

# STABILIZING AUTOMORPHISMS OF QUANTUM AFFINE SPACE

ETHAN JENSEN<sup>1</sup> AND ANNE V. SHEPLER<sup>2</sup>

ABSTRACT. We examine the graded automorphism groups of quantum affine spaces and classify these groups for spaces of dimension 7 or less. Using permutation actions on partitions, we investigate cases when the group decomposes as a product of graded automorphism groups of smaller dimensional spaces, and we describe the groups arising from the Kronecker tensor product of independent quantum parameter matrices.

## 1. INTRODUCTION

Determining the automorphism groups of algebras remains a challenging task, with difficulties even in the case of commutative polynomial rings over fields. Consider, for example, Nagata's 1972 wildness conjecture on the automorphism group of  $\mathbb{C}[x_1, x_2, x_3]$  proved in 2004 by Shestakov and Umirbaev [17], see also Kraft [11]. Recent attention has turned to automorphism groups of noncommutative algebras viewed as coordinate rings. We investigate the graded automorphism group of quantum affine spaces. These are finitely generated algebras with each pair of generators commuting up to a nonzero scalar.

The graded automorphisms of a quantum affine space  $S_{\mathbf{q}}(V)$  give critical information on the ungraded automorphisms. Here  $V \cong \mathbb{K}^n$  is the vector space spanned by generators over a field  $\mathbb{K}$  and  $\mathbf{q}$  is a matrix of quantum scalars recording the noncommutative multiplication. Often the group of all automorphisms  $\text{Aut}(S_{\mathbf{q}}(V))$  coincides with the group of graded automorphisms  $\text{Aut}_{\text{gr}}(S_{\mathbf{q}}(V))$ , see Ceken, Palmieri, Wang, and Zhang [5, Section 3] and Yakimov [18, Corollary 3.7]. More generally,  $\text{Aut}(S_{\mathbf{q}}(V))$  is the semidirect product of the unipotent automorphisms  $\text{Aut}_{\text{uni}}(S_{\mathbf{q}}(V))$  with  $\text{Aut}_{\text{gr}}(S_{\mathbf{q}}(V))$  unless  $\mathbf{q}$  contains a row of all 1's, see [5, Lemma 3.2]. The groups  $\text{Aut}_{\text{gr}}(S_{\mathbf{q}}(V))$  have been classified in low dimensions, see see Alev and Dumas [3] and Levandovskyy and Shepler [12]. Explicit classifications in higher dimensions have been lacking. See also Alev and Chamarie [1] and Artamonov and Wisbauer [2].

We write the quantum affine space as  $S_{\mathbf{q}}(V) = \mathbb{K}_{\mathbf{q}}[v_1, \dots, v_n]$  generated by  $v_1, \dots, v_n$  with relations  $v_j v_i = q_{ij} v_i v_j$  for  $q_{ij}$  in  $\mathbb{K}^*$  with  $q_{ij} q_{ji} = 1 = q_{ii}$  with matrix of quantum scalars  $\mathbf{q} = \{q_{ij}\}$ , also known as the skew polynomial ring. We use the terms *quantum affine space* and *skew polynomial algebra* interchangeably; they both refer to the algebra  $S_{\mathbf{q}}(V)$ . Various results give the graded automorphism group in special cases, for example,  $\text{Aut}_{\text{gr}}(S_{\mathbf{q}}(V)) = \text{Aut}(S_{\mathbf{q}}(V)) \cong (\mathbb{K}^{\times})^n$  when the  $q_{ij}$  are generic, see [1] and [18]. See [1] and [5] for the cases at the other extreme when the quantum scalars  $q_{ij}$  agree and/or are all  $\pm 1$ . Bazlov and Berenstein [4] described  $\text{Aut}_{\text{gr}}(S_{\mathbf{q}}(V))$  as a product of subgroups of  $\text{GL}(V)$  with nontrivial overlap.

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1. Dept. of Mathematics, University of North Texas, Denton, Texas, ethan.jensen@my.unt.edu.

2. Corresponding author, Dept. of Mathematics, University of North Texas, Denton, Texas, ashepler@unt.edu.

Jin [9] recently gave an elegant theorem describing  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  as a semidirect product of general linear groups with a quotient of permutation groups. We give a short and direct proof of a slightly different description using the Splitting Lemma. This characterization aids in a classification of these groups up to  $\dim V \leq 7$ . We give the semidirect product structure explicitly in terms of a maximal subgroup whose orbits refine a particular partition of  $[n]^2 = \{(i, j) : 1 \leq i, j \leq n\}$ . Graded automorphisms must permute blocks of  $\{1, \dots, n\}$  of the same size, where each block indexes identical rows of  $\mathfrak{q}$ . We leverage the preservation of block size to obtain an efficient algorithm (computationally inexpensive) used in the classification.

We use the semidirect product structure to decompose  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  as a direct product of quantum affine spaces of potentially lower dimension (i.e., fewer generators) in [Theorem 4.2](#) (see also [Proposition 2.16](#)). This decomposition implies that the set of groups  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  for all  $\mathfrak{q}$  and  $V$  is closed under direct products, see [Corollary 4.5](#). In addition, we give the graded automorphisms of  $S_{\mathfrak{q} \otimes \mathfrak{q}'}(V \otimes V')$  formed from the Kronecker product of two quantum parameter matrices  $\mathfrak{q}$  and  $\mathfrak{q}'$  sufficiently independent in [Theorem 5.4](#).

One asks which groups may arise as the graded automorphism group of a quantum affine space. This is determined jointly by the base field  $\mathbb{K}$  and the diagonal action of subgroups of the symmetric group  $\mathfrak{S}_n$  on  $[n]^2$ . With small assumptions on the cardinality and characteristic of  $\mathbb{K}$ , we identify several infinite families of groups that appear, namely, all groups of the form  $(\mathbb{K}^\times)^n \rtimes G$  for  $G$  the symmetric group  $\mathfrak{S}_n$ , the dihedral group  $D_{2n}$ , or a cyclic group. Every graded automorphism group is either monomial or determined by monomial groups acting in lower dimension, see [Proposition 6.4](#), so such families are helpful for classification results. We also explain why the stabilizing permutation group of a graded automorphism group must appear infinitely often in any classification, see [Remark 5.5](#) and [Corollary 5.6](#).

**Outline.** In [Section 2](#), we consider the structure of the group of graded automorphisms of skew polynomial rings. We identify in [Section 3](#) certain countably infinite families that appear as such groups. In [Section 4](#), we decompose  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  as a direct product of graded automorphism groups of subalgebras of  $S_{\mathfrak{q}}(V)$  generated by fewer variables, and in [Section 5](#), we consider skew polynomial rings arising from the Kronecker product (tensor product) of quantum parameter matrices. Lastly, we classify in [Section 6](#) the groups  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  for  $\dim V \leq 7$ .

**Conventions and notation.** We fix a field  $\mathbb{K}$  throughout of arbitrary characteristic. By a partition, we mean set partition unless otherwise indicated, and we write  $[n]$  for  $\{1, \dots, n\}$ . For  $V \cong \mathbb{K}^n$  with fixed basis  $\{v_1, \dots, v_n\}$  and  $B \subset [n]$ , we write  $V_B = \text{Span}_{\mathbb{K}}\{v_i : i \in B\}$ .

## 2. QUANTUM AFFINE SPACE AND GRADED AUTOMORPHISMS

We give here different descriptions for the graded automorphism group of quantum affine space.

**Quantum affine space.** An  $n \times n$  matrix  $\mathfrak{q} = \{q_{ij}\}$  with entries in  $\mathbb{K}$  is a *system of quantum parameters* or a *quantum parameter matrix* when

$$q_{ii} = q_{ij} q_{ji} = 1 \quad \text{for } 1 \leq i, j \leq n.$$

We fix a  $n \times n$  quantum parameter matrix  $\mathfrak{q}$  and a finite dimensional vector space  $V$  over  $\mathbb{K}$  with basis  $v_1, \dots, v_n$ . The *skew polynomial algebra*  $S_{\mathfrak{q}}(V)$  (also called the *quantum polynomial ring*

or *quantum affine space*) associated to  $\mathfrak{q}$  is the  $\mathbb{K}$ -algebra generated by  $v_1, \dots, v_n$  with relations  $v_j v_i = q_{ij} v_i v_j$ :

$$S_{\mathfrak{q}}(V) = \mathbb{K}\langle v_1, \dots, v_n \rangle / (v_j v_i - q_{ij} v_i v_j : 1 \leq i, j \leq n).$$

We view  $S_{\mathfrak{q}}(V)$  as a graded algebra with  $\deg v_i = 1$  for all  $i$ , see [8], and write  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  for the ring of graded automorphisms of  $S_{\mathfrak{q}}(V)$ , i.e., automorphisms that preserve degree. Every graded automorphism defines a linear transformation on  $V$  by restriction, and we view  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  as a subgroup of  $\text{GL}(n, \mathbb{K})$ . Conversely, [Lemma 2.1](#) below describes which elements of  $\text{GL}(n, \mathbb{K})$  extend to graded automorphisms on  $S_{\mathfrak{q}}(V)$  with respect to the basis  $v_1, \dots, v_n$ , see also [4] and [9]. We write  $h = (h_{ij})$  in  $\text{GL}(n, \mathbb{K})$  for  $h v_j = \sum_i h_{ij} v_i$ .

**Lemma 2.1.** [12, Lemma 3.2] *A matrix  $h \in \text{GL}(n, \mathbb{K})$  lies in  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  if and only if*

$$h_{i\ell} h_{jm} = 0 \quad \text{or} \quad q_{ij} = q_{\ell m} \quad \text{for all} \quad 1 \leq i, j, m, \ell \leq n.$$

**Monomial automorphisms.** Let  $\mathbb{G}_n$  denote the group of monomial matrices in  $\text{GL}(n, \mathbb{K})$ , i.e., invertible matrices with exactly  $n$  nonzero entries. Then  $\mathbb{G}_n$  admits a decomposition  $(\mathbb{K}^\times)^n \rtimes \mathfrak{S}_n$  for  $(\mathbb{K}^\times)^n$  identified with the diagonal matrices in  $\text{GL}(n, \mathbb{K})$  and the symmetric group  $\mathfrak{S}_n$  identified with the permutation matrices acting on  $V$  by permutation of basis elements  $v_i$ . The group  $\mathfrak{S}_n$  acts on the set of  $n \times n$  quantum parameter matrices by  $\sigma : \mathfrak{q} \mapsto \mathfrak{q}'$  where  $q'_{ij} = q_{\sigma(i)\sigma(j)}$ . A monomial matrix  $h = d\sigma$  for  $d$  in  $\text{GL}(n, \mathbb{K})$  diagonal and  $\sigma$  in  $\mathfrak{S}_n$  lies in  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  exactly when  $\mathfrak{q}$  is invariant under this action of  $\sigma$ , in which case we call  $h$  a *monomial automorphism*.

**Remark 2.2.** Note that  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  contains only monomial automorphisms if and only if  $\mathfrak{q}$  has no identical rows, see [10, Lemma 3.5e]. This implies that if  $q_{ij} \neq 1$  for all  $i \neq j$ , then  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  is a monomial group of matrices, see [16, Lemma 3.4], [5, Lemma 3.2], and [5, Prop 3.9].

**Remark 2.3.** We will use the semi-direct product structure arising from the Splitting Lemma: If  $\pi : G \rightarrow G'$  and  $\iota : G' \rightarrow G$  are group homomorphisms with  $\pi\iota = \text{Id}$ , then the exact sequence  $1 \rightarrow \text{Ker } \pi \hookrightarrow G \xrightarrow{\pi} G' \rightarrow 1$  of groups splits and

$$G \cong \text{Ker } \pi \rtimes G'$$

under the map  $g \mapsto (g\iota\pi(g^{-1}), \pi(g))$  with multiplication  $(g, \sigma)(h, \tau) = (g\iota(\sigma)h\iota(\sigma)^{-1}, \sigma\tau)$ .

**Structure of Graded Automorphism Group.** Jin [9] elegantly presented the graded automorphism group  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  as a semidirect product. We give a short proof of an equivalent structure using the Splitting Lemma for groups (see [Remark 2.3](#)). We use this approach to decompose the group of graded automorphisms in terms of potentially smaller graded automorphism groups, see [Proposition 2.16](#), [Theorem 4.2](#), and [Theorem 5.4](#). Our formulation allows us to directly leverage the fact that the acting permutation group permutes blocks of the same size, making it easy to express as a subgroup of a product of symmetric groups read off immediately from the matrix  $\mathfrak{q}$ . Using this fact, we recharacterize the graded automorphism group in terms of maximal subgroups of a permutation group with an orbit condition.

The graded automorphism group is a semidirect product of two types of automorphisms: The first preserves the subspaces of  $V$  spanned by basis elements corresponding to identical rows in  $\mathfrak{q}$  and the second permutes these subspaces. We fix a  $n \times n$  quantum parameter matrix  $\mathfrak{q}$  defining  $S_{\mathfrak{q}}(V)$  and make heavy use of a partition from [10, Definition 3.1]:

**Definition 2.4.** Let  $\mathcal{B}_q$  be the partition of  $[n] = \{1, \dots, n\}$  with  $i \sim j$  if the  $i$ -th and  $j$ -th rows of  $q$  are identical, i.e.,  $q_{im} = q_{jm}$  for all  $m$ , in which case we say  $i$  and  $j$  lie in the same block of  $q$ .

We decompose  $V$  as a direct sum  $V = \bigoplus_{B \in \mathcal{B}_q} V_B$  where each  $V_B = \text{Span}\{v_i : i \in B\}$ . We identify  $\text{GL}(V_B)$  with the subgroup of  $\text{GL}(V)$  that acts via  $\text{GL}(V_B)$  on  $V_B$  and fixes basis elements  $v_i$  with  $i$  not in  $B$ . Kirkman, Kuzmanovich, and Zhang [10, Lemma 3.2] show that the groups  $\text{GL}(V_B)$  are subgroups of  $\text{Aut}_{\text{gr}}(S_q(V))$  for all blocks  $B \in \mathcal{B}_q$ . For  $B, C \subset [n]$ , we consider the minor matrix of size  $|B| \times |C|$

$$(2.5) \quad \mathbf{q}_{BC} = (q_{ij})_{i \in B, j \in C}.$$

Note that this minor has all entries identical whenever  $B$  and  $C$  are blocks of  $q$  and that  $\mathbf{q}_{BB}$  is a quantum parameter matrix for any  $B \subset [n]$ . For  $r = |\mathcal{B}_q|$ , we identify the symmetric group  $\mathfrak{S}_r$  with the group of permutations of the blocks of  $q$  so that  $\mathfrak{S}_r$  acts on  $\mathcal{B}_q$ .

For  $\sigma$  in  $\mathfrak{S}_r$  preserving block size, i.e., with  $|\sigma(B)| = |B|$  for all blocks  $B$  of  $q$ , let  $\sigma q$  be the  $n \times n$  matrix whose minor matrices are given by (compare with [4])

$$(\sigma q)_{BC} = \mathbf{q}_{\sigma(B)\sigma(C)} \quad \text{for all } B, C \in \mathcal{B}_q.$$

As  $q$  is a quantum parameter matrix, so is  $\sigma q$ . We consider the subgroup of permutations that fix  $q$ :

**Definition 2.6.** The *stabilizing permutation group* of  $\text{Aut}_{\text{gr}}(S_q(V))$  is the group of permutations

$$\begin{aligned} \text{Stab}(q) &= \{\sigma \in \mathfrak{S}_r : \sigma \text{ preserves block size and } \sigma q = q\} \\ &= \{\sigma \in \mathfrak{S}_r : \mathbf{q}_{BC} = \mathbf{q}_{\sigma(B)\sigma(C)} \text{ for all blocks } B, C \text{ of } q\}. \end{aligned}$$

We view  $\text{Stab}(q)$  as the *stabilizing automorphisms* of quantum affine space under an injection recorded with the next lemma.

