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DISTRIBUTION OF THE MAXIMA OF RANDOM TAKAGI FUNCTIONS

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Abstract. This paper concerns the maximum value and the set of maximum points of a random version of Takagi's continuous, nowhere differentiable function. Let $F(x) := \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} \varepsilon_n \phi(2^{n-1}x)$, $x \in \mathbf{R}$, where $\varepsilon_1, \varepsilon_2, \dots$ are independent, identically distributed random variables taking values in $\{-1, 1\}$, and ϕ is the "tent map" defined by $\phi(x) = 2\,\mathrm{dist}\,(x, \mathbf{Z})$. Let $p := \mathrm{P}\ (\varepsilon_1 = 1)$, $M := \max\left\{ F(x) : x \in \mathbf{R}\right\}$, and $\mathcal{M} := \left\{ x \in [0,1) : F(x) = M \right\}$. An explicit expression for M is given in terms of the sequence $\{\varepsilon_n\}$, and it is shown that the probability distribution μ of M is purely atomic if $p < \frac{1}{2}$, and is singular continuous if $p \geqq \frac{1}{2}$. In the latter case, the Hausdorff dimension and the multifractal spectrum of μ are determined. It is shown further that the set $\mathcal{M}$ is finite almost surely if $p < \frac{1}{2}$, and is topologically equivalent to a Cantor set almost surely if $p \geqq \frac{1}{2}$. The distribution of the cardinality of $\mathcal{M}$ is determined in the first case, and the almost-sure Hausdorff dimension of $\mathcal{M}$ is shown to be $(2p-1)/2p$ in the second case. The distribution of the leftmost point of $\mathcal{M}$ is also given. Finally, some of the results are extended to the more general functions $\sum a^{n-1} \varepsilon_n \phi(2^{n-1}x)$, where $0 < a < 1$.

Key words and phrases: Takagi function, singular distribution, random Cantor set, multifractal spectrum, Hausdorff dimension, set of maximum points, random walk, first passage time.

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1. Introduction and main results

Takagi's nowhere differentiable continuous function (see [10]) is given by

$$T(x) := \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} \phi(2^{n-1}x),$$

where ϕ is the "tent map" defined by $\phi(x) = 2\,\mathrm{dist}\,(x, \mathbf{Z})$. It was first shown by Kahane [5] that $\max T(x) = \frac{4}{3}$, and the set of points in $[0,1)$ where the maximum is attained is a Cantor set of Hausdorff dimension $\frac{1}{2}$. Moreover, the maximum points are precisely those numbers x whose binary expansion $x = (0.x_1x_2x_3\dots)_2$ satisfies $x_{2j-1} + x_{2j} = 1$ for every $j \in \mathbf{N}$. In particular, the leftmost maximum point in $[0,1)$ is $x_0 = \frac{1}{3} = (0.010101\dots)_2$.

The aim of this paper is to study the maxima of the random function

$$F(x) := \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} \varepsilon_n \phi(2^{n-1}x), \qquad x \in \mathbf{R},$$

where $\{\varepsilon_n\}_{n\in\mathbf{N}}$ are independent, identically distributed (i.i.d.) random variables on some probability space $(\Omega, \mathcal{A}, \mathrm{P})$ taking values in $\{-1, 1\}$. Observe that every realization of F vanishes at integer x; is 1-periodic and continuous; and, by a result of Kono [6, Theorem 2], is nowhere differentiable.

Let

$$p := \mathrm{P}\,(\varepsilon_1 = 1), \qquad q := \mathrm{P}\,(\varepsilon_1 = -1) = 1 - p.$$

The two degenerate cases $p = 1$ and $p = 0$ yield $F(x) = T(x)$ and $F(x) = -T(x)$, respectively. We assume here that $0 < p < 1$, and will be interested in the stochastic behavior of the maximum value

$$M := \max\left\{F(x) : x \in \mathbf{R}\right\},$$

the set of maximum points on one full period,

$$\mathcal{M} := \left\{x \in [0,1) : F(x) = M\right\},$$

and the leftmost member of this set,

$$X_0 := \min \mathcal{M}.$$

This introduction gives an overview of some of the main results of the paper; the proofs and related results are developed in later sections.

