h)$ by the scalar $\chi_h(g) = \det[g]_{V/V^h}$ and $\alpha(u \wedge v) = 0$ for any $u \in V^h$ and $v \in V$, so

$$\alpha({}^g v_1 \wedge {}^g v_2) = \chi_h(g) \cdot \alpha(v_1 \wedge v_2) \tag{8.5}$$

and

$$\begin{aligned}\alpha(v_1\wedge v_2)-\chi_h(g)\cdot{}^{g^{-1}}(\alpha(v_1\wedge v_2))+V_h&=\alpha(v_1\wedge v_2)-{}^{g^{-1}}(\chi_h(g)\cdot\alpha(v_1\wedge v_2))+V_h\\ &=\alpha(v_1\wedge v_2)-{}^{g^{-1}}(\alpha({}^{g}v_1\wedge{}^{g}v_2))+V_h=(\alpha-{}^{g^{-1}}\alpha)(v_1\wedge v_2)+V_h\,.\end{aligned}\tag{8.6}$$

Then by Lemma 5.3(2), $(\alpha-{}^{g^{-1}}\alpha)(v_1\wedge v_2)+V_h=0$, and the image of $\Phi$ is $\chi_h$-invariant.

Now we construct an inverse to $\Phi$,

$$\Phi':\mathrm{HH}^2_{-1}(h)\longleftarrow(V/V_h)^{\chi_h}\,,\qquad\big((\lambda\otimes hg)\oplus(\alpha\otimes h)\big)+\mathrm{B}^2_{-1}(h)\leftarrow\!\shortmid v+V_h\,,$$

as follows using the projection map $\pi_h^\perp:V\to(V_h)^\perp$. For any coset $v+V_h$ in $(V/V_h)^{\chi_h}$, define the map $\alpha:\bigwedge^2V\to V$ by setting

$$\alpha(u\wedge w)=0\quad\text{for all } u\in V^h,\ w\in V\qquad\text{and}\qquad\alpha(v_1\wedge v_2)=\pi_h^\perp(v)\in(V_h)^\perp\,.\tag{8.7}$$

Note that this is independent of choice of coset representative $v$. Before defining $\lambda$, we observe that $(\alpha-{}^{g^{-1}}\alpha)(v_1\wedge v_2)$ lies in $V_h$: Eq. (8.7) implies that Eqs. (8.5) and (8.6) hold and thus

$$\begin{aligned}(\alpha-{}^{g^{-1}}\alpha)(v_1\wedge v_2)+V_h&=\alpha(v_1\wedge v_2)-\chi_h(g)\,{}^{g^{-1}}(\alpha(v_1\wedge v_2))+V_h\\ &=\pi_h^\perp(v)-\chi_h(g)\,{}^{g^{-1}}\big(\pi_h^\perp(v)\big)+V_h=0\end{aligned}$$

as $v+V_h=\pi_h^\perp(v)+V_h$ is $\chi_h$-invariant and $G$ preserves $V_h$ set-wise. Thus we may write

$$(\alpha-{}^{g^{-1}}\alpha)(v_1\wedge v_2)=a\,(v_1-{}^{h}v_1)-b\,(v_2-{}^{h}v_2)\qquad\text{for some } a,b\in\mathbb{F}\,.$$

Define a linear function $\lambda:V\to\mathbb{F}$ by setting $\lambda\equiv0$ on $V^h$, $\lambda(v_2)=a$, and $\lambda(v_1)=b$. We argue that $(\lambda\otimes hg)\oplus(\alpha\otimes h)$ is a cocycle in $\mathrm{Z}^2_{-1}(h)$ by checking the three conditions of Lemma 5.3. Since $\lambda\equiv0$ on $V^h$ and $\operatorname{Im}T\subset V^G\subset V^h$, Lemma 5.3(1) holds. We next verify Lemma 5.3(2), i.e.,

$$(\alpha-{}^{g^{-1}}\alpha)(u\wedge w)=\lambda(w)(u-{}^{h}u)-\lambda(u)(w-{}^{h}w)\qquad\text{for all } u,w\in V\,.$$

This holds for $u=v_1$ and $w=v_2$ by construction of $\lambda$. It also holds when $u\in V^h$ and $w\in V$: The right-hand side vanishes since $u-{}^{h}u=0$ and $\lambda(u)=0$ and the left-hand side vanishes as well since $\alpha(u\wedge w)=0=\alpha({}^{g}u\wedge{}^{g}w)$ by the construction of $\alpha$ as both $u$ and ${}^{g}u$ lie in $V^h$. Thus Lemma 5.3(2) is satisfied. Lemma 5.3(3) is satisfied by Lemma 6.6. So $(\lambda\otimes hg)\oplus(\alpha\otimes h)$ is a cocycle.