**Lemma 2.7.** *There exists an injective group homomorphism  $\iota : \text{Stab}(q) \rightarrow \text{Aut}_{\text{gr}}(S_q(V))$  with image  $\prod_{B \in \mathcal{B}_q} \text{Hom}(V_B, V_{\sigma(B)})$ .*

*Proof.* For  $\sigma$  in  $\text{Stab}(q)$ , let  $\iota(\sigma)$  be the unique invertible linear map on  $V$  which permutes the basis elements  $v_i$  of  $V$ , sends  $V_B$  to  $V_{\sigma(B)}$  for each block  $B$  of  $q$ , and preserves the order  $v_1 < v_2 < \dots < v_n$  within each block, using  $|B| = |\sigma(B)|$ , so that  $\iota(\sigma)$  lies in  $\prod_{B \in \mathcal{B}_q} \text{Hom}(V_B, V_{\sigma(B)})$ . **Lemma 2.1** implies that  $\iota(\sigma)$  lies in  $\text{Aut}_{\text{gr}}(S_q(V))$  since if  $\iota(\sigma)(v_\ell) = v_i$  and  $\iota(\sigma)(v_m) = v_j$ , then  $q_{\ell m} = q_{ij}$  as  $q_{BC} = q_{\sigma(B)\sigma(C)}$  for  $\ell, m$  in respective blocks  $B, C$  of  $q$ . Finally, to see that  $\iota$  is a homomorphism, consider  $\sigma'$  in  $\text{Stab}(q)$ . The composition  $\iota(\sigma)\iota(\sigma')$  also permutes basis vectors of  $V$  and preserves the order in each block, so by uniqueness, it agrees with  $\iota(\sigma\sigma')$ .  $\square$

**Proposition 2.8.** *There exists a surjective group homomorphism  $\pi : \text{Aut}_{\text{gr}}(S_q(V)) \rightarrow \text{Stab}(q)$  with  $\text{Ker } \pi = \prod_{B \in \mathcal{B}_q} \text{GL}(V_B)$  and  $\pi \iota = \text{Id}$  for  $\iota$  the map of **Lemma 2.7**.*

*Proof.* For  $h = (h_{ij}) \in \text{Aut}_{\text{gr}}(S_q(V))$ , define a function  $\pi(h) : \mathcal{B}_q \rightarrow \mathcal{B}_q$  which permutes blocks by

$$\pi(h)(B) = \{i : h_{ij} \neq 0 \text{ for some } j \in B\}.$$

We argue  $\pi(h)$  has the advertised codomain. First note that  $\pi(h)(B)$  is a subset of a block for each block  $B$ . Indeed, for  $i$  and  $\ell$  in  $\pi(h)(B)$ ,  $h_{ij} \neq 0 \neq h_{\ell m}$  for some  $j, m \in B$ . To see that  $i$  and  $\ell$  lie in the same block, fix an index  $s$  and find  $t$  with  $h_{st} \neq 0$ . By **Lemma 2.1**,  $q_{is} = q_{jt}$  as  $h_{ij}h_{st} \neq 0$ ,  $q_{jt} = q_{mt}$  as  $j$  and  $m$  lie in the same block, and  $q_{mt} = q_{\ell s}$  as  $h_{\ell m}h_{st} \neq 0$ . A similar argument shows that  $\pi(h)(B)$  is an entire block. Indeed, if  $\pi(h)(B) \subset C$  for  $C$  a block and  $\ell \in C$ ,

take  $m$  with  $h_{\ell m} \neq 0$  and  $i \in \pi(h)(B)$  so that  $h_{ij} \neq 0$  for some  $j \in B$ . Then for all  $t$ , and  $h_{st} \neq 0$ ,  $q_{jt} = q_{is} = q_{\ell s} = q_{mt}$  (using  $h_{ij}h_{st} \neq 0 \neq h_{\ell m}h_{st}$ ) and hence rows  $j$  and  $m$  of  $\mathfrak{q}$  are identical. Thus  $m \in B$  and  $\ell \in \pi(h)(B)$ .

Note that  $\pi(h)$  takes each block of  $\mathfrak{q}$  to a block of the same size. Otherwise, some block would be sent to a smaller block, and, after reindexing,  $h$  would be a block matrix with all entries 0 above and below a non-square block (minor submatrix) forcing  $\det h = 0$ . Next notice that for  $i \in B' = \pi(h)(B)$  and  $\ell \in C' = \pi(h)(C)$ , there exist  $j \in B$  and  $m \in C$  with  $h_{ij}h_{\ell m} \neq 0$ . Then  $q_{i\ell} = q_{jm}$  by [Lemma 2.1](#), so  $\mathfrak{q}_{BC} = \mathfrak{q}_{B'C'}$ .

To see that  $\pi : \text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \rightarrow \text{Stab}(\mathfrak{q})$ ,  $h \mapsto \pi(h)$ , is a group homomorphism, fix  $g, h$  in  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  and  $i \in \pi(gh)(B)$ . Then  $(gh)_{ij} \neq 0$  for some  $j \in B$  and so  $g_{i\ell} \neq 0 \neq h_{\ell j}$  for some  $\ell$ . Then  $\ell \in \pi(h)(B)$  and so  $i \in \pi(g)(\pi(h)(B))$ . Thus,  $\pi(gh)(B) = \pi(g)\pi(h)(B)$ . A quick check confirms that  $\pi \iota = \text{Id}$  and that  $\pi$  has the kernel claimed.  $\square$

**Semidirect product structure.** We obtain a semidirect product by letting  $\text{Stab}(\mathfrak{q})$  act on  $\prod_{B \in \mathcal{B}_{\mathfrak{q}}} \text{GL}(V_B)$  by conjugation via the map of [Lemma 2.7](#): For  $g, h$  in  $\prod_{B \in \mathcal{B}_{\mathfrak{q}}} \text{GL}(V_B)$  and  $\sigma, \tau$  in  $\text{Stab}(\mathfrak{q})$ ,

$$(g, \sigma)(h, \tau) = (g \iota(\sigma) h \iota(\sigma)^{-1}, \sigma \tau) \quad \text{in} \quad \left( \prod_{B \in \mathcal{B}_{\mathfrak{q}}} \text{GL}(V_B) \right) \rtimes \text{Stab}(\mathfrak{q}).$$

The composition in the first coordinate indeed lies in  $\prod_B \text{GL}(V_B)$ : For a fixed block  $B$  of  $\mathfrak{q}$ ,

$$V_B \xrightarrow{\iota(\sigma)^{-1}} V_{\sigma^{-1}(B)} \xrightarrow{h} V_{\sigma^{-1}(B)} \xrightarrow{\iota(\sigma)} V_B \xrightarrow{g} V_B.$$

We obtain a short proof of an alternate formulation of the main result of Jin [\[9\]](#) using the Splitting Lemma. We use this approach to establish [Proposition 2.16](#) and [Theorem 4.2](#) and the classification in [Section 6](#).

**Corollary 2.9.** *For a skew polynomial algebra  $S_{\mathfrak{q}}(V)$ ,*

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \cong \left( \prod_{B \in \mathcal{B}_{\mathfrak{q}}} \text{GL}(V_B) \right) \rtimes \text{Stab}(\mathfrak{q}).$$

*Proof.* [Remark 2.3](#) with the maps  $\iota : \text{Stab}(\mathfrak{q}) \rightarrow \text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  and  $\pi : \text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \rightarrow \text{Stab}(\mathfrak{q})$  of [Lemma 2.7](#) and [Proposition 2.8](#) give the isomorphism  $g \mapsto (g \cdot \iota \pi(g^{-1}), \pi(g))$ .  $\square$

**Orbit Condition.** We now characterize the graded automorphism group of a skew polynomial algebra in terms of subgroups of the symmetric group maximal with respect to an orbit condition. The maximality condition explains why so few permutation groups appear as factors of the semidirect product in [Corollary 2.9](#). It also allows one to rule out groups in a classification of graded automorphism groups, see [Section 6](#).

Recall that  $\mathcal{B}_{\mathfrak{q}}$  is the partition of  $[n]$  given by grouping identical rows of  $\mathfrak{q}$  and we fix  $r = |\mathcal{B}_{\mathfrak{q}}|$  as the number of distinct rows with  $\mathfrak{S}_r$  acting on  $\mathcal{B}_{\mathfrak{q}}$  by permutations. We use the diagonal action of  $\mathfrak{S}_r$  on  $[r]^2 = [r] \times [r]$  and show that the stabilizing permutation group  $\text{Stab}(\mathfrak{q})$  is maximal among all subgroups  $G$  of  $\mathfrak{S}_r$  with the same orbits, i.e., with  $[r]^2/G = [r]^2/\text{Stab}(\mathfrak{q})$ .

**Lemma 2.10.** *For any partition  $\mathcal{P}$  of  $[r]^2$ , the set of subgroups  $\{G \subset \mathfrak{S}_r : [r]^2/G \text{ refines } \mathcal{P}\}$  forms a lattice under subgroup inclusion.*

*Proof.* The orbits of the trivial group on  $[r]^2$  are the sets  $\{(i, j)\}$  giving a refinement of any partition of  $[r]^2$ , so the set of subgroups is nonempty. If  $G$  and  $H$  are subgroups with  $[r]^2/G$  and  $[r]^2/H$  refining  $\mathcal{P}$ , then  $[r]^2/\langle G, H \rangle$  and  $[r]^2/(G \cap H)$  also refine  $\mathcal{P}$ .  $\square$

We now give a description of the graded automorphism groups useful for the classification in [Section 6](#). We write any partition  $\lambda$  of  $n$  as  $(\lambda_1^{m_1} \dots \lambda_s^{m_s})$  with  $\lambda_1, \dots, \lambda_s$  the distinct parts and  $m_i$  the multiplicity of  $\lambda_i$ , i.e., number of parts equal to  $\lambda_i$ . Recall the minor matrices defined by blocks of  $\mathfrak{q}$ , see [Eq. \(2.5\)](#).

**Theorem 2.11.** *For a quantum parameter matrix  $\mathfrak{q}$  with blocks  $B_1, \dots, B_r$ , let  $\mathcal{P}$  be the partition of  $[r]^2$  given by  $(i, j) \sim (\ell, m)$  if  $\mathfrak{q}_{B_i B_j} = \mathfrak{q}_{B_\ell B_m}$ . Then*

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \cong \left( \prod_{i=1}^s \text{GL}(\lambda_i, \mathbb{K})^{m_i} \right) \rtimes G$$

for some integer partition  $\lambda = (\lambda_1^{m_1} \dots \lambda_s^{m_s})$  of  $\dim V$  with  $\sum_i m_i = r$  and  $G$  the maximal subgroup of  $\mathfrak{S}_r$  such that the orbit space  $[r]^2/G$  refines  $\mathcal{P}$ . Here,  $G = \text{Stab}(\mathfrak{q})$ , and  $G$  acts by a subgroup of  $\prod_{i=1}^s \mathfrak{S}_{m_i}$  where each  $\mathfrak{S}_{m_i}$  permutes the  $m_i$  copies of  $\text{GL}(\lambda_i, \mathbb{K})$ .

*Proof.* We use [Lemma 2.10](#) and [Corollary 2.9](#). For  $\sigma \in \mathfrak{S}_r$ , the partition given by the orbits  $[r]^2/\langle \sigma \rangle$  refines  $\mathcal{P}$  exactly when  $\mathfrak{q}_{BC} = \mathfrak{q}_{\sigma(B)\sigma(C)}$  for all blocks  $B, C$  of  $\mathcal{B}_{\mathfrak{q}}$ . Thus,

$$G = \text{Stab}(\mathfrak{q}) = \{ \sigma \in \mathfrak{S}_r : \sigma \text{ preserves block size and } \sigma\mathfrak{q} = \mathfrak{q} \} = \{ \sigma \in \mathfrak{S}_r : [r]^2/\langle \sigma \rangle \text{ refines } \mathcal{P} \}$$

is maximal among the groups with orbits refining  $\mathcal{P}$ , giving the first statement.

For each block  $B \in \mathcal{B}_{\mathfrak{q}}$ , we identify  $\text{GL}(V_B)$  with  $\text{GL}(|B|, \mathbb{K})$  and define the partition  $\lambda$  of  $n$  corresponding to the block sizes:  $\lambda = (\lambda_1^{m_1}, \dots, \lambda_s^{m_s})$ , where the distinct parts  $\lambda_i$  give the distinct sizes of blocks of  $\mathfrak{q}$  and  $m_i$  is the number of blocks of size  $\lambda_i$ . Since each  $\sigma$  in  $G$  preserves block size,  $G$  must be a subgroup of  $\prod_{i=1}^s \mathfrak{S}_{m_i}$  up to conjugation in  $\mathfrak{S}_r$ .  $\square$

**Remark 2.12.** Two extreme cases of [Theorem 2.11](#) arise. When  $\mathfrak{q}$  has no duplicate rows, all the blocks have size 1 giving a generalization of [\[5, Proposition 3.9\]](#):

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \cong (\mathbb{K}^\times)^n \rtimes \text{Stab}(\mathfrak{q}).$$

In contrast, when the block sizes of  $\mathfrak{q}$  are all distinct recorded by a partition  $\lambda$ ,  $\text{Stab}(\mathfrak{q})$  is trivial:

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \cong \prod_{\lambda_i} \text{GL}(\lambda_i, \mathbb{K}).$$

We now consider some examples.

**Example 2.13.** Say  $|\mathbb{K}| > 3$  and let  $\mathfrak{q} = \begin{pmatrix} A & B & C \\ C & A & B \\ B & C & A \end{pmatrix}$  for  $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ ,  $B = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$ , and  $C = \begin{pmatrix} a^{-1} & a^{-1} \\ a^{-1} & a^{-1} \end{pmatrix}$  for  $a \in \mathbb{K}$  with  $a \notin \{0, 1, -1\}$ . Then  $\mathcal{B}_{\mathfrak{q}} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$  and [Theorem 2.11](#) implies that  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \cong D \rtimes \text{Stab}(\mathfrak{q})$  where  $\text{Stab}(\mathfrak{q}) \subset \mathfrak{S}_3$  and

$$D = \left\{ \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix} : M_i \in \text{GL}(2, \mathbb{K}) \right\} \cong (\text{GL}(2, \mathbb{K}))^3.$$

A straightforward calculation shows that  $\text{Stab}(\mathfrak{q}) = \langle (123) \rangle$  and thus

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) = \left\{ \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & M_3 \\ M_1 & 0 & 0 \\ 0 & M_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & M_2 & 0 \\ 0 & 0 & M_3 \\ M_1 & 0 & 0 \end{pmatrix} : M_i \in \text{GL}(2, \mathbb{K}) \right\}.$$

**Example 2.14.** Let  $V \cong \mathbb{K}^8$  with  $\mathbb{Q} \subset \mathbb{K}$  and let

$$\mathfrak{q} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & -1 & -1 & 1 & 3 & \frac{1}{2} \\ 2 & 1 & 1 & 2 & 2 & 2 & 4 & 1 \\ 2 & 1 & 1 & 2 & 2 & 2 & 4 & 1 \\ -1 & \frac{1}{2} & \frac{1}{2} & 1 & 1 & -1 & 3 & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} & 1 & 1 & -1 & 3 & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} & -1 & -1 & 1 & 3 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & \frac{1}{4} \\ 2 & 1 & 1 & 2 & 2 & 2 & 4 & 1 \end{pmatrix}.$$

Here,  $\mathcal{B}_{\mathfrak{q}} = \{\{1, 6\}, \{2, 3, 8\}, \{4, 5\}, \{7\}\}$  and [Corollary 2.9](#) implies that

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) = (\text{GL}(V_{\{1,6\}}) \times \text{GL}(V_{\{2,3,8\}}) \times \text{GL}(V_{\{4,5\}}) \times \text{GL}(V_{\{7\}})) \rtimes \text{Stab}(\mathfrak{q}).$$

A quick computation verifies that  $\text{Stab}(\mathfrak{q}) = \langle (1\ 3) \rangle$ , which swaps rows 1 and 6 with 4 and 5. Thus

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) = \text{GL}(8, \mathbb{K}) \cap \left\{ \begin{pmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{pmatrix}, \begin{pmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{pmatrix} \right\}.$$

**Example 2.15.** For  $\mathbb{K} = \mathbb{F}_5$ ,  $\dim V = 9$ ,  $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$ ,  $B = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ ,  $C = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}$ , and  $\mathfrak{q} = \begin{pmatrix} A & B & C \\ C & A & B \\ B & C & A \end{pmatrix}$ ,

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \cong (\mathbb{K}^{\times})^9 \rtimes [((1\ 2\ 3))^3 \rtimes \langle (1\ 2\ 3) \rangle] \cong [(\mathbb{K}^{\times})^3 \rtimes \langle (1\ 2\ 3) \rangle]^3 \rtimes \langle (1\ 2\ 3) \rangle.$$

**Constant off-diagonal minors.** The last examples leads us to consider quantum parameter matrices with constant matrix minors. Given a quantum parameter matrix  $\mathfrak{q}$  and a partition  $\mathcal{P}$  of  $[n]$  with blocks  $D$  and  $E$ , we call the submatrix  $\mathfrak{q}_{DE}$  a *diagonal  $\mathcal{P}$ -minor* when  $D = E$  and an *off-diagonal  $\mathcal{P}$ -minor* when  $D \neq E$ . Note that  $\mathfrak{q}$  is constant on the off-diagonal minors for the block-circle decomposition partition, see [\[10, Proposition 4.3b\]](#).