Notice that this cocycle is a distinguished coset representative as in Proposition 7.1. It is then straightforward to check that $\Phi\Phi'=1$. □

**Remark 8.8.** We recover from the last theorem the description of the graded deformation cohomology $\mathrm{HH}^2_{-1}(A)$ for $A=S(V)\rtimes G$ in the nonmodular setting. Indeed, for $G$ cyclic with $|G|$ and $\operatorname{char}\mathbb{F}$ coprime and $\mathbb{F}$ algebraically closed, Theorem 8.1 gives Proposition 3.2. In this setting, $\operatorname{Im}T=V^G$ and $V/V_h\cong V^h$ as an $\mathbb{F}G$-module. In addition, if $h$ is a reflection, then $h$ is diagonalizable with $\chi_h(h)\neq1$ (see Eq. (2.2)) so $\mathbb{F}^{\chi_h}=0$. Finally, as $h$ acts on $V/V_h$ and on $(V^h)^*$ as the identity and $\chi_h(h)\neq1$, no element of $V/V_h\otimes(V^h)^*$ can be $\chi_h$-invariant so the middle summand of Theorem 8.1 vanishes. We give a generalization of this phenomenon in the next result.

**Corollary 8.9.** *Suppose $G \subset \mathrm{GL}(V)$ is a cyclic group acting on $V = \mathbb{F}^n$. Let $|G| = p^k r$ for some $k$ with $r$ and $\operatorname{char}\mathbb{F}$ coprime and suppose $\mathbb{F}$ contains a primitive $r$-th root of unity. Then the space of infinitesimal graded deformations of $A = S(V) \rtimes G$ is isomorphic as an $\mathbb{F}$-vector space to*

$$\mathrm{HH}^2_{-1}(A) \;\cong\; (V^G)^* \oplus \big(V \otimes \textstyle\bigwedge^2 V^*\big)^G \oplus \bigoplus_{\substack{h\in G\\ \operatorname{codim} V^h=1\\ \det(h)=1}} \big(\mathbb{F} \oplus \big(V/V_h \otimes (V^h)^*\big)\big)^{\chi_h} \oplus \bigoplus_{\substack{h\in G\\ \operatorname{codim} V^h=2\\ \det(h)=1}} (V/V_h)^{\chi_h}\,.$$

*Proof.* As $\mathbb{F}$ has a primitive $r$-th root-of-unity, we may choose a Jordan canonical form for the generator $g$ of $G$. Say $\operatorname{codim} V^h = 1$ for some $h = g^i$ in $G$ with $\det h \neq 1$. Then $h$ is diagonalizable with a single non-1 eigenvalue. Since a single Jordan block $B$ of $g$ has only one eigenvalue $\xi$ with multiplicity the size of the block and $B^i$ has only one eigenvalue $\xi^i$ with the same multiplicity, there must be a $1\times 1$ Jordan block of $G$ corresponding to an eigenvector $g$ and $h$ share not in $V^h$. Thus $G$ preserves set-wise both $V^h$ and the choice of vector space complement $(V^h)^\perp = V_h = \mathrm{Im}(1-h)$. We identify $V/V_h$ with $V^h$ as an $\mathbb{F}G$-module and observe as in Remark 8.8 that $\chi_h(h) \neq 1$ yet $h$ fixes $\mathbb{F} \oplus \big(V/V_h \otimes (V^h)^*\big)$ point-wise, so 0 is the only $\chi_h$-invariant element of this space. We use a similar argument when $\operatorname{codim} V^h = 2$ with $\det h \neq 1$: $h$ must have two non-1 eigenvalues and thus there must be one or two Jordan blocks of $g$ corresponding to $V_h$, a vector space complement to $V^h$; we again identify $V/V_h$ with $V^h$ as an $\mathbb{F}G$-module and note that $\chi_h(h) = \det h$ to conclude that the space of $\chi_h$-invariants is the zero space. □

## 9. Applications to deformation theory

Here we demonstrate how to use the description of infinitesimal graded deformations in Theorem 8.1 to obtain explicit graded deformations in a particular setting. We use three resolutions of $A = S(V) \rtimes G$ for a cyclic group $G$ acting on $V \cong \mathbb{F}^n$ using the twisted product resolutions of [30]:

- Resolution $X_\bullet$ is a twisting of a periodic resolution for $\mathbb{F}G$ with the Koszul resolution for $S(V)$ (see Section 4),
- Resolution $Y_\bullet$ is a twisting of the bar resolution for $\mathbb{F}G$ and the Koszul resolution for $S(V)$ (see [28, 30] for the differentials),
- Resolution $Z_\bullet$ is the bar resolution of $A$.

Each is an $A^e$-free resolution of $A$, and we make a choice of chain maps between the three resolutions:

$$\begin{tikzcd}
X_\bullet: & \cdots \arrow[r] & X_3 \arrow[r] \arrow[d, shift right] & X_2 \arrow[r] \arrow[d, shift right] & \cdots & \text{(for finding cohomology representatives)} \\
Y_\bullet: & \cdots \arrow[r] & Y_3 \arrow[r] \arrow[u, shift right] \arrow[d, shift right] & Y_2 \arrow[r] \arrow[u, shift right] \arrow[d, shift right] & \cdots & \text{(for computing Gerstenhaber brackets)} \\
Z_\bullet: & \cdots \arrow[r] & Z_3 \arrow[r] \arrow[u, shift right] & Z_2 \arrow[r] \arrow[u, shift right] & \cdots & \text{(for describing deformations)}
\end{tikzcd}$$

Resolutions $Y_\bullet$ and $Z_\bullet$ here may be used for any finite group $G$. Resolution $X_\bullet$ is reserved for a finite cyclic group $G$.

**Cyclic transvection groups.** We now turn to the unipotent cyclic groups acting on $V = \mathbb{F}^2$, i.e., the cyclic transvection groups. Consider $G = \langle g\rangle \subset \mathrm{GL}(V)$ with $\operatorname{char}\mathbb{F} = p > 0$ and $g$ a nondiagonalizable reflection. Then $|G| = p$ and there is a basis $v_1, v_2$ of $V$ so that

$$g(v_1) = v_1 \quad\text{and}\quad g(v_2) = v_1 + v_2, \qquad\text{i.e.,}\quad g = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \quad\text{and}\quad G = \left\{\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)\right\} \subset \mathrm{GL}_n(\mathbb{F})\,.$$

Here $S(V)=\mathbb{F}[v_1,v_2]$ and $\operatorname{Im} T=\{0\}$ by Lemma 6.2 as $g$ itself is a nondiagonalizable reflection. For each $h$ in $G$, $\operatorname{codim} V^h \le 1$ and $V^h = V_h = V^G = \mathbb{F}v_1$ whereas $V/V_h \cong \mathbb{F}v_2$. Note that $\chi_h : G \to \mathbb{F}^\times$ is the trivial character in this setting as each $h'$ in $G$ acts trivially on each $V/V_h$. For $A = S(V)\rtimes G$, Theorem 8.1 thus implies that

$$\begin{aligned}\mathrm{HH}^2_{-1}(A) \;&\cong\; \underbrace{(V^G)^* \oplus (V\otimes \textstyle\bigwedge^2 V^*)^G}_{\text{contribution of } 1_G} \;\oplus\; \bigoplus_{\substack{h\in G\\ h\neq 1_G}} \underbrace{\big(\mathbb{F}\oplus (V/V^G \otimes (V^G)^*)\big)^G}_{\text{contribution of reflections}} \\ &=\; (\mathbb{F}v_1)^* \oplus (\mathbb{F}v_1 \otimes \mathbb{F}v_1\wedge v_2) \;\oplus\; \bigoplus_{\substack{h\in G\\ h\neq 1_G}} \big(\mathbb{F}\oplus (V/\mathbb{F}v_1 \otimes (\mathbb{F}v_1)^*)\big) \;\cong\; \mathbb{F}^{2p}\,. \end{aligned} \tag{9.1}$$

**Lifting a Hochschild cocycle to a deformation.** Consider $1+(v_2+\mathbb{F}v_1)\otimes v_1^*$ in the summand $\mathbb{F}\oplus (V/\mathbb{F}v_1\otimes(\mathbb{F}v_1)^*)$ of $\mathrm{HH}^2_{-1}(A)$ in Eq. (9.1) corresponding to $h=g$ where $v_1^*$ is the map $v_1\mapsto 1$. Using the isomorphism in the proof of Theorem 8.1, we identify this element with the cocycle $\gamma_X$ on $X_\bullet$ in $\mathrm{C}^2_{-1}(A) = (V^*\otimes \mathbb{F}G)\oplus(V\otimes \bigwedge^2 V^*\otimes \mathbb{F}G)$ given by

$$\gamma_X(v_1)=g^2, \qquad \gamma_X(v_2)=g^2, \qquad \text{and} \qquad \gamma_X(v_2\wedge v_1) = v_2\otimes g\,.$$