In the next proposition, we consider the induced action of  $\text{Stab}(\mathfrak{q})$  on the set of partitions  $\mathcal{P}$  of  $[n]$  refined by  $\mathcal{B}_{\mathfrak{q}}$ : For  $\sigma$  in  $\text{Stab}(\mathfrak{q})$ , set  $\sigma\mathcal{P}$  to be the partition of  $[n]$  whose blocks are  $\sigma(B_1) \cup \dots \cup \sigma(B_m)$  for  $B_1 \cup \dots \cup B_m$  a block of  $\mathcal{P}$ , where the  $B_i$  are blocks of  $\mathcal{B}_{\mathfrak{q}}$ .

**Proposition 2.16.** *Suppose  $\mathfrak{q}$  is an  $n \times n$  quantum parameter matrix constant on off-diagonal  $\mathcal{P}$ -minors for a partition  $\mathcal{P}$  of  $[n]$  refined by  $\mathcal{B}_{\mathfrak{q}}$ . If  $\text{Stab}(\mathfrak{q})$  fixes  $\mathcal{P}$ , then*

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \cong \prod_{D \in \mathcal{P}} \text{Aut}_{\text{gr}}(S_{\mathfrak{q}_{DD}}(V_D)) \rtimes \text{Stab}_{\mathcal{P}}(\mathfrak{q})$$

for  $\text{Stab}_{\mathcal{P}}(\mathfrak{q}) = \{\sigma \in \mathfrak{S}_{|\mathcal{P}|} : |D| = |\sigma(D)| \text{ and } \mathfrak{q}_{\sigma(D)\sigma(E)} = \mathfrak{q}_{DE} \text{ for all blocks } D, E \text{ of } \mathcal{P}\}$ .

*Proof.* We first define a group homomorphism  $\psi : \text{Stab}(\mathfrak{q}) \rightarrow \text{Stab}_{\mathcal{P}}(\mathfrak{q})$ . Fix  $\sigma$  in  $\text{Stab}(\mathfrak{q})$ . As  $\sigma\mathcal{P} = \mathcal{P}$ , for any block  $D = B_1 \cup \dots \cup B_m$  of  $\mathcal{P}$  with  $B_i \in \mathcal{B}_{\mathfrak{q}}$ ,

$$\sigma(D) = \sigma(B_1) \cup \dots \cup \sigma(B_m)$$

is again a block of  $\mathcal{P}$  with  $|\sigma(D)| = |D|$  as  $\sigma$  permutes the blocks of  $\mathfrak{q}$  of the same size. We may thus identify  $\sigma$  with an element of  $\mathfrak{S}_{|\mathcal{P}|}$ . Since  $\mathfrak{q}_{BC} = \mathfrak{q}_{\sigma(B)\sigma(C)}$  for all blocks  $B, C$  of  $\mathfrak{q}$ , it follows that  $\mathfrak{q}_{DE} = \mathfrak{q}_{\sigma(D)\sigma(E)}$  for all blocks  $D, E$  of  $\mathcal{P}$  and  $\sigma$  lies in  $\text{Stab}_{\mathcal{P}}(\mathfrak{q})$ . We obtain the advertised map  $\psi$  and compose with the projection  $\pi$  from [Proposition 2.8](#): Define

$$\pi' = \psi \circ \pi : \text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \xrightarrow{\pi} \text{Stab}(\mathfrak{q}) \xrightarrow{\psi} \text{Stab}_{\mathcal{P}}(\mathfrak{q}).$$

Note that  $\text{Ker } \pi' \subset \prod_{D \in \mathcal{P}} \text{GL}(V_D)$  since  $\pi'(h)(D) = \{i : h_{ij} \neq 0 \text{ for } j \in D\}$  by construction.

Using [Lemma 2.1](#), as  $\mathfrak{q}$  is constant on off-diagonal  $\mathcal{P}$ -minors, one may verify that

$$\text{Ker}(\pi') = \text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \cap \prod_{D \in \mathcal{P}} \text{GL}(V_D) = \prod_{D \in \mathcal{P}} \text{Aut}_{\text{gr}}(S_{\mathfrak{q}_{DD}}(V_D)).$$

We define an injective group homomorphism  $\iota' : \text{Stab}_{\mathcal{P}}(\mathfrak{q}) \rightarrow \text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$ . For each  $\sigma \in \text{Stab}_{\mathcal{P}}(\mathfrak{q})$ , set  $\iota'(\sigma)$  to be the unique invertible linear map on  $V$  which permutes the basis elements of  $V$ , sends  $V_D$  to  $V_{\sigma(D)}$ , and preserves the order  $v_1 < v_2 < \dots < v_n$  of basis elements within each block of  $\mathcal{P}$ , using the fact that  $|D| = |\sigma(D)|$ . We verify that  $\iota'(\sigma)$  lies in  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  using the argument in the proof of [Lemma 2.7](#).

A straightforward check confirms that  $\pi' \iota'$  is the identity, and the result follows from the Splitting Lemma, see [Remark 2.3](#).  $\square$

**Remark 2.17.** For any parameter matrix  $\mathfrak{q}$ , there are two partitions which satisfy the condition in [Proposition 2.16](#) trivially, namely  $\mathcal{B}_{\mathfrak{q}}$  itself and  $[n]$ . Using the partition  $\mathcal{P} = \mathcal{B}_{\mathfrak{q}}$ , we recover the result in [Corollary 2.9](#). At the other extreme, using  $\mathcal{P} = [n]$  gives the trivial statement  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \cong \text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \rtimes \mathfrak{S}_1$ . Thus, we seek a partition coarser than  $\mathcal{B}_{\mathfrak{q}}$  and finer than  $[n]$  satisfying the hypothesis of [Proposition 2.16](#). In [Example 2.15](#), one such example is the partition  $\mathcal{P} = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\}$  with  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \cong \text{Aut}_{\text{gr}}(S_{\mathfrak{q}'}(V))^3 \rtimes \langle (123) \rangle$  for  $\mathfrak{q}' = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$ .

**Corollary 2.18.** *Say  $\mathcal{P}$  is a partition of  $[n]$  refining  $\mathcal{B}_{\mathfrak{q}}$  and each off-diagonal  $\mathcal{P}$ -minor of  $\mathfrak{q}$  is constant and shares no entries with any diagonal  $\mathcal{P}$ -minor. Then*

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \cong \prod_{D \in \mathcal{P}} \text{Aut}_{\text{gr}}(S_{\mathfrak{q}_{DD}}(V_D)) \rtimes \text{Stab}_{\mathcal{P}}(\mathfrak{q})$$

where  $\text{Stab}_{\mathcal{P}}(\mathfrak{q}) = \{\sigma \in \mathfrak{S}_{[n]} : |\sigma(D)| = |D| \text{ and } \mathfrak{q}_{\sigma(D)\sigma(E)} = \mathfrak{q}_{DE} \text{ for all } D, E \in \mathcal{P}\}$ .

*Proof.* The claim follows from [Proposition 2.16](#) after verifying that  $\text{Stab}(\mathfrak{q})$  fixes  $\mathcal{P}$ . Take  $\sigma \in \text{Stab}(\mathfrak{q})$  and  $D \in \mathcal{P}$ . Since  $D$  is a disjoint union of blocks of  $\mathfrak{q}$  and  $\sigma$  preserves the sizes of those blocks,  $|\sigma(D)| = |D|$ . Thus  $\mathfrak{q}_{DD} = \mathfrak{q}_{\sigma(D)\sigma(D)}$  as  $\sigma$  stabilizes  $\mathfrak{q}$ . Since  $\mathfrak{q}_{DD}$  shares no entries with off-diagonal  $\mathcal{P}$ -minors, it follows that  $\sigma(D)$  is a subset of a single block of  $\mathcal{P}$ . Then  $\sum_{D \in \mathcal{P}} |\sigma(D)| \leq \sum_{D \in \mathcal{P}} |D|$ , which implies  $\sigma(D)$  is again a block of  $\mathcal{P}$  for each block  $D$  of  $\mathcal{P}$ . Thus  $\sigma\mathcal{P} = \mathcal{P}$ .  $\square$

**Remark 2.19.** We write  $\text{Aut}_{\text{uni}}(S_{\mathfrak{q}}(V))$  for the set of unipotent automorphisms of  $S_{\mathfrak{q}}(V)$ , i.e.,  $\phi$  in  $\text{Aut}(S_{\mathfrak{q}}(V))$  such that  $\phi(v_i)$  is  $v_i$  plus terms of graded degree  $> 1$  for all  $i$ , see Ceken, Palmieri, Wang, Zhang [\[5\]](#). If  $\mathfrak{q}$  has no rows of all 1's, then by [\[5, Lemma 3.2\]](#)

$$\text{Aut}(S_{\mathfrak{q}}(V)) = \text{Aut}_{\text{uni}}(S_{\mathfrak{q}}(V)) \rtimes \text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)),$$

and if there are no nontrivial algebraic relations among the parameters  $q_{ij}$ , then  $\text{Aut}_{\text{uni}}(S_{\mathfrak{q}}(V))$  is trivial and  $\text{Aut}(S_{\mathfrak{q}}(V)) = \text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  by [\[5, Theorem 3.4\]](#). See also Yakimov [\[18, Corollary 3.7\]](#).

## 3. INFINITE FAMILIES OF GRADED AUTOMORPHISM GROUPS

In this section, we exhibit certain families of monomial graded automorphism groups. We will see in [Proposition 6.4](#) that nonmonomial graded automorphism groups are determined by monomial groups acting in lower dimension, so these families are helpful for classification results. These families have the form  $\{\mathbb{K}^n \rtimes G_n\}$  for  $G_n$  a subgroup of  $\mathfrak{S}_n$ , with each group arising as  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(\mathbb{K}^n))$  for some  $n \times n$  quantum parameter matrix  $\mathfrak{q}$ .

**Proposition 3.1.** *For  $n \geq 1$  and  $\text{char}(\mathbb{K}) \neq 2$ , there is a unique  $n \times n$  quantum parameter matrix  $\mathfrak{q}$  with  $\text{Stab}(\mathfrak{q}) = \mathfrak{S}_n$ . This yields the isomorphism*

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(\mathbb{K}^n)) \cong (\mathbb{K}^\times)^n \rtimes \mathfrak{S}_n.$$

*In fact,  $\mathfrak{S}_n$  is the only 2-transitive permutation group acting on  $[n]$  which is  $\text{Stab}(\mathfrak{q})$  for some quantum parameter matrix  $\mathfrak{q}$ .*

*Proof.* By the 2-transitivity of  $\mathfrak{S}_n$ ,  $\text{Stab}(\mathfrak{q}) = \mathfrak{S}_n$  if and only if the rows of  $\mathfrak{q}$  are distinct and  $q_{ij} = q_{\ell m}$  for all  $i \neq j$  and  $\ell \neq m$ . Thus,  $\text{Stab}(\mathfrak{q}) = \mathfrak{S}_n$  exactly when  $q_{ij} = -1$  for all  $i \neq j$  since  $q_{ji} = q_{ij} = q_{ji}^{-1}$  implies  $q_{ij} = -1$  for all  $i \neq j$  when the rows are distinct, and the isomorphism follows from [Corollary 2.9](#). For the second claim, we use [Theorem 2.11](#) after noting that  $[n]^2/G = [n]^2/\mathfrak{S}_n$  for any 2-transitive group  $G$ .  $\square$

**Remark 3.2.** For  $n > 3$ , [Proposition 3.1](#) shows that no quantum parameter matrix  $\mathfrak{q}$  exists with  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \cong \prod_{B \in \mathcal{B}_{\mathfrak{q}}} \text{GL}(V_B) \rtimes \text{Alt}_n$  for  $V = \mathbb{K}^n$  and  $\text{Stab}(\mathfrak{q}) = \text{Alt}_n$ , the alternating group on  $n$  elements, as  $\text{Alt}_n$  is 2-transitive on  $[n]$ .

We now consider the dihedral group  $D_{2n}$  of order  $2n$ .

**Proposition 3.3.** *For  $\text{char}(\mathbb{K}) \neq 2$  and  $n \geq 4$ , there exists a quantum parameter matrix  $\mathfrak{q}$  such that  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(\mathbb{K}^n)) \cong (\mathbb{K}^\times)^n \rtimes D_{2n}$ .*

*Proof.* For  $n = 4$ , let  $\mathfrak{q} = \begin{pmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$ . Since all the rows of  $\mathfrak{q}$  are distinct, by [Corollary 2.9](#),

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(\mathbb{K}^4)) = (\mathbb{K}^\times)^4 \rtimes \{\sigma \in \mathfrak{S}_4 : q_{ij} = q_{\sigma(i)\sigma(j)} \text{ for } 1 \leq i, j \leq 4\}.$$

A short calculation shows that the group on the right is generated by  $(1234)$  and  $(13)$  and thus

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(\mathbb{K}^4)) \cong (\mathbb{K}^\times)^4 \rtimes \langle (1234), (13) \rangle \cong (\mathbb{K}^\times)^4 \rtimes D_8.$$

For  $n > 4$ , let  $\mathfrak{q}$  be the  $n \times n$  quantum parameter matrix in which every entry is 1 except for the superdiagonal and subdiagonal entries, and the top-right and bottom-left entries, which are  $-1$ :

$$\mathfrak{q} = \begin{pmatrix} 1 & -1 & 1 & \cdots & 1 & -1 \\ -1 & 1 & -1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \ddots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & -1 \\ -1 & 1 & 1 & \cdots & -1 & 1 \end{pmatrix}.$$

As the rows of  $\mathfrak{q}$  are distinct,  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(\mathbb{K}^n)) \cong (\mathbb{K}^\times)^n \rtimes \text{Stab}(\mathfrak{q})$  by [Corollary 2.9](#), where  $\text{Stab}(\mathfrak{q})$  is  $\{\sigma \in \mathfrak{S}_n : q_{ij} = q_{\sigma(i)\sigma(j)} \text{ for } 1 \leq i, j \leq n\}$ . In the following calculations, we take indices mod  $n$ . The group  $D_{2n}$  is generated by  $(12 \cdots n)$  and  $\tau := (1n)(2n-1) \cdots$ , the product of 2-cycles defined by  $\tau(i) = 1 - i \text{ mod } n$ . These lie in  $\text{Stab}(\mathfrak{q})$  as  $\mathfrak{q}$  is constant on super diagonals and is symmetric.

Conversely, suppose  $\sigma$  lies in  $\text{Stab}(\mathfrak{q})$ . Then  $-1 = q_{i(i+1)} = q_{\sigma(i)\sigma(i+1)}$  so our choice of  $\mathfrak{q}$  implies that  $\sigma(i+1) - \sigma(i) \equiv \pm 1 \text{ mod } n$  for  $1 \leq i < n$ . But  $\sigma(i+1) - \sigma(i) \equiv 1$  implies  $\sigma(i+2) - \sigma(i+1) \equiv 1$

as well, otherwise  $\sigma(i+2) \equiv \sigma(i+1) - 1 \equiv \sigma(i) + 1 - 1 \equiv \sigma(i)$  which is impossible. Likewise  $\sigma(i+1) - \sigma(i) \equiv -1$  implies  $\sigma(i+2) - \sigma(i) \equiv -1$  as well. Then as  $\sigma(i)$  is the telescoping sum  $\sigma(1) + (\sigma(2) - \sigma(1)) + \dots + (\sigma(i) - \sigma(i-1))$ , either  $\sigma(i) \equiv \sigma(1) + (i-1) \pmod n$  for all  $1 \leq i \leq n$  or else  $\sigma(i) \equiv \sigma(1) - (i-1) \pmod n$  for all  $1 \leq i \leq n$ . In the first case,  $\sigma$  is a power of  $(12 \cdots n)$ , and in the second, a power of  $(12 \cdots n)$  multiplied by  $\tau$ . In either case,  $\sigma$  lies in  $D_{2n}$ .  $\square$

Lastly, we turn to cyclic groups.

**Proposition 3.4.** *For  $n \geq 1$ , let  $G$  be a cyclic subgroup of  $\mathfrak{S}_n$ . For  $|\mathbb{K}|$  sufficiently large, with  $\text{char}(\mathbb{K}) \neq 2$  whenever  $|G|$  is even, there is a quantum parameter matrix  $\mathfrak{q}$  with  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(\mathbb{K}^n)) \cong (\mathbb{K}^\times)^n \rtimes G$ .*

*Proof.* Suppose  $\sigma$  in  $\mathfrak{S}_n$  generates  $G$ . First note that  $\sigma \mathfrak{q} = \mathfrak{q}$  if and only if  $(\tau \sigma \tau^{-1})(\tau \mathfrak{q}) = \tau \mathfrak{q}$  for all  $\tau$  in  $\mathfrak{S}_n$ , with  $\tau \mathfrak{q}$  another quantum parameter matrix. Thus, without loss of generality, we may assume  $\sigma$  has a disjoint cycle decomposition

$$\sigma = (12 \cdots m_1)(m_1 + 1 \cdots m_2) \cdots (m_{r-1} + 1 \cdots n)$$

for some  $m_i$ . For  $1 \leq i, j \leq n$ , set  $c_i$  to be the length of the cycle containing  $i$  and write  $i \sim j$  for  $i$  and  $j$  in same orbit, i.e., in same cycle, so  $\sigma(i) \equiv i + 1 \pmod{c_i}$ . For  $1 \leq i \leq j \leq n$  with  $i \sim j$ , select  $q_{ij} \neq 0$  satisfying the rules  $q_{ij} = 1$  if  $i = j$ ,  $q_{ij} = -1$  if  $j - i = c_i/2$ , and, for all  $\ell \leq m$  with  $i \sim \ell \sim m$ ,  $q_{ij} = q_{\ell m}$  exactly when  $j - i \equiv m - \ell \pmod{c_i}$ . After these have been chosen, for  $i \not\sim j$ , select  $q_{ij} \neq 0$  satisfying the rule that, for all  $\ell \leq m$ ,  $q_{ij} = q_{\ell m}$  exactly when  $i \sim \ell$ ,  $j \sim m$ , and  $j - i \equiv m - \ell \pmod{\text{gcd}(c_i, c_j)}$ . Set  $q_{ji} = q_{ij}^{-1}$  and assume further that the  $q_{ij}$  are chosen so that  $q_{ij} \neq q_{m\ell}$  for all  $i \leq j$  with  $i \not\sim j$  and all  $\ell \leq m$  to guarantee that  $\text{Stab}(\mathfrak{q})$  is not too large. Observe that  $\mathfrak{q} = (q_{ij})$  is then a quantum parameter matrix.

We now argue that  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(\mathbb{K}^n)) \cong (\mathbb{K}^\times)^n \rtimes G$ . All rows of  $\mathfrak{q}$  are distinct, since 1 only appears as an entry of  $\mathfrak{q}$  on the diagonal. Thus  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(\mathbb{K}^n)) \cong (\mathbb{K}^\times)^n \rtimes \text{Stab}(\mathfrak{q})$  by [Corollary 2.9](#). We argue that  $G = \text{Stab}(\mathfrak{q})$ . For fixed  $i < j$ ,  $i \sim \sigma(i)$  and  $j \sim \sigma(j)$ . Furthermore,  $\sigma(i) - i \equiv 1 \pmod{c_i}$  and  $\sigma(j) - j \equiv 1 \pmod{c_j}$  and so  $\sigma(j) - \sigma(i) \equiv j - i \pmod{\text{gcd}(c_i, c_j)}$ . Thus,  $q_{ij} = q_{\sigma(i)\sigma(j)}$  and  $\langle \sigma \rangle \subset \text{Stab}(\mathfrak{q})$ . Conversely, suppose  $\tau \in \text{Stab}(\mathfrak{q})$ . Then  $q_{\tau(i)\tau(j)} = q_{ij}$  and  $\tau(j) - \tau(i) \equiv j - i \pmod{\text{gcd}(c_i, c_j)}$  so

$$\tau(j) - j \equiv \tau(i) - i \pmod{\text{gcd}(c_i, c_j)} \quad \text{for all } i, j.$$

By the Chinese Remainder Theorem for non co-prime moduli, see [\[14\]](#), there exists a positive integer  $N$  (unique modulo  $\text{lcm}(c_1, \dots, c_n) = |G|$ ) with  $\tau(i) - i \equiv N \pmod{c_i}$  for all  $i$ . Hence  $\tau = \sigma^N \in G$ . Thus,  $\text{Stab}(\mathfrak{q}) = G$  and so  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(\mathbb{K}^n)) \cong (\mathbb{K}^\times)^n \rtimes G$ .  $\square$

**Example 3.5.** For  $G$  the cyclic group generated by  $(123456)(78)$ ,  $|\mathbb{K}| \geq 15$ , and  $\text{char}(\mathbb{K}) \neq 2$ , the last proof gives  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(\mathbb{K}^9)) = (\mathbb{K}^\times)^9 \rtimes G$  for  $a, b, c, d, e, f$  and their inverses distinct in  $\mathbb{K}$  and

$$\mathfrak{q} = \left( \begin{array}{cccccc|cc|c} 1 & a & b & -1 & b^{-1} & a^{-1} & c & d & e \\ a^{-1} & 1 & a & b & -1 & b^{-1} & d & c & e \\ b^{-1} & a^{-1} & 1 & a & b & -1 & c & d & e \\ -1 & b^{-1} & a^{-1} & 1 & a & b & d & c & e \\ b & -1 & b^{-1} & a^{-1} & 1 & a & c & d & e \\ a & b & -1 & b^{-1} & a^{-1} & 1 & d & c & e \\ \hline c^{-1} & d^{-1} & c^{-1} & d^{-1} & c^{-1} & d^{-1} & 1 & -1 & f \\ d^{-1} & c^{-1} & d^{-1} & c^{-1} & d^{-1} & c^{-1} & -1 & 1 & f \\ \hline e^{-1} & e^{-1} & e^{-1} & e^{-1} & e^{-1} & e^{-1} & f^{-1} & f^{-1} & 1 \end{array} \right).$$

## 4. DIRECT PRODUCT DECOMPOSITION

We now decompose the graded automorphism group  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  of a quantum affine space into graded automorphism groups of lower dimensional spaces using results in [Section 2](#), and we verify that the set of graded automorphism groups is closed under direct products. We again fix  $V \cong \mathbb{K}^n$  and an  $n \times n$  quantum parameter matrix  $\mathfrak{q}$ . Various partitions of the set  $[n]$  indexing the rows of  $\mathfrak{q}$  describe automorphisms. For example, the partition  $\mathcal{B}_{\mathfrak{q}}$  on  $[n]$  identifies the ‘‘elementary automorphisms’’ of [\[10\]](#), and the block-circle decomposition of  $[n]$  in [\[10\]](#) identifies the important class of mystic reflection subgroups of  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$ . We construct here a partition  $\mathcal{P}$  of  $[n]$  giving a decomposition of  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  into a direct product of groups  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}_{DD}}(V))$  ranging over the blocks  $D$  of  $\mathcal{P}$ .

The group  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  is completely determined by the blocks of  $\mathfrak{q}$  and the stabilizing permutation group  $\text{Stab}(\mathfrak{q})$ . However, computing  $\text{Stab}(\mathfrak{q})$  concretely may be computationally prohibitive. Indeed,  $\text{Stab}(\mathfrak{q})$  can be found with an algorithm of time complexity  $O(n!)$ , and we seek an alternative description for  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  that bypasses that calculation. The partition  $\mathcal{P}$  introduced here only requires an algorithm of time complexity  $O(n^2)$ , see [Remark 4.3](#).

Observe that every element  $\sigma \in \text{Stab}(\mathfrak{q})$  moves a row of  $\mathfrak{q}$  to another row which is a permutation of the first. Moreover, if these two rows differ at a column, then  $\sigma$  can not fix that column. This motivates the next definition.

**Definition 4.1.** Define an equivalence relation  $\mathcal{R}$  on  $[n]$  by taking the symmetric closure of the relation with  $i \sim \ell$  whenever there is a chain  $i = i_0, i_1, \dots, i_p = \ell$  for some  $p \geq 0$  with, for all  $1 \leq s \leq p$ ,

- (1) the row of  $\mathfrak{q}$  indexed by  $i_{s-1}$  is a permutation of the row indexed by  $i_s$ , or
- (2)  $q_{i_{s-1}i_s} \neq q_{i_t i_s}$  for some  $0 \leq t < s$ .

**Theorem 4.2.** *Let  $\mathfrak{q}$  be a quantum parameter matrix. The graded automorphism group breaks into a direct product over blocks of the partition  $\mathcal{P}$  of  $[n]$  given by the equivalence classes of  $\mathcal{R}$ :*

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \cong \prod_{D \in \mathcal{P}} \text{Aut}_{\text{gr}}(S_{\mathfrak{q}_{DD}}(V_D)).$$

*Proof.* We verify that the partition  $\mathcal{P}$  satisfies the hypothesis of [Proposition 2.16](#) and that the group  $G = \{\sigma \in \mathfrak{S}_{|\mathcal{P}|} : |\sigma(D)| = D \text{ and } \mathfrak{q}_{\sigma(D)\sigma(E)} = \mathfrak{q}_{DE} \text{ for all blocks } D, E \text{ of } \mathcal{P}\}$  given there is trivial.

First notice that  $\mathcal{P}$  is refined by  $\mathcal{B}_{\mathfrak{q}}$  since if two rows of  $\mathfrak{q}$  are equal, then their indices are equivalent under  $\mathcal{R}$  and lie in the same block of  $\mathcal{P}$ . Next we observe that  $\mathfrak{q}$  is constant on the off-diagonal  $\mathcal{P}$ -minors. To see this, take  $D, E \in \mathcal{P}$  with  $D \neq E$  and let  $i, \ell \in D$  and  $j, m \in E$ . Then  $q_{im} = q_{\ell m}$  since otherwise a chain  $i = i_0, i_1, \dots, \ell$  or  $\ell = \ell_0, \ell_1, \dots, i$  as in [Definition 4.1](#) could be extended to a chain also satisfying the condition of [Definition 4.1](#) by appending  $m$  to the end, which would force  $i$  and  $m$  in the same equivalence class of  $\mathcal{R}$  and  $D = E$ . Similarly,  $q_{im} = q_{ij}$ , and thus  $q_{ij} = q_{im} = q_{\ell m}$ . So  $\mathfrak{q}$  is constant on the off-diagonal  $\mathcal{P}$ -minor  $\mathfrak{q}_{DE}$  of  $\mathfrak{q}$ .

Let  $\sigma \in \text{Stab}(\mathfrak{q})$ . Define  $\tau \in \mathfrak{S}_n$  to be the unique permutation preserving the order of indices within each block with  $\tau(B) = \sigma(B)$  for all blocks  $B$  of  $\mathfrak{q}$ . If the row indexed by  $i$  is a permutation of the row indexed by  $j$ , say by  $\tau'$ , then  $\mathfrak{q}_{\tau(i)\tau(m)} = \mathfrak{q}_{im} = \mathfrak{q}_{j\tau'(m)} = \mathfrak{q}_{\tau(j)\tau\tau'(m)}$ , and so the rows indexed by  $\tau(i)$  and  $\tau(j)$  are equal up to a permutation as well. Similarly, if  $\mathfrak{q}_{im} \neq \mathfrak{q}_{jm}$ , then  $\mathfrak{q}_{\tau(i)\tau(m)} \neq \mathfrak{q}_{\tau(j)\tau(m)}$ . Thus, whenever a chain  $i = i_0, i_1, \dots, i_m = j$  exists as in [Definition 4.1](#),  $\tau(i) = \tau(i_0), \tau(i_1), \dots, \tau(i_m) = j$  is also a chain satisfying the definition. Thus  $\sigma(D) \in \mathcal{P}$  for all  $D \in \mathcal{P}$ .

To see that  $G$  is trivial, take  $\sigma \in G$  and a block  $D$  of  $\mathcal{P}$ . Then  $\mathfrak{q}_{\sigma(D)\sigma(E)} = \mathfrak{q}_{DE}$  for all blocks  $E$  of  $\mathcal{P}$ , which implies that each row indexed by an element of  $D$  is a permutation of a row indexed by an element of  $\sigma(D)$ . Thus the elements of  $D$  are equivalent to those of  $\sigma(D)$  under the relation  $\mathcal{R}$ , and  $D = \sigma(D)$  by the definition of  $\mathcal{P}$ .  $\square$

**Remark 4.3.** Let  $\mathcal{P}$  denote the partition giving the equivalence classes of  $\mathcal{R}$  (Definition 4.1). From the proof of Theorem 4.2, we see that  $\mathcal{P}$  satisfies three key properties:

- (1)  $\mathcal{P}$  is refined by  $\mathcal{B}_{\mathfrak{q}}$ ;
- (2) indices of rows of  $\mathfrak{q}$  that are permutations of each other lie in the same block of  $\mathcal{P}$ ;
- (3) for any two blocks  $D, E$  of  $\mathcal{P}$  with  $D \neq E$ , the  $\mathcal{P}$ -minor  $\mathfrak{q}_{DE}$  is constant.

Observe that  $\mathcal{P}$  is the finest partition with these properties. Indeed, we construct  $\mathcal{P}$  with the following two step process. First, we construct the partition of  $[n]$  whose blocks index rows of  $\mathfrak{q}$  which are permutations of each other. We do this by constructing the corresponding multiset of entries for each row, hashing these multisets, and then grouping indices by hash value. This has  $O(n^2)$  time complexity for multiset construction and  $O(n)$  time complexity for the grouping.

For the second step, we successively merge blocks  $D, E$  whenever  $\mathfrak{q}_{DE}$  is not a constant matrix. The algorithm terminates with a partition  $\mathcal{P}$  such that the  $\mathcal{P}$ -minor  $\mathfrak{q}_{DE}$  is constant for all distinct blocks  $D, E$  of  $\mathcal{P}$ . This step has  $O(n^2)$  time complexity since the number of comparisons is bounded by the number of entries of  $\mathfrak{q}$ . This results in an algorithm generating  $\mathcal{P}$  with  $O(n^2)$  time complexity overall.

**Example 4.4.** Suppose  $a, b, c, d, f$  and their inverses are distinct in  $\mathbb{K}$  and

$$\mathfrak{q} = \begin{pmatrix} 1 & -1 & a & b & c & c & f & f \\ -1 & 1 & b & a & c & c & f & f \\ a^{-1} & b^{-1} & 1 & -1 & c & c & d & f \\ b^{-1} & a^{-1} & -1 & 1 & c & c & f & d \\ c^{-1} & c^{-1} & c^{-1} & c^{-1} & 1 & -1 & c^{-1} & c^{-1} \\ c^{-1} & c^{-1} & c^{-1} & c^{-1} & -1 & 1 & c^{-1} & c^{-1} \\ f^{-1} & f^{-1} & d^{-1} & f^{-1} & c & c & 1 & -1 \\ f^{-1} & f^{-1} & f^{-1} & d^{-1} & c & c & -1 & 1 \end{pmatrix}.$$

Note that any  $\sigma \in \text{Stab}(\mathfrak{q})$  can only permute blocks whose rows are permutations of each other, so it must fix the sets  $\{1, 2\}$ ,  $\{3, 4\}$ ,  $\{5, 6\}$ , and  $\{7, 8\}$ . We turn to the off-diagonal minors. If  $\sigma$  interchanges 1 and 2, then the block  $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$  is preserved which implies that  $\sigma$  interchanges 3 and 4. But then the block  $\begin{pmatrix} d & f \\ f & d \end{pmatrix}$  is preserved so  $\sigma$  must interchange 7 and 8. So  $\text{Stab}(\mathfrak{q}) \subset \langle (12)(34)(78), (56) \rangle$ . The corresponding partition  $\mathcal{P}$  of Theorem 4.2 has blocks  $D = \{1, 2, 3, 4, 7, 8\}$  and  $E = \{5, 6\}$ , and  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \cong \text{Aut}_{\text{gr}}(S_{\mathfrak{q}_{DD}}(V_D)) \times \text{Aut}_{\text{gr}}(S_{\mathfrak{q}_{EE}}(V_E))$ .

Theorem 4.2 implies that the set of graded automorphism groups is closed under direct products.

**Corollary 4.5.** Let  $V_1$  and  $V_2$  be vector spaces with  $\dim V_1 = n_1$  and  $\dim V_2 = n_2$ . Let  $\mathfrak{q}_1$  and  $\mathfrak{q}_2$  be  $n_1 \times n_1$  and  $n_2 \times n_2$  quantum parameter matrices, respectively. For  $|\mathbb{K}|$  sufficiently large, there exists a quantum parameter matrix  $\mathfrak{q}$  of size  $(n_1 + n_2) \times (n_1 + n_2)$  such that

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V_1 \oplus V_2)) \cong \text{Aut}_{\text{gr}}(S_{\mathfrak{q}_1}(V_1)) \times \text{Aut}_{\text{gr}}(S_{\mathfrak{q}_2}(V_2)).$$

*Proof.* We take a basis for  $V = V_1 \oplus V_2$  by appending a basis for  $V_2$  onto a basis for  $V_1$  after embedding  $V_i$  in  $V$ . Select  $a$  not in  $\{0, -1, 1\}$  nor an entry of both  $\mathfrak{q}_1$  and  $\mathfrak{q}_2$ . We set  $\mathfrak{q} = \begin{pmatrix} \mathfrak{q}_1 & A \\ A' & \mathfrak{q}_2 \end{pmatrix}$

where  $A$  is the  $n_1 \times n_2$  constant matrix with all entries  $a$  and  $A'$  is the  $n_2 \times n_1$  constant matrix with all entries  $a^{-1}$ . Then  $\mathfrak{q}$  is a quantum parameter matrix corresponding to  $V_1 \oplus V_2$ .