We lift $\gamma_X$ to a specific 2-cocycle $\gamma=\gamma_Y$ on the bar-twisted-Koszul resolution,

$$\gamma : Y_2 = (\mathbb{F}G\otimes \mathbb{F}G)\oplus(\mathbb{F}G\otimes V)\oplus(\textstyle\bigwedge^2 V) \to S(V)\otimes \mathbb{F}G,$$

by applying a choice of chain map $X_\bullet\to Y_\bullet$: For $0\le i,j<|G|$, calculations give

$$\gamma(g^i\otimes g^j)=0\,,\quad \gamma(g^i\otimes v_1) = i\, g^{i+1}\,,\quad \gamma(g^i\otimes v_2) = \tbinom{i+1}{2}\, g^{i+1}\,,\quad \text{and } \gamma(v_2\wedge v_1) = v_2\otimes g\,. \tag{9.2}$$

We argue that $\gamma$ lifts to a graded deformation of $\mathbb{F}[v_1,v_2]\rtimes G$. We consider the lifting conditions for $\gamma$ on $Y_\bullet$ ([29, Theorem 5.3]): As $\dim V=2$, $\gamma$ lifts to a deformation if and only if

$$[\gamma,\gamma]=0 \text{ as a cochain on } Y_3.$$

Here, [ , ] is the Gerstenhaber bracket on Hochschild cohomology lifted to the resolution $Y_\bullet$. Since $\gamma$ has degree $-1$, the square bracket $[\gamma,\gamma]$ is a 3-cochain of degree $-2$. Then as $\dim V=2$, we need only check the value of the square bracket on $h\otimes v_1\wedge v_2$ in $\mathbb{F}G\otimes\bigwedge^2 V$ for $h$ in $G$. We use the formula on the right-hand side of [29, Theorem 6.1(2)]:

$$\begin{aligned}[\gamma,\gamma](g^i\otimes v_1\wedge v_2) &= \gamma(\gamma(g^i\otimes v_2)\otimes v_1) - \gamma(\gamma(g^i\otimes v_1)\otimes v_2) + \gamma(g^i\otimes v_2)\,g \\ &= \tbinom{i+1}{2}\,\gamma(g^{i+1}\otimes v_1) - i\,\gamma(g^{i+1}\otimes v_2) + \tbinom{i+1}{2}\, g^{i+2} \\ &= \tbinom{i+1}{2}(i+1)\, g^{i+2} - i\tbinom{i+2}{2} g^{i+2} + \tbinom{i+1}{2}\, g^{i+2}\,.\end{aligned}$$

A computation shows that the right-hand side is zero and thus $[\gamma,\gamma]=0$ as a cochain. Hence $\gamma$ in $\mathrm{HH}^2(A)$ is not just an infinitesimal graded deformation, but the infinitesimal (first multiplication map) of a (formal) graded deformation of $A=S(V)\rtimes G$.

**Graded deformation as an explicit Drinfeld orbifold algebra.** We now give that graded deformation explicitly. For a pair of linear parameter functions

$$\lambda : \mathbb{F}G\otimes V\to \mathbb{F}G \qquad \text{and} \qquad \kappa : \textstyle\bigwedge^2 V \to V\otimes \mathbb{F}G\,,$$

let $\mathcal{H}_{\lambda,\kappa}$ be the $\mathbb{F}$-algebra (see [29], for example) generated by $\mathbb{F}G$ and $V$ with defining relations

$$hu - {}^h u h = \lambda(h\otimes u) \quad \text{and} \quad uv - vu = \kappa(u\wedge v) \quad \text{for all } h\in G \text{ and } u,v\in V\,.$$

For $\gamma$ defined in Eq. (9.2), now consider the algebra $\mathcal{H}_{\lambda,\alpha}$ defined with specific parameters

$$\lambda = \gamma\big|_{\mathbb{F}G\otimes V} \quad \text{and} \quad \alpha = \gamma\big|_{\bigwedge^2 V}\,, \qquad \text{so } \gamma = \lambda \oplus \alpha\,.$$

We identify $\alpha$ with $\kappa^L$ and set $\kappa^C$ to zero in [29, Theorem 6.1] and check the six conditions of that theorem to conclude that $\mathcal{H}_{\lambda,\alpha}$ satisfies the PBW property and is isomorphic to $\mathbb{F}[v_1, v_2] \rtimes G$ as a vector space. Thus the $\mathbb{F}$-algebra $\mathcal{H}_{\lambda,\alpha}$ generated by $\mathbb{F}G$ and $V$ with relations

$$gv_1 - v_1 g = g^2, \quad gv_2 - v_1 g - v_2 g = 0, \quad \text{and} \quad v_2 v_1 - v_1 v_2 = v_2 g$$

is a PBW deformation of $\mathbb{F}[v_1, v_2] \rtimes G$. This analog of a universal enveloping algebra is called a *Drinfeld orbifold algebra*.

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