Write  $\mathcal{P}_1, \mathcal{P}_2$ , and  $\mathcal{P}$  for the partitions of  $[n_1], [n_2]$ , and  $[n_1 + n_2]$  corresponding to  $\mathfrak{q}_1, \mathfrak{q}_2$ , and  $\mathfrak{q}$ , respectively, arising from the decomposition given by [Theorem 4.2](#).

Fix indices  $i, j, m \in [n_1 + n_2]$ . Since  $A$  and  $A'$  are constant matrices, rows  $i$  and  $j$  of  $\mathfrak{q}$  are permutations of each other if and only if they correspond to permuted rows both in  $\mathfrak{q}_1$  or both in  $\mathfrak{q}_2$ . Therefore,  $\mathcal{P} = \mathcal{P}_1 \sqcup \mathcal{P}_2$ .

Since  $\mathfrak{q}_{DD} = (\mathfrak{q}_i)_{DD}$  for  $D \in \mathcal{P}_i$ ,

$$\prod_{D \in \mathcal{P}} \text{Aut}_{\text{gr}}(S_{\mathfrak{q}_{DD}}((V_1 \oplus V_2)_D)) \cong \prod_{D \in \mathcal{P}_1} \text{Aut}_{\text{gr}}(S_{(\mathfrak{q}_1)_{DD}}(V_D)) \times \prod_{D \in \mathcal{P}_2} \text{Aut}_{\text{gr}}(S_{(\mathfrak{q}_2)_{DD}}(V_D))$$

and thus  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V_1 \oplus V_2)) \cong \text{Aut}_{\text{gr}}(S_{\mathfrak{q}_1}(V_1)) \times \text{Aut}_{\text{gr}}(S_{\mathfrak{q}_2}(V_2))$  by [Theorem 4.2](#).  $\square$

**Remark 4.6.** The requirement that  $|\mathbb{K}|$  be sufficiently large in the last proposition is only to guarantee some  $a \in \mathbb{K}^\times$  is not an entry of both  $\mathfrak{q}_1$  and  $\mathfrak{q}_2$  with  $a \neq a^{-1}$ . [Section 6](#) offers a more detailed discussion on how the cardinality of  $\mathbb{K}$  determines which graded automorphism groups appear for some quantum parameter matrix  $\mathfrak{q}$  with entries in  $\mathbb{K}$ .

## 5. KRONECKER PRODUCTS OF QUANTUM PARAMETERS

We investigate here the graded automorphism group arising from the Kronecker product of quantum parameter matrices. This gives a useful tool for constructing quantum affine spaces with desirable automorphism groups.

We fix two quantum parameter matrices  $\mathfrak{q} \in \text{Mat}(n, \mathbb{K})$  and  $\mathfrak{q}' \in \text{Mat}(n', \mathbb{K})$  for integers  $n, n' \geq 1$  and index each entry of  $\mathfrak{q} \otimes \mathfrak{q}'$  in  $\text{Mat}(nn', \mathbb{K})$  by a pair of indices  $(i, i')$  and  $(j, j')$ , setting

$$(5.1) \quad (\mathfrak{q} \otimes \mathfrak{q}')_{(i, i'), (j, j')} = q_{ij} q'_{i'j'}.$$

Note that  $\mathfrak{q} \otimes \mathfrak{q}'$  is again a quantum parameter matrix and defines the skew polynomial algebra  $S_{\mathfrak{q} \otimes \mathfrak{q}'}(V \otimes V')$  with relations

$$(v_j \otimes w_m)(v_i \otimes w_\ell) = q_{ij} q'_{\ell m} (v_i \otimes w_\ell)(v_j \otimes w_m) \quad \text{for } 1 \leq i, j \leq n \text{ and } 1 \leq \ell, m \leq n'$$

for basis  $v_1, \dots, v_n$  of  $V$  and basis  $w_1, \dots, w_{n'}$  of  $V'$ , using the basis  $v_i \otimes w_\ell$  of  $V \otimes V'$ . Generically, a pair of quantum parameter matrices  $\mathfrak{q}$  and  $\mathfrak{q}'$  are independent in the sense that they share no entries nor even products of pairs of entries except 1, i.e.,

$$\{ab : a, b \text{ entries of } \mathfrak{q}\} \cap \{cd : c, d \text{ entries of } \mathfrak{q}'\} = \{1\}.$$

We formalize this notion in the next definition.

**Definition 5.2.** We say that  $\mathfrak{q}$  and  $\mathfrak{q}'$  are *multiplicatively independent* when  $\mathfrak{q} \otimes \mathfrak{q}$  and  $\mathfrak{q}' \otimes \mathfrak{q}'$  have 1 as their only common entry.

We use the indexing convention of [Eq. \(5.1\)](#) to describe blocks of  $\mathfrak{q} \otimes \mathfrak{q}'$ :

**Lemma 5.3.** *The partition  $\mathcal{B}_{\mathfrak{q} \otimes \mathfrak{q}'}$  refines the product partition  $\{B \times B' : B \in \mathcal{B}_{\mathfrak{q}}, B' \in \mathcal{B}_{\mathfrak{q}'}\}$  of  $[n] \times [n']$ . If  $\mathfrak{q}$  and  $\mathfrak{q}'$  are multiplicatively independent, then these partitions of  $[n] \times [n']$  are equal.*

*Proof.* Suppose  $i$  and  $\ell$  are in the same block of  $\mathfrak{q}$  and  $i'$  and  $\ell'$  are in the same block of  $\mathfrak{q}'$ . Then  $q_{im} q'_{i'm'} = q_{\ell m} q'_{\ell'm'}$  and thus  $(\mathfrak{q} \otimes \mathfrak{q}')_{(i,i')(m,m')} = (\mathfrak{q} \otimes \mathfrak{q}')_{(\ell,\ell')(m,m')}$  for all  $(m, m') \in [n] \times [n']$  so that  $(i, i')$  and  $(\ell, \ell')$  are in the same block of  $\mathfrak{q} \otimes \mathfrak{q}'$ . Refinement in the opposite direction uses multiplicative independence: For all  $(m, m')$ , if  $q_{im} q'_{i'm'} = q_{\ell m} q'_{\ell'm'}$ , then  $q_{im} q_{m\ell} = q'_{i'm'} q'_{m'\ell'}$ , and hence  $q_{im} q_{m\ell} = 1 = q'_{i'm'} q'_{m'\ell'}$  so  $q_{im} = q_{\ell m}$  and  $q'_{i'm'} = q'_{\ell'm'}$ .  $\square$

We now describe the graded automorphism group arising from the tensor product of two quantum parameter matrices with multiplicatively independent entries.

**Theorem 5.4.** *For  $\mathfrak{q}$  and  $\mathfrak{q}'$  multiplicatively independent,  $\text{Stab}(\mathfrak{q} \otimes \mathfrak{q}') \cong \text{Stab}(\mathfrak{q}) \times \text{Stab}(\mathfrak{q}')$  and*

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q} \otimes \mathfrak{q}'}(V \otimes V')) \cong \left( \prod_{B \in \mathcal{B}_{\mathfrak{q}}, B' \in \mathcal{B}_{\mathfrak{q}'}} \text{GL}(V_B \otimes V'_{B'}) \right) \rtimes (\text{Stab}(\mathfrak{q}) \times \text{Stab}(\mathfrak{q}')).$$

*Proof.* By Lemma 5.3,  $\mathcal{B}_{\mathfrak{q} \otimes \mathfrak{q}'} = \{B \times B' : B \in \mathcal{B}_{\mathfrak{q}}, B' \in \mathcal{B}_{\mathfrak{q}'}\}$ . The subspace of  $V \otimes V'$  corresponding to the block  $B \times B'$  is  $(V \otimes V')_{B \times B'} = V_B \otimes V_{B'}$  with our indexing convention. Corollary 2.9 then gives

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q} \otimes \mathfrak{q}'}(V \otimes V')) \cong \left( \prod_{B \in \mathcal{B}_{\mathfrak{q}}, B' \in \mathcal{B}_{\mathfrak{q}'}} \text{GL}(V_B \otimes V'_{B'}) \right) \rtimes \text{Stab}(\mathfrak{q} \otimes \mathfrak{q}').$$

Recall  $\text{Stab}(\mathfrak{q} \otimes \mathfrak{q}') = \{\tau \in \mathfrak{S}_{rr'} : \tau \text{ preserves block sizes for } \mathfrak{q} \otimes \mathfrak{q}' \text{ and } \tau(\mathfrak{q} \otimes \mathfrak{q}') = \mathfrak{q} \otimes \mathfrak{q}'\}$  for  $r = |\mathcal{B}_{\mathfrak{q}}|$  and  $r' = |\mathcal{B}_{\mathfrak{q}'}|$ . Here preserving block sizes means  $|\tau(B, B')| = |B||B'|$  for all blocks  $B$  of  $\mathfrak{q}$  and  $B'$  of  $\mathfrak{q}'$ , in which case  $\tau(\mathfrak{q} \otimes \mathfrak{q}')_{(B,B')(C,C')}$  is  $(\mathfrak{q} \otimes \mathfrak{q}')_{\tau(B,B')\tau(C,C')}$ . We identify  $\mathfrak{S}_r \times \mathfrak{S}_{r'}$  with a subgroup of  $\mathfrak{S}_{rr'}$ , and, for  $\sigma \in \mathfrak{S}_r$  and  $\sigma' \in \mathfrak{S}_{r'}$  preserving blocks sizes of  $\mathfrak{q}$  and  $\mathfrak{q}'$ , respectively, we define the quantum parameter matrix

$$(\sigma, \sigma')(\mathfrak{q} \otimes \mathfrak{q}') = \sigma \mathfrak{q} \otimes \sigma' \mathfrak{q}'.$$

We argue that  $\text{Stab}(\mathfrak{q}) \times \text{Stab}(\mathfrak{q}') = \text{Stab}(\mathfrak{q} \otimes \mathfrak{q}')$  under this identification.

If  $(\sigma, \sigma') \in \text{Stab}(\mathfrak{q}) \times \text{Stab}(\mathfrak{q}')$ , then  $(\sigma, \sigma')(\mathfrak{q} \otimes \mathfrak{q}') = \sigma \mathfrak{q} \otimes \sigma' \mathfrak{q}' = \mathfrak{q} \otimes \mathfrak{q}'$  so that  $(\sigma, \sigma') \in \text{Stab}(\mathfrak{q} \otimes \mathfrak{q}')$ .

For the opposite inclusion, let  $\tau \in \text{Stab}(\mathfrak{q} \otimes \mathfrak{q}')$ . Define  $\pi_1, \pi_2$  to be the projection functions onto the first and second coordinates, respectively. We claim that for any fixed  $B'$  in  $\mathcal{B}_{\mathfrak{q}'}$ , the function  $\sigma : \mathcal{B}_{\mathfrak{q}} \rightarrow \mathcal{B}_{\mathfrak{q}}$  defined by  $B \mapsto \pi_1 \tau(B, B')$  is independent of choice of  $B'$ .

To verify the claim, we fix  $B', C'$  in  $\mathcal{B}_{\mathfrak{q}'}$  and  $B$  in  $\mathcal{B}_{\mathfrak{q}}$ , and show  $\pi_1 \tau(B, B') = \pi_1 \tau(B, C')$ . Write  $\tau(B, B') = (D, D')$  and  $\tau(B, C') = (F, F')$  for some blocks  $D, D', F, F'$ . Take any  $(M, M')$  in  $\mathcal{B}_{\mathfrak{q}} \times \mathcal{B}'_{\mathfrak{q}}$  and set  $(L, L') = \tau^{-1}(M, M')$ . Then

$$(\mathfrak{q} \otimes \mathfrak{q}')_{(B,B')(L,L')} = (\mathfrak{q} \otimes \mathfrak{q}')_{(D,D')(M,M')} \quad \text{and} \quad (\mathfrak{q} \otimes \mathfrak{q}')_{(B,C')(L,L')} = (\mathfrak{q} \otimes \mathfrak{q}')_{(F,F')(M,M')}.$$

Since each minor matrix considered has all identical entries, we take a sample entry from each. Say  $b, b', c', d, d', f, f', \ell, \ell', m, m'$  lie in  $B, B', C', D, D', F, F', L, L', M, M'$ , respectively. The first equality in the display gives  $q_{b\ell} q'_{b'\ell'} = q_{dm} q'_{d'm'}$  which forces  $q_{b\ell} = q_{dm}$  by the multiplicative independence of  $\mathfrak{q}$  and  $\mathfrak{q}'$ . A similar argument using the second equality confirms that  $q_{b\ell} = q_{fm}$ , and hence  $q_{dm} = q_{fm}$ . As we may take  $m$  to be arbitrary,  $D = F$ . Thus  $\sigma = \pi_1 \tau(-, B') : \mathcal{B}_{\mathfrak{q}} \rightarrow \mathcal{B}_{\mathfrak{q}}$  does not depend on choice of  $B'$  in  $\mathcal{B}_{\mathfrak{q}'}$ .

Likewise, we may define a function  $\sigma' : \mathcal{B}_{\mathfrak{q}'} \rightarrow \mathcal{B}_{\mathfrak{q}'}$  defined by  $B' \mapsto \pi_2 \tau(B, B')$  for a fixed block  $B$  of  $\mathcal{B}_{\mathfrak{q}}$  and show that  $\sigma'$  does not depend on choice of  $B$ .

Then  $\tau : \mathcal{B}_{\mathfrak{q} \otimes \mathfrak{q}'} \rightarrow \mathcal{B}_{\mathfrak{q} \otimes \mathfrak{q}'}$  may be written as  $\tau = (\sigma, \sigma')$ . As  $\tau$  is bijective, both  $\sigma$  and  $\sigma'$  must be bijective. For example, if  $\sigma(B_1) = \sigma(B_2)$ , then for any block  $C'$ ,  $\tau(B_1, C') = (\sigma(B_1), \sigma'(C')) =$

$(\sigma(B_2), \sigma'(C')) = \tau(B_2, C')$  and  $B_1 = B_2$  as  $\tau$  is one-to-one. Hence  $\sigma$  and  $\sigma'$  are permutations of the blocks of  $\mathfrak{q}$  and  $\mathfrak{q}'$ , respectively.

We next argue that  $\sigma$  lies in  $\text{Stab}(\mathfrak{q})$  and  $\sigma'$  lies in  $\text{Stab}(\mathfrak{q}')$ . First observe that both  $\sigma$  and  $\sigma'$  preserve block size. Indeed, for all blocks  $B$  in  $\mathcal{B}_{\mathfrak{q}}$  and  $B'$  in  $\mathcal{B}_{\mathfrak{q}'}$ ,  $|B||B'| = |\tau(B, B')| = |\sigma(B)||\sigma'(B')|$  as  $\tau \in \text{Stab}(\mathfrak{q} \otimes \mathfrak{q}')$ . Thus if either  $\sigma$  or  $\sigma'$  preserve block size, then so does the other. And if neither  $\sigma$  nor  $\sigma'$  preserve block size, then there are blocks  $B$  and  $B'$  sent by  $\sigma$  and  $\sigma'$ , respectively, to larger blocks, implying that  $|B||B'| = |\tau(B, B')| = |\sigma(B)||\sigma'(B')| > |B||B'|$ . Since  $\tau \in \text{Stab}(\mathfrak{q} \otimes \mathfrak{q}')$ , for any blocks  $B, L$  of  $\mathfrak{q}$  and  $B', L'$  of  $\mathfrak{q}'$ ,

$$(\mathfrak{q} \otimes \mathfrak{q}')_{(B, B')(L, L')} = (\mathfrak{q} \otimes \mathfrak{q}')_{\tau(B, B')\tau(L, L')} = (\mathfrak{q} \otimes \mathfrak{q}')_{(\sigma(B), \sigma'(B'))(\sigma(L), \sigma'(L'))}$$

which implies that  $\mathfrak{q}_{BL} = \mathfrak{q}_{\sigma(B)\sigma(L)}$  and  $\mathfrak{q}'_{B'L'} = \mathfrak{q}'_{\sigma'(B')\sigma'(L')}$  since  $\mathfrak{q}$  and  $\mathfrak{q}'$  are multiplicatively independent and  $\sigma$  and  $\sigma'$  preserve block sizes. Thus  $\tau = (\sigma, \sigma')$  lies in  $\text{Stab}(\mathfrak{q}) \times \text{Stab}(\mathfrak{q}')$  and hence  $\text{Stab}(\mathfrak{q} \otimes \mathfrak{q}') \subset \text{Stab}(\mathfrak{q}) \times \text{Stab}(\mathfrak{q}')$ .  $\square$

**Remark 5.5.** For  $|\mathbb{K}|$  sufficiently large and any  $n \times n$  quantum parameter matrix  $\mathfrak{q}$ , [Theorem 5.4](#) implies that there is an  $m \times m$  quantum parameter matrix  $\mathfrak{q}'$  for all positive multiples  $m$  of  $n$  with stabilizing permutation group  $\text{Stab}(\mathfrak{q}') \cong \text{Stab}(\mathfrak{q})$ . Indeed, we may find  $\mathfrak{q}''$  of appropriate size with  $\mathfrak{q}$  and  $\mathfrak{q}''$  multiplicatively independent and  $\text{Stab}(\mathfrak{q}'')$  trivial by [Proposition 3.4](#).

The last theorem gives infinite families of graded automorphism groups that all share the same stabilizing permutation group.

**Corollary 5.6.** *For positive integers  $n, m$ , and  $n_1, \dots, n_r$ , and a permutation group  $G$ , there exists a quantum parameter matrix  $\mathfrak{q} \in \text{Mat}(n, \mathbb{K})$  with  $\text{Stab}(\mathfrak{q}) = G$ , giving*

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(\mathbb{K}^n)) \cong \prod_i \text{GL}(n_i, \mathbb{K}) \rtimes G,$$

*if and only if there exists a quantum parameter matrix  $\mathfrak{q}' \in \text{Mat}(mn, \mathbb{K})$  with  $\text{Stab}(\mathfrak{q}') = G$ , giving*

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}'}(\mathbb{K}^{mn})) \cong \prod_i \text{GL}(m \cdot n_i, \mathbb{K}) \rtimes G.$$

*Proof.* Suppose  $\mathfrak{q}$  is an  $n \times n$  quantum parameter matrix with  $G = \text{Stab}(\mathfrak{q})$  giving the semidirect product indicated by [Theorem 2.11](#). Let  $\mathfrak{q}' = \mathfrak{q} \otimes \mathbb{1}_m$  for  $\mathbb{1}_m$  the  $m \times m$  matrix whose every entry is 1. Since  $\mathbb{1}_m$  and  $\mathfrak{q}$  are multiplicatively independent, and  $\text{Stab}(\mathbb{1}_m)$  is trivial, [Theorem 5.4](#) implies that  $\text{Stab}(\mathfrak{q}) \cong \text{Stab}(\mathfrak{q}')$ . By [Lemma 5.3](#), since  $\mathcal{B}_{\mathbb{1}_m}$  has only one block,  $\mathcal{B}_{\mathfrak{q}'}$  is in bijection with  $\mathcal{B}_{\mathfrak{q}}$ , and each block of  $\mathcal{B}_{\mathfrak{q}'}$  has  $m$  times as many elements as the corresponding block of  $\mathcal{B}_{\mathfrak{q}}$ . Thus,

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}'}(\mathbb{K}^{nm})) \cong \prod_i \text{GL}(m \cdot n_i, \mathbb{K}) \rtimes G.$$

Conversely, fix a quantum parameter matrix  $\mathfrak{q}' \in \text{Mat}(mn, \mathbb{K})$  with the above graded automorphism group. Then every block of  $\mathcal{B}_{\mathfrak{q}'}$  has size a multiple of  $m$ . As all entries in  $\mathfrak{q}'_{BC}$  are equal for all blocks  $B, C \in \mathcal{B}_{\mathfrak{q}'}$ , there is a  $n \times n$  quantum parameter matrix  $\mathfrak{q}$  with  $\mathfrak{q}' = \mathfrak{q} \otimes \mathbb{1}_m$  and a similar argument verifies that  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(\mathbb{K}^n))$  is as claimed.  $\square$

**Example 5.7.** Observe  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \cong (\mathbb{K}^\times)^3 \rtimes \mathfrak{S}_3$  and  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}'}(V)) \cong (\text{GL}(3, \mathbb{K}))^3 \rtimes \mathfrak{S}_3$  for

$$\mathfrak{q} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathfrak{q}' = \mathfrak{q} \otimes \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

## 6. CLASSIFICATION OF GRADED AUTOMORPHISM GROUPS OF LOW DIMENSION

We classify graded automorphism groups of quantum affine spaces up to  $\dim V \leq 7$ . Since two skew polynomial rings are isomorphic as algebras if and only if their quantum parameters differ by a permutation applied to indices, see Gaddis [6, Theorem 7.4], we classify these groups up to a permutation of the basis elements of  $V$ . Determining the classification reduces to computing the possibilities for the stabilizing permutation group  $\text{Stab}(\mathfrak{q})$  of an  $n \times n$  quantum parameter matrix  $\mathfrak{q}$  up to conjugation in  $\mathfrak{S}_n$ . Not every subgroup of the symmetric group can appear as a stabilizing permutation group. For example, of the 96 conjugacy classes of subgroups of  $\mathfrak{S}_7$ , only 53 appear. This may be explained by the following corollary of [Theorem 2.11](#).

**Corollary 6.1.** *Consider a skew polynomial algebra  $S_{\mathfrak{q}}(V)$  with  $r$  the number of distinct rows of  $\mathfrak{q}$ . The group of graded automorphism is  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \cong \left(\prod_{i=1}^r \text{GL}(n_i, \mathbb{K})\right) \rtimes G$  for a group  $G \subset \mathfrak{S}_r$  with the property that  $H \subset G$  whenever  $[r]^2/H = [r]^2/G$  for all  $H \subset \mathfrak{S}_r$ . Here,  $G = \text{Stab}(\mathfrak{q})$ .*

We used the software GAP [7] to give conjugacy classes of subgroups of  $\mathfrak{S}_n$  and a depth-first search algorithm in Python [15] to compute the graded automorphism groups of all possible  $\mathfrak{q}$  matrices, using [Corollary 6.1](#) to rule out groups when helpful. The core linear algebra operations were performed with NumPy [13].

**Maximality condition is not sufficient.** Not every group  $G$  with the maximality property of [Corollary 6.1](#) is the stabilizing permutation group of some  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$ , as we see in the next example.

**Example 6.2.** Suppose  $G = \langle (12)(34), (13)(24) \rangle \subset \mathfrak{S}_r$  for  $r \geq 4$ . Then  $G$  satisfies the maximality condition of [Corollary 6.1](#). We argue  $G \neq \text{Stab}(\mathfrak{q})$  for every quantum parameter matrix  $\mathfrak{q}$ . Fix  $\mathfrak{q}$  with blocks  $B_1, \dots, B_r$  and  $G = \text{Stab}(\mathfrak{q})$ . Define a new  $4 \times 4$  matrix  $\mathfrak{q}'$  with  $q'_{ij}$  the entry in  $\mathfrak{q}_{B_i B_j}$  for  $i, j \leq 4$ . Note that  $\mathfrak{q}_{B_i B_m} = \mathfrak{q}_{B_j B_m}$  for  $i, j \leq 4$  and  $m > 4$  as  $G$  is transitive on  $\{1, 2, 3, 4\}$  but fixes  $m$ . This implies that  $\mathfrak{q}'$  is a quantum parameter matrix with distinct rows and  $\text{Stab}(\mathfrak{q}') = G$ . Thus

$$\mathfrak{q}' = \begin{pmatrix} 1 & a & b & c \\ a & 1 & c & b \\ b & c & 1 & a \\ c & b & a & 1 \end{pmatrix} \quad \text{for some } a, b, c \in \{1, -1\}.$$

Note  $c \neq b$  else  $(12)$  would lie in  $G$ . If  $(b, c) = (1, -1)$ , then  $a \neq 1$ , else  $(23)$  would lie in  $G$ , and  $a \neq -1$ , else rows 1 and 3 would be identical. Similarly impossible is the case  $(b, c) = (-1, 1)$ . Hence there is no quantum parameter matrix  $\mathfrak{q}$  with  $G = \text{Stab}(\mathfrak{q})$  and  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V)) \cong \prod_{i=1}^r \text{GL}(n_i, \mathbb{K}) \rtimes G$ .

**Tracking minimal field size.** Whether or not a group may arise as  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(\mathbb{K}^n))$  for some quantum parameter matrix  $\mathfrak{q}$  depends on the cardinality of the field  $\mathbb{K}$ , see [Proposition 3.4](#) for example: The field must be large enough for  $\mathfrak{q}$  to contain enough distinct entries. For example, if  $\mathbb{K} = \mathbb{F}_2$ , then the only quantum parameter matrix is the trivial matrix  $\mathfrak{q} = \mathbf{1}_n$  with  $S_{\mathfrak{q}}(V) \cong \mathbb{K}[v_1, \dots, v_n]$  and  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(\mathbb{K}^n)) \cong \text{GL}(n, \mathbb{K})$ . A sharp lower bound on  $|\mathbb{K}|$  is calculated for each group. For  $n > 7$ , if  $G$  indeed does appear as  $\text{Stab}(\mathfrak{q})$  for some  $\mathfrak{q}$ , the number of orbits of  $G$  on  $[n]^2$  is an upper bound on the minimum cardinality of  $\mathbb{K}^\times$  required, but we are unaware of any formulas for these bounds.

**Example 6.3.** Let  $n = 4$  and  $H = (\mathbb{K}^\times)^4 \rtimes \langle (123) \rangle$ . We argue that 4 is the minimal cardinality of  $\mathbb{K}$  to guarantee  $H$  arises as the graded automorphism group of some skew polynomial algebra. We may rule out  $|\mathbb{K}| = 2$ , and there are four possibilities for  $\mathfrak{q}$  when  $|\mathbb{K}| = 3$ , namely

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

which give rise to respective graded automorphism groups

$$\mathrm{GL}(4, \mathbb{K}), \quad \mathrm{GL}(3, \mathbb{K}) \times \mathbb{K}^\times, \quad (\mathbb{K}^\times)^4 \rtimes \langle (12), (123) \rangle, \quad \text{and} \quad (\mathbb{K}^\times)^4 \rtimes \mathfrak{S}_4.$$

Of these, none are isomorphic to  $H$ , so the minimal cardinality of  $\mathbb{K}$  is at least 4. A simple computation verifies that  $\mathrm{Aut}_{\mathrm{gr}}(S_{\mathfrak{q}}(V)) \cong H$  for  $\mathbb{K} = \{0, 1, a, a^{-1}\} \cong \mathbb{F}_4$  and  $\mathfrak{q} = \begin{pmatrix} 1 & a & a^{-1} & a \\ a^{-1} & 1 & a & a \\ a & a^{-1} & 1 & a \\ a^{-1} & a^{-1} & a^{-1} & 1 \end{pmatrix}$ .

**Nonmonomial groups classified using monomial groups of lower dimension.** By [Theorem 2.11](#), we may assign to each quantum parameter matrix  $\mathfrak{q}$  a partition  $\lambda$  of  $n = \dim V$  with  $\mathrm{Aut}_{\mathrm{gr}}(S_{\mathfrak{q}}(V)) \cong \left( \prod_i \mathrm{GL}(\lambda_i, \mathbb{K})^{m_i} \right) \rtimes G$  for  $G = \mathrm{Stab}(\mathfrak{q})$ , the stabilizing permutation group, and  $\lambda = (\lambda_1^{m_1} \dots \lambda_s^{m_s})$ . Here,  $\lambda_1, \dots, \lambda_s$  are the distinct parts of  $\lambda$  with  $m_i$  the multiplicity of  $\lambda_i$  so that the length of  $\lambda$  is  $r = \sum_{i=1}^s m_i$ .

Every *monomial* graded automorphism group has the form  $(\mathbb{K}^\times)^n \rtimes G$  corresponding to the partition  $\lambda = (1, \dots, 1)$ . For *nonmonomial* graded automorphism groups corresponding to a fixed  $\lambda$ , the stabilizing permutation groups  $G$  all arise from monomial graded automorphism groups in lower dimension  $r = \ell(\lambda)$ , as explained with the next proposition.

**Proposition 6.4.** *Suppose  $|\mathbb{K}|$  is sufficiently large and let  $\lambda = (\lambda_1^{m_1} \dots \lambda_s^{m_s})$  be a partition of  $n$  of length  $r$ . Let  $G$  be a subgroup of  $\mathfrak{S}_r$ . The following are equivalent:*

- *There is a  $n \times n$  quantum parameter matrix  $\mathfrak{q}$  with block sizes corresponding to  $\lambda$  with  $\mathrm{Stab}(\mathfrak{q}) = G$ .*
- *There is an  $r \times r$  quantum parameter matrix  $\mathfrak{q}'$  with  $\mathrm{Stab}(\mathfrak{q}') = G$  a subgroup of  $\prod_{i=1}^s \mathfrak{S}_{m_i}$  in  $\mathfrak{S}_r$  (up to conjugation) with  $\mathrm{Aut}_{\mathrm{gr}}(S_{\mathfrak{q}'}(V))$  monomial.*

*Proof.* We use [Theorem 2.11](#). Take  $\mathfrak{q}$  as described and label its blocks  $B_1, \dots, B_r$  so that each  $\mathfrak{q}_{B_i B_j}$  has all entries the same. Let  $\mathfrak{q}'$  be the  $r \times r$  quantum parameter matrix whose  $ij$ -th entry is the entry in  $\mathfrak{q}_{B_i B_j}$ . In other words,  $\mathfrak{q}'$  is the matrix obtained by removing duplicate rows and columns of  $\mathfrak{q}$  and reindexing. Let  $\mathcal{P}$  be the partition of  $[r]$  with two indices  $i, j$  in the same block exactly when  $|B_i| = |B_j|$ . We write  $\mathcal{P} = \{E_1, E_2, \dots, E_s\}$  and replace entries in  $\mathfrak{q}'$  so that entries that were equal remain equal and entries that were distinct remain distinct in each off-diagonal minor  $\mathfrak{q}'_{E_i E_j}$  and so that each of these minors has distinct entries from the rest of  $\mathfrak{q}'$ . Then  $\sigma \in \mathfrak{S}_r$  lies in  $\mathrm{Stab}(\mathfrak{q}')$  exactly when  $\sigma(i)$  and  $i$  are in the same block of  $\mathcal{P}$  and  $\mathfrak{q}'_{ij} = \mathfrak{q}'_{\sigma(i)\sigma(j)}$  for all  $i, j$ . This happens exactly when  $|B_i| = |\sigma(B_i)|$  and  $\mathfrak{q}_{B_i B_j} = \mathfrak{q}_{\sigma(B_i)\sigma(B_j)}$  for all  $i, j$ . Therefore  $\mathrm{Stab}(\mathfrak{q}') = \mathrm{Stab}(\mathfrak{q}) = G$ . Finally, since  $\mathfrak{q}'$  has distinct rows, the graded automorphism group corresponding to  $\mathfrak{q}'$  is monomial.

Now assume  $\mathfrak{q}'$  as described in the statement is given. Then  $\mathfrak{q}'$  must have distinct rows as  $\mathrm{Aut}_{\mathrm{gr}}(S_{\mathfrak{q}'}(V))$  is monomial. We construct  $\mathfrak{q}$  from  $\mathfrak{q}'$  by repeating  $\lambda_i$  times the  $i$ -th row of  $\mathfrak{q}'$  to create a matrix of size  $n \times r$  and then adding the columns necessary to obtain a  $n \times n$  quantum parameter matrix  $\mathfrak{q}$  as claimed with  $\mathrm{Stab}(\mathfrak{q}) = \mathrm{Stab}(\mathfrak{q}')$ .  $\square$

**Example 6.5.** Say  $\mathfrak{q}$  is an  $8 \times 8$  quantum parameter matrix corresponding to the partition  $\lambda = (1, 1, 2, 2, 2) = (1^2 \ 2^3)$  so  $\ell(\lambda) = 5$ . We find in the  $\dim V = 5$  table the rows with  $\text{Stab } \mathfrak{q}'$  a subgroup of  $\mathfrak{S}_2 \times \mathfrak{S}_3$  identified with a subgroup of  $\mathfrak{S}_5$  up to conjugation in  $\mathfrak{S}_5$ , as there must be two blocks of the smallest size and three of the next smallest size. Thus, up to reindexing,  $\text{Stab}(\mathfrak{q}) = G$  and

$$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(\mathbb{K}^8)) \cong ((\mathbb{K}^\times)^2 \times \text{GL}(2, \mathbb{K})^3) \rtimes G$$

for  $G \subset \mathfrak{S}_2 \times \mathfrak{S}_3$  one of 8 groups, namely,  $1$ ,  $1 \times \langle (123) \rangle$ ,  $1 \times \mathfrak{S}_3$ ,  $\mathfrak{S}_2 \times 1$ ,  $\mathfrak{S}_2 \times \langle (12) \rangle$ ,  $\mathfrak{S}_2 \times \langle (123) \rangle$ ,  $\langle (12) \times (12) \rangle$ , and  $\mathfrak{S}_2 \times \mathfrak{S}_3$  for  $1$  the identity group. All of these 8 groups arise for some  $\mathfrak{q}$ .

**Listing the monomial groups, counting the nonmonomial groups.** For  $\dim V \leq 5$ , we give each graded automorphism group  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  explicitly. For  $\dim V = 6$  and  $\dim V = 7$ , we give the monomial graded automorphism groups  $(\mathbb{K}^\times)^m \rtimes G$  by listing the stabilizing permutation groups  $G = \text{Stab}(\mathfrak{q})$  that arise explicitly. We count the nonmonomial graded automorphism groups in the classification using [Proposition 6.4](#), which explains how to write them out explicitly using earlier tables in order to complete the classification. We omit explicit tables for these nonmonomial groups for brevity.

**The explicit classification.** Below, we classify the matrix groups that arise as graded automorphism groups for skew polynomial algebras with  $\dim V \leq 7$ . We give the classification up to a permutation of the basis elements, i.e., up to a permutation of indices of the quantum parameter matrices. In the tables,  $(\mathbb{K}^\times)^m \times \text{GL}(\ell, \mathbb{K})$  indicates a group of block diagonal matrices, which sometimes appears in a semidirect product with a permutation group acting by permuting the blocks. There are exactly

- 3 groups for  $\dim V = 2$ ,
- 6 groups for  $\dim V = 3$ ,
- 15 groups for  $\dim V = 4$ ,
- 25 groups for  $\dim V = 5$ ,
- 65 groups for  $\dim V = 6$ , and
- 105 groups for  $\dim V = 7$

that arise as  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  for some quantum parameter matrix  $\mathfrak{q}$ . For  $\dim V = 3$ , this recovers [\[12, Theorem 11.1\]](#). Aside from a handful of permutation groups of small order, the classification for  $\dim V$  equal to 4, 5, 6, and 7 was determined by exhaustive computation, taking only a brief amount of time with current computer capabilities.

In each table below, we list the groups in the classification for a fixed dimension of  $V$  and indicate a sample  $\mathfrak{q}$  giving that group as  $\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$  along with the stabilizing permutation group  $G = \text{Stab}(\mathfrak{q})$  as in [Theorem 2.11](#). We also indicate the minimum cardinality of  $\mathbb{K}$  required. For each matrix  $\mathfrak{q}$  listed, we assume  $a, b, c, d, e, f$  are pairwise distinct scalars in  $\mathbb{K}^*/\{\pm 1\}$ .

We also indicate in each table the orbits of  $G$  on  $[n]^2$  for  $n = \dim V$  using the fact that  $G$  permutes blocks of  $\mathfrak{q}$  of the same size: For each possible block size, we reindex so the blocks of that size are  $B_1, \dots, B_m$  and give an  $m \times m$  matrix indicating the orbits of  $G$  on  $[m] \times [m]$ ; the entries range from 1 to the number of orbits with  $ij$ -th and  $k\ell$ -th entries equal whenever  $(i, j)$  and  $(k, \ell)$  lie in the same orbit.

TABLE 1. Graded Automorphism Groups for  $\dim V = 2$ 

$\text{Aut}_{\text{gr}}(S_q(V))$	$q$	$\text{Stab } q$	Orbits	Min $ \mathbb{K} $
$(\mathbb{K}^\times)^2$	$\begin{pmatrix} 1 & a \\ a^{-1} & 1 \end{pmatrix}$	$\langle e \rangle$	$[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}]$	4
$(\mathbb{K}^\times)^2 \rtimes \langle(12)\rangle$	$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$	$\langle(12)\rangle$	$[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}]$	3
$\text{GL}(2, \mathbb{K})$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\langle e \rangle$	$[1]$	2

TABLE 2. Graded Automorphism Groups for  $\dim V = 3$ 

$\text{Aut}_{\text{gr}}(S_q(V))$	$q$	$\text{Stab } q$	Orbits	Min $ \mathbb{K} $
$(\mathbb{K}^\times)^3$	$\begin{pmatrix} 1 & a & b \\ a^{-1} & 1 & c \\ b^{-1} & c^{-1} & 1 \end{pmatrix}$	$\langle e \rangle$	$[\begin{smallmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{smallmatrix}]$	4
$(\mathbb{K}^\times)^3 \rtimes \langle(12)\rangle$	$\begin{pmatrix} 1 & -1 & a \\ -1 & 1 & a \\ a^{-1} & a^{-1} & 1 \end{pmatrix}$	$\langle(12)\rangle$	$[\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 4 & 4 & 1 \end{smallmatrix}]$	3
$(\mathbb{K}^\times)^3 \rtimes \langle(123)\rangle$	$\begin{pmatrix} 1 & a & a^{-1} \\ a^{-1} & 1 & a \\ a & a^{-1} & 1 \end{pmatrix}$	$\langle(123)\rangle$	$[\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{smallmatrix}]$	4
$(\mathbb{K}^\times)^3 \rtimes \mathfrak{S}_3$	$\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$	$\mathfrak{S}_3$	$[\begin{smallmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{smallmatrix}]$	3
$\mathbb{K}^\times \times \text{GL}(2, \mathbb{K})$	$\begin{pmatrix} 1 & a & a \\ a^{-1} & 1 & 1 \\ a^{-1} & 1 & 1 \end{pmatrix}$	$\langle e \rangle$	$[1], [1]$	3
$\text{GL}(3, \mathbb{K})$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\langle e \rangle$	$[1]$	2

For  $\dim V = 4$ , the classification comprises 15 graded automorphism groups.

TABLE 3. Graded Automorphism Groups for  $\dim V = 4$

$\text{Aut}_{\text{gr}}(S_q(V))$	$\mathfrak{q}$	$\text{Stab } \mathfrak{q}$	Orbits	Min $ \mathbb{K} $
$(\mathbb{K}^\times)^4$	$\begin{pmatrix} 1 & a & b & c \\ a^{-1} & 1 & d & e \\ b^{-1} & d^{-1} & 1 & f \\ c^{-1} & e^{-1} & f^{-1} & 1 \end{pmatrix}$	$\langle e \rangle$	$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$	5
$(\mathbb{K}^\times)^4 \rtimes \langle (12) \rangle$	$\begin{pmatrix} 1 & -1 & b & c \\ -1 & 1 & b & c \\ b^{-1} & b^{-1} & 1 & d \\ c^{-1} & c^{-1} & d^{-1} & 1 \end{pmatrix}$	$\langle (12) \rangle$	$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \\ 5 & 5 & 6 & 7 \\ 8 & 8 & 9 & 10 \end{bmatrix}$	3
$(\mathbb{K}^\times)^4 \rtimes \langle (123) \rangle$	$\begin{pmatrix} 1 & a & a^{-1} & b \\ a^{-1} & 1 & a & b \\ a & a^{-1} & 1 & b \\ b^{-1} & b^{-1} & b^{-1} & 1 \end{pmatrix}$	$\langle (123) \rangle$	$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \\ 2 & 3 & 1 & 4 \\ 5 & 5 & 5 & 6 \end{bmatrix}$	4
$(\mathbb{K}^\times)^4 \rtimes \langle (123), (12) \rangle$	$\begin{pmatrix} 1 & -1 & -1 & b \\ -1 & 1 & -1 & b \\ -1 & -1 & 1 & b \\ b^{-1} & b^{-1} & b^{-1} & 1 \end{pmatrix}$	$\langle (123), (12) \rangle$	$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 1 & 2 & 3 \\ 2 & 2 & 1 & 3 \\ 4 & 4 & 4 & 5 \end{bmatrix}$	3
$(\mathbb{K}^\times)^4 \rtimes \langle (12)(34) \rangle$	$\begin{pmatrix} 1 & \pm 1 & a & b \\ \pm 1 & 1 & b & a \\ a^{-1} & b^{-1} & 1 & \pm 1 \\ b^{-1} & a^{-1} & \pm 1 & 1 \end{pmatrix}$	$\langle (12)(34) \rangle$	$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 5 & 6 & 7 & 8 \\ 6 & 5 & 8 & 7 \end{bmatrix}$	3
$(\mathbb{K}^\times)^4 \rtimes \langle (12), (34) \rangle$	$\begin{pmatrix} 1 & -1 & a & a \\ -1 & 1 & a & a \\ a^{-1} & a^{-1} & 1 & -1 \\ a^{-1} & a^{-1} & -1 & 1 \end{pmatrix}$	$\langle (12), (34) \rangle$	$\begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 1 & 3 & 3 \\ 4 & 4 & 5 & 6 \\ 4 & 4 & 6 & 5 \end{bmatrix}$	5
$(\mathbb{K}^\times)^4 \rtimes \langle (1234) \rangle$	$\begin{pmatrix} 1 & a & \pm 1 & a^{-1} \\ a^{-1} & 1 & a & \pm 1 \\ \pm 1 & a^{-1} & 1 & a \\ a & \pm 1 & a^{-1} & 1 \end{pmatrix}$	$\langle (1234) \rangle$	$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}$	5
$(\mathbb{K}^\times)^4 \rtimes \langle (1234), (13) \rangle$	$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$	$\langle (1234), (13) \rangle$	$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 1 & 2 \\ 2 & 3 & 2 & 1 \end{bmatrix}$	3
$(\mathbb{K}^\times)^4 \rtimes \mathfrak{S}_4$	$\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$	$\mathfrak{S}_4$	$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 \end{bmatrix}$	3
$(\mathbb{K}^\times)^2 \times \text{GL}(2, \mathbb{K})$	$\begin{pmatrix} 1 & a & b & b \\ a^{-1} & 1 & b & b \\ b^{-1} & b^{-1} & 1 & 1 \\ b^{-1} & b^{-1} & 1 & 1 \end{pmatrix}$	$\langle e \rangle$	$[12], [1]$	4
$((\mathbb{K}^\times)^2 \times \text{GL}(2, \mathbb{K})) \rtimes \langle (12) \rangle$	$\begin{pmatrix} 1 & -1 & a & a \\ -1 & 1 & a & a \\ a^{-1} & a^{-1} & 1 & 1 \\ a^{-1} & a^{-1} & 1 & 1 \end{pmatrix}$	$\langle (12) \rangle$	$[12], [1]$	3
$(\text{GL}(2, \mathbb{K}))^2$	$\begin{pmatrix} 1 & 1 & a & a \\ 1 & 1 & a & a \\ a^{-1} & a^{-1} & 1 & 1 \\ a^{-1} & a^{-1} & 1 & 1 \end{pmatrix}$	$\langle e \rangle$	$[12]$	4
$(\text{GL}(2, \mathbb{K}))^2 \rtimes (12)$	$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$	$\langle (12) \rangle$	$[21]$	3
$\mathbb{K}^\times \times \text{GL}(3, \mathbb{K})$	$\begin{pmatrix} 1 & a & a & a \\ a^{-1} & 1 & 1 & 1 \\ a^{-1} & 1 & 1 & 1 \\ a^{-1} & 1 & 1 & 1 \end{pmatrix}$	$\langle e \rangle$	$[1], [1]$	3
$\text{GL}(4, \mathbb{K})$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\langle e \rangle$	$[1]$	2

For  $\dim V = 5$ , we list monomial graded automorphism groups in the first table and nonmonomial groups in the second table. There are total of 25 graded automorphism groups.

TABLE 4. Monomial Graded Automorphism Groups for  $\dim V = 5$

$\text{Aut}_{\text{gr}}(S_q(V))$	$q$	$\text{Stab } q$	Orbits	Min $ \mathbb{K} $
$(\mathbb{K}^\times)^5$	$\begin{pmatrix} 1 & a & b & c & d \\ a^{-1} & 1 & e & f & g \\ b^{-1} & e^{-1} & 1 & h & i \\ c^{-1} & f^{-1} & h^{-1} & 1 & j \\ d^{-1} & g^{-1} & i^{-1} & j^{-1} & 1 \end{pmatrix}$	$\langle e \rangle$	$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{bmatrix}$	5
$(\mathbb{K}^\times)^5 \rtimes \langle (12) \rangle$	$\begin{pmatrix} 1 & -1 & a & b & c \\ -1 & 1 & a & b & c \\ a^{-1} & a^{-1} & 1 & d & e \\ b^{-1} & b^{-1} & d^{-1} & 1 & f \\ c^{-1} & c^{-1} & e^{-1} & f^{-1} & 1 \end{pmatrix}$	$\langle (12) \rangle$	$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \\ 6 & 6 & 7 & 8 & 9 \\ 10 & 10 & 11 & 12 & 13 \\ 14 & 14 & 15 & 16 & 17 \end{bmatrix}$	3
$(\mathbb{K}^\times)^5 \rtimes \langle (123) \rangle$	$\begin{pmatrix} 1 & a & a^{-1} & c & d \\ a^{-1} & 1 & a & c & d \\ a & a^{-1} & 1 & c & d \\ c^{-1} & c^{-1} & c^{-1} & 1 & f \\ d^{-1} & d^{-1} & d^{-1} & f^{-1} & 1 \end{pmatrix}$	$\langle (123) \rangle$	$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \\ 6 & 6 & 6 & 7 & 8 \\ 9 & 9 & 9 & 10 & 11 \end{bmatrix}$	4
$(\mathbb{K}^\times)^5 \rtimes \langle (1234) \rangle$	$\begin{pmatrix} 1 & a & -1 & a^{-1} & d \\ a^{-1} & 1 & a & -1 & d \\ -1 & a^{-1} & 1 & a & d \\ a & -1 & a^{-1} & 1 & d \\ d^{-1} & d^{-1} & d^{-1} & d^{-1} & 1 \end{pmatrix}$	$\langle (1234) \rangle$	$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 3 & 5 \\ 3 & 4 & 1 & 2 & 5 \\ 2 & 3 & 4 & 1 & 5 \\ 6 & 6 & 6 & 6 & 7 \end{bmatrix}$	5
$(\mathbb{K}^\times)^5 \rtimes \langle (12)(34) \rangle$	$\begin{pmatrix} 1 & \pm 1 & a & b & c \\ \pm 1 & 1 & b & a & c \\ a^{-1} & b^{-1} & 1 & \pm 1 & d \\ b^{-1} & a^{-1} & \pm 1 & 1 & d \\ c^{-1} & c^{-1} & d^{-1} & d^{-1} & 1 \end{pmatrix}$	$\langle (12)(34) \rangle$	$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 7 & 6 & 9 & 8 & 10 \\ 11 & 11 & 12 & 12 & 13 \end{bmatrix}$	3
$(\mathbb{K}^\times)^5 \rtimes \langle (12), (34) \rangle$	$\begin{pmatrix} 1 & \pm 1 & a & a & b \\ \pm 1 & 1 & a & a & b \\ a^{-1} & a^{-1} & 1 & \pm 1 & d \\ a^{-1} & a^{-1} & \pm 1 & 1 & d \\ b^{-1} & b^{-1} & d^{-1} & d^{-1} & 1 \end{pmatrix}$	$\langle (12), (34) \rangle$	$\begin{bmatrix} 1 & 2 & 3 & 3 & 4 \\ 2 & 1 & 3 & 3 & 4 \\ 5 & 5 & 6 & 7 & 8 \\ 5 & 5 & 7 & 6 & 8 \\ 9 & 9 & 10 & 10 & 11 \end{bmatrix}$	3
$(\mathbb{K}^\times)^5 \rtimes \langle (12), (123) \rangle$	$\begin{pmatrix} 1 & -1 & -1 & a & b \\ -1 & 1 & -1 & a & b \\ -1 & -1 & 1 & a & b \\ a^{-1} & a^{-1} & a^{-1} & 1 & c \\ b^{-1} & b^{-1} & b^{-1} & c^{-1} & 1 \end{pmatrix}$	$\langle (12), (123) \rangle$	$\begin{bmatrix} 1 & 2 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 3 & 4 \\ 5 & 5 & 5 & 6 & 7 \\ 8 & 8 & 8 & 9 & 10 \end{bmatrix}$	3
$(\mathbb{K}^\times)^5 \rtimes \langle (1234), (13) \rangle$	$\begin{pmatrix} 1 & 1 & -1 & 1 & a \\ 1 & 1 & 1 & -1 & a \\ -1 & 1 & 1 & 1 & a \\ 1 & -1 & 1 & 1 & a \\ a^{-1} & a^{-1} & a^{-1} & a^{-1} & 1 \end{pmatrix}$	$\langle (1234), (13) \rangle$	$\begin{bmatrix} 1 & 2 & 3 & 2 & 4 \\ 2 & 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 & 4 \\ 2 & 3 & 2 & 1 & 4 \\ 5 & 5 & 5 & 5 & 6 \end{bmatrix}$	3
$(\mathbb{K}^\times)^5 \rtimes \langle (1234), (12) \rangle$	$\begin{pmatrix} 1 & -1 & -1 & -1 & a \\ -1 & 1 & -1 & -1 & a \\ -1 & -1 & 1 & -1 & a \\ -1 & -1 & -1 & 1 & a \\ a^{-1} & a^{-1} & a^{-1} & a^{-1} & 1 \end{pmatrix}$	$\langle (1234), (12) \rangle$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 3 \\ 2 & 1 & 2 & 2 & 3 \\ 2 & 2 & 1 & 2 & 3 \\ 2 & 2 & 2 & 1 & 3 \\ 4 & 4 & 4 & 4 & 5 \end{bmatrix}$	3
$(\mathbb{K}^\times)^5 \rtimes \langle (12), (345) \rangle$	$\begin{pmatrix} 1 & \pm 1 & a & a & a \\ \pm 1 & 1 & a & a & a \\ a^{-1} & a^{-1} & 1 & b & b^{-1} \\ a^{-1} & a^{-1} & b^{-1} & 1 & b \\ a^{-1} & a^{-1} & b & b^{-1} & 1 \end{pmatrix}$	$\langle (12), (345) \rangle$	$\begin{bmatrix} 1 & 2 & 3 & 3 & 3 \\ 2 & 1 & 3 & 3 & 3 \\ 4 & 4 & 5 & 6 & 7 \\ 4 & 4 & 7 & 5 & 6 \\ 4 & 4 & 6 & 7 & 5 \end{bmatrix}$	5
$(\mathbb{K}^\times)^5 \rtimes \langle (12), (123)(45) \rangle$	$\begin{pmatrix} 1 & -1 & -1 & a & a \\ -1 & 1 & -1 & a & a \\ -1 & -1 & 1 & a & a \\ a^{-1} & a^{-1} & a^{-1} & 1 & -1 \\ a^{-1} & a^{-1} & a^{-1} & -1 & 1 \end{pmatrix}$	$\langle (12), (123)(45) \rangle$	$\begin{bmatrix} 1 & 2 & 2 & 3 & 3 \\ 2 & 1 & 2 & 3 & 3 \\ 2 & 2 & 1 & 3 & 3 \\ 4 & 4 & 4 & 5 & 6 \\ 4 & 4 & 4 & 6 & 5 \end{bmatrix}$	3
$(\mathbb{K}^\times)^5 \rtimes \langle (12345) \rangle$	$\begin{pmatrix} 1 & a & b & b^{-1} & a^{-1} \\ a^{-1} & 1 & a & b & b^{-1} \\ b^{-1} & a^{-1} & 1 & a & b \\ b & b^{-1} & a^{-1} & 1 & a \\ a & b & b^{-1} & a^{-1} & 1 \end{pmatrix}$	$\langle (12345) \rangle$	$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$	4
$(\mathbb{K}^\times)^5 \rtimes \langle (12345), (15)(24) \rangle$	$\begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix}$	$\langle (12345), (15)(24) \rangle$	$\begin{bmatrix} 1 & 2 & 3 & 3 & 2 \\ 2 & 1 & 2 & 3 & 3 \\ 3 & 2 & 1 & 2 & 3 \\ 3 & 3 & 2 & 1 & 2 \\ 2 & 3 & 3 & 2 & 1 \end{bmatrix}$	3
$(\mathbb{K}^\times)^5 \rtimes \mathfrak{S}_5$	$\begin{pmatrix} 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}$	$\mathfrak{S}_5$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 & 2 \\ 2 & 2 & 1 & 2 & 2 \\ 2 & 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 2 & 1 \end{bmatrix}$	3

TABLE 5. Nonmonomial Graded Automorphism Groups for  $\dim V = 5$ 

$\text{Aut}_{\text{gr}}(S_{\mathfrak{q}}(V))$	$\mathfrak{q}$	$\text{Stab}_{\mathfrak{q}}$	Orbits	Min $ \mathbb{K} $
$(\mathbb{K}^{\times})^3 \times \text{GL}(2, \mathbb{K})$	$\begin{pmatrix} 1 & a & b & c & c \\ a^{-1} & 1 & d & e & e \\ b^{-1} & d^{-1} & 1 & f & f \\ c^{-1} & e^{-1} & f^{-1} & 1 & 1 \\ c^{-1} & e^{-1} & f^{-1} & 1 & 1 \end{pmatrix}$	$\langle e \rangle$	$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, [1]$	4
$((\mathbb{K}^{\times})^3 \times \text{GL}(2, \mathbb{K})) \rtimes \langle (12) \rangle$	$\begin{pmatrix} 1 & -1 & a & b & b \\ -1 & 1 & a & b & b \\ a^{-1} & a^{-1} & 1 & c & c \\ b^{-1} & b^{-1} & c^{-1} & 1 & 1 \\ b^{-1} & b^{-1} & c^{-1} & 1 & 1 \end{pmatrix}$	$\langle (12) \rangle$	$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 4 & 4 & 5 \end{bmatrix}, [1]$	5
$((\mathbb{K}^{\times})^3 \times \text{GL}(2, \mathbb{K})) \rtimes \langle (123) \rangle$	$\begin{pmatrix} 1 & a & a^{-1} & b & b \\ a^{-1} & 1 & a & b & b \\ a & a^{-1} & 1 & b & b \\ b^{-1} & b^{-1} & b^{-1} & 1 & 1 \\ b^{-1} & b^{-1} & b^{-1} & 1 & 1 \end{pmatrix}$	$\langle (123) \rangle$	$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, [1]$	4
$((\mathbb{K}^{\times})^3 \times \text{GL}(2, \mathbb{K})) \rtimes \mathfrak{S}_3$	$\begin{pmatrix} 1 & -1 & -1 & a & a \\ -1 & 1 & -1 & a & a \\ -1 & -1 & 1 & a & a \\ a^{-1} & a^{-1} & a^{-1} & 1 & 1 \\ a^{-1} & a^{-1} & a^{-1} & 1 & 1 \end{pmatrix}$	$\mathfrak{S}_3$	$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, [1]$	3
$\mathbb{K}^{\times} \times (\text{GL}(2, \mathbb{K}))^2$	$\begin{pmatrix} 1 & a & a & b & b \\ a^{-1} & 1 & 1 & c & c \\ a^{-1} & 1 & 1 & c & c \\ b^{-1} & c^{-1} & c^{-1} & 1 & 1 \\ b^{-1} & c^{-1} & c^{-1} & 1 & 1 \end{pmatrix}$	$\langle e \rangle$	$[1], \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$	3
$(\mathbb{K}^{\times} \times (\text{GL}(2, \mathbb{K}))^2) \rtimes \mathfrak{S}_2$	$\begin{pmatrix} 1 & a & a & b & b \\ a^{-1} & 1 & 1 & -1 & -1 \\ a^{-1} & 1 & 1 & -1 & -1 \\ b^{-1} & -1 & -1 & 1 & 1 \\ b^{-1} & -1 & -1 & 1 & 1 \end{pmatrix}$	$\mathfrak{S}_2$	$[1], \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$	3
$(\mathbb{K}^{\times})^2 \times \text{GL}(3, \mathbb{K})$	$\begin{pmatrix} 1 & a & b & b & b \\ a^{-1} & a & c & c & c \\ b^{-1} & c^{-1} & 1 & 1 & 1 \\ b^{-1} & c^{-1} & 1 & 1 & 1 \\ b^{-1} & c^{-1} & 1 & 1 & 1 \end{pmatrix}$	$\langle e \rangle$	$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, [1]$	3
$((\mathbb{K}^{\times})^2 \times \text{GL}(3, \mathbb{K})) \rtimes \langle (12) \rangle$	$\begin{pmatrix} 1 & -1 & a & a & a \\ -1 & 1 & a & a & a \\ a^{-1} & a^{-1} & 1 & 1 & 1 \\ a^{-1} & a^{-1} & 1 & 1 & 1 \\ a^{-1} & a^{-1} & 1 & 1 & 1 \end{pmatrix}$	$\langle (12) \rangle$	$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, [1]$	3
$\text{GL}(2, \mathbb{K}) \times \text{GL}(3, \mathbb{K})$	$\begin{pmatrix} 1 & 1 & a & a & a \\ 1 & 1 & a & a & a \\ a^{-1} & a^{-1} & 1 & 1 & 1 \\ a^{-1} & a^{-1} & 1 & 1 & 1 \\ a^{-1} & a^{-1} & 1 & 1 & 1 \end{pmatrix}$	$\langle e \rangle$	$[1], [1]$	3
$\mathbb{K}^{\times} \times \text{GL}(4, \mathbb{K})$	$\begin{pmatrix} 1 & a & a & a & a \\ a^{-1} & 1 & 1 & 1 & 1 \\ a^{-1} & 1 & 1 & 1 & 1 \\ a^{-1} & 1 & 1 & 1 & 1 \\ a^{-1} & 1 & 1 & 1 & 1 \end{pmatrix}$	$\langle e \rangle \times \langle e \rangle$	$[1], [1]$	3
$\text{GL}(5, \mathbb{K})$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$	$\langle e \rangle$	$[1]$	2

**Dimension six.** We now assume  $\dim V = 6$ . There are total of 65 graded automorphism groups. There are 11 partitions of 6, with the partition  $(1, 1, 1, 1, 1, 1)$  corresponding to monomial automorphisms. The next table provides the number of groups corresponding to the fixed block decomposition given by each partition. We give the monomial groups explicitly below.

TABLE 6. Graded Automorphism Groups for  $\dim V = 6$  organized by partition

$\lambda$	$(1^6)$	$(1^4 2)$	$(1^2 2^2)$	$(2^3)$	$(1^3 3)$	$(1 2 3)$	$(3^2)$	$(1^2 4)$	$(2 4)$	$(1 5)$	$(6)$
count	36	9	4	4	4	1	2	2	1	1	1

For  $|\mathbb{K}| = 3$  there are 21 monomial automorphism groups; they have stabilizing permutation groups

$$\langle e \rangle, \langle (12) \rangle, \langle (12)(34)(56) \rangle, \langle (12)(34) \rangle, \langle (12)(34), (56) \rangle, \langle (12), (34) \rangle, \langle (15)(34), (56)(24) \rangle, \langle (13)(24), (12)(34)(56) \rangle, \langle (12), (23) \rangle, \langle (1234)(56), (24) \rangle, \langle (1234), (24) \rangle, \langle (123), (34) \rangle, \langle (15)(24), (12345) \rangle, \langle (12), (23), (45) \rangle, \langle (1234), (45) \rangle, \langle (123456), (13) \rangle, \langle (12), (234)(56) \rangle, \langle (123456), (163452) \rangle, \langle (123456), (1346), (25) \rangle, \langle (1234), (13), (56) \rangle, \langle (12345), (56) \rangle.$$

For  $|\mathbb{K}| \geq 4$  and  $\text{char } \mathbb{K} = 2$ , there are 12 monomial automorphism groups: 4 groups arise from the  $|\mathbb{K}| = 3$  case and have stabilizing permutation groups

$$\langle e \rangle, \langle (12)(34) \rangle, \langle (12)(34)(56) \rangle, \langle (123)(456), (12)(45) \rangle,$$

and 8 additional groups have stabilizing permutation groups

$$\langle (123) \rangle, \langle (123)(456) \rangle, \langle (1234)(56) \rangle, \langle (1234) \rangle, \langle (12345) \rangle, \langle (123456) \rangle, \langle (123), (456) \rangle, \langle (12)(34)(56), (135) \rangle.$$

For  $|\mathbb{K}| \geq 5$  and  $\text{char } \mathbb{K} \neq 2$ , there are 36 monomial automorphism groups:  $21 + 8 = 29$  with stabilizing permutation groups as in the cases  $|\mathbb{K}| = 3$  and  $|\mathbb{K}| = 4$  and 7 with stabilizing permutation groups

$$\langle (123)(456), (34)(25)(16) \rangle, \langle (123)(45) \rangle, \langle (12), (34), (56) \rangle, \langle (12), (3456) \rangle, \langle (123), (456), (45) \rangle, \langle (123456), (14) \rangle, \langle (23), (123)(456), (56) \rangle.$$

**Dimension seven.** We now assume  $\dim V = 7$ . There are a total of 105 graded automorphism groups. These arise from the 15 partitions of 7, with the partition  $(1, 1, 1, 1, 1, 1, 1)$  corresponding to monomial automorphism groups. The next two tables provides a count of the number of graded automorphism groups corresponding to the fixed block decomposition given by each partition.

TABLE 7. Graded Automorphism Groups for  $\dim V = 7$  organized by partition

$\lambda$	$(1^7)$	$(1^5 2)$	$(1^3 2^2)$	$(1 2^3)$	$(1^4 3)$	$(1^2 2 3)$	$2^2 3$	$(1 3^2)$
count	53	14	8	4	9	2	2	2

  

$\lambda$	$(1^3 4)$	$(1 2 4)$	$(3 4)$	$(1^2 5)$	$(2 5)$	$(1 6)$	$(7)$
count	4	1	1	2	1	1	1

For  $|\mathbb{K}| = 3$ , there are 31 monomial automorphism groups; they have stabilizing permutation groups  $\langle e \rangle$ ,  $\langle (12) \rangle$ ,  $\langle (12)(34) \rangle$ ,  $\langle (12)(34)(56) \rangle$ ,  $\langle (12), (34) \rangle$ ,  $\langle (13)(24), (12)(34)(56) \rangle$ ,  $\langle (12), (23) \rangle$ ,  $\langle (12)(34), (56) \rangle$ ,  $\langle (12)(45), (23)(56) \rangle$ ,  $\langle (13), (1234) \rangle$ ,  $\langle (12), (34), (56) \rangle$ ,  $\langle (13), (1234)(56) \rangle$ ,  $\langle (12)(34), (14)(52) \rangle$ ,  $\langle (12), (23), (45)(67) \rangle$ ,  $\langle (12), (23), (45) \rangle$ ,  $\langle (16)(35), (12)(34)(56) \rangle$ ,  $\langle (1234567), (17)(26)(35) \rangle$ ,  $\langle (13), (1234), (56) \rangle$ ,  $\langle (12)(35), (12345), (67) \rangle$ ,  $\langle (123), (34) \rangle$ ,  $\langle (12), (34)(56)(57) \rangle$ ,  $\langle (12), (12345) \rangle$ ,  $\langle (12), (13), (45), (46) \rangle$ ,  $\langle (1234), (14), (56) \rangle$ ,  $\langle (12)(34), (1546)(23), (14) \rangle$ ,  $\langle (13), (1234), (56), (67) \rangle$ ,  $\langle (12)(34)(56), (23), (45) \rangle$ ,  $\langle (3456)(17), (12)(345) \rangle$ ,  $\langle (123)(45)(67), (124)(35) \rangle$ ,  $\langle (12345), (56) \rangle$ ,  $\langle (123456), (67) \rangle$ .

For  $|\mathbb{K}| = 4$ , there are 16 monomial automorphism groups: 4 from the  $|\mathbb{K}| = 3$  case, namely, those with stabilizing permutation groups

$$\langle e \rangle, \langle (12)(34) \rangle, \langle (12)(34)(56) \rangle, \langle (123)(456), (12)(45) \rangle,$$

and 12 additional groups with stabilizing permutation groups

$$\langle (123) \rangle, \langle (123)(456) \rangle, \langle (1234) \rangle, \langle (1234)(56) \rangle, \langle (12345) \rangle, \langle (123)(45)(67) \rangle, \langle (123456) \rangle, \langle (1234567) \rangle, \langle (123), (456) \rangle, \langle (1234)(567) \rangle, \langle (135), (123456) \rangle, \langle (1234567), (235)(476) \rangle.$$

For  $|\mathbb{K}| \geq 5$ , there are 53 monomial automorphism groups:  $31 + 12 = 43$  with stabilizing permutation groups as in the cases  $|\mathbb{K}| = 3$  and  $|\mathbb{K}| = 4$  above and 10 additional groups with types

$$\begin{aligned} &\langle (16)(25)(34), (123456) \rangle, \langle (135)(24) \rangle, \langle (12), (3456) \rangle, \langle (12345)(67) \rangle, \\ &\langle (12), (345), (67) \rangle, \langle (12), (23), (456) \rangle, \langle (123)(456), (14) \rangle, \\ &\langle (13), (1234), (567) \rangle, \langle (1234), (56), (67) \rangle, \langle (123), (456), (67) \rangle. \end{aligned}$$

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## 8. DECLARATIONS

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, DENTON, TEXAS 76203, USA

*Email address:* [ethan.jensen@unt.edu](mailto:ethan.jensen@unt.edu)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, DENTON, TEXAS 76203, USA

*Email address:* [ashepler@unt.edu](mailto:ashepler@unt.edu)