A natural first question is how M and X_0 are determined by the sequence $\{\varepsilon_n\}$. This can be answered in terms of the first passage times of the random walk $\{S_n\}$, where

$$S_0 := 0, \quad \text{and} \quad S_n := \varepsilon_1 + \cdots + \varepsilon_n, \quad n \in \mathbf{N}.$$

Let

$$\tau_j := \inf\{n : S_n = j\}, \quad j \in \mathbf{N},$$

where the infimum of an empty set is taken to be $+\infty$.

THEOREM 1.1. *With the convention that* $(1/2)^\infty \equiv 0$, *we have*

$$M = 2\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{\tau_{2k-1}} \quad \textit{and} \quad X_0 = \sum_{k=1}^{\infty} \left[\left(\frac{1}{2}\right)^{\tau_{2k-1}} - \left(\frac{1}{2}\right)^{\tau_{2k}}\right]. \tag{1}$$

The second expression shows the generalization of the pattern found in the leftmost maximum point of Takagi's function: in the binary expansion of X_0, the first τ_1 digits are 0's, the next $\tau_2 - \tau_1$ digits are 1's, the next $\tau_3 - \tau_2$ are 0's again, and so on.

Since τ_{2k-1} is odd, Theorem 1.1 implies that M takes values in the set

$$\mathcal{C} := \left\{ \sum_{n=0}^{\infty} \omega_n 4^{-n} : \omega_n \in \{0,1\} \ \forall\, n \right\} = \left\{ \frac{4}{3} \sum_{n=1}^{\infty} \hat{\omega}_n 4^{-n} : \hat{\omega}_n \in \{0,3\} \ \forall\, n \right\}.$$

Thus, $\mathcal{C}$ is the "middle half" Cantor set on $\left[0, \frac{4}{3}\right]$, and $\dim_H \mathcal{C} = \frac{1}{2}$.

Let μ denote the probability distribution of M. That is,

$$\mu(B) := \mathrm{P}\,(M \in B), \qquad B \in \mathrm{Borels}\,(\mathbf{R}).$$

If $B = \{x\}$, we shall write $\mu(x)$ instead of $\mu\big(\{x\}\big)$. Denote the support of μ by $\mathrm{suppt}\,(\mu)$, and recall that the Hausdorff dimension of μ is defined by

$$\dim_H \mu = \inf\big\{\dim_H E : E \in \mathrm{Borels}\,(\mathbf{R}),\ \mu(E) = 1\big\}.$$

For $j \in \mathbf{Z}_+$, define

$$p_j := \frac{1}{j+1}\binom{2j}{j} p^{j+1} q^{j-1}. \tag{2}$$

THEOREM 1.2 (The distribution of M). (a) *The support of* μ *is* $\mathcal{C}$. *In particular,* $\dim_H \mathrm{suppt}\,(\mu) = \frac{1}{2}$.

(b) *If* $p < \frac{1}{2}$, *then* μ *is purely atomic, and* μ *is specified completely by*

$$\mu(0) = 1 - \frac{p}{q} \tag{3}$$

and

$$\mu\biggl(\sum_{i=1}^{k} 4^{-n_i}\biggr) = p_{n_1} p_{n_2 - n_1} \cdots p_{n_k - n_{k-1}} \left(1 - \frac{p}{q}\right) \tag{4}$$

for any choice of $k \in \mathbf{N}$ *and integers* $0 \leqq n_1 < n_2 < \cdots < n_k$.

(c) *If* $p \geqq \frac{1}{2}$, *then* μ *is singular continuous, and*

$$\dim_H \mu = -\frac{p-q}{\log 4} \sum_{j=1}^{\infty} p_j \log p_j. \tag{5}$$

The last equation shows that the dimension of μ varies continuously with p. It is zero when $p = \frac{1}{2}$ and when $p = 1$, and is maximized around $p \approx .755$, at which point $\dim_H \mu \approx .493$. This is just slightly below the dimension of the support of μ.

In the case $p \geqq \frac{1}{2}$, more information about the measure μ is contained in its multifractal spectrum, which describes the local scaling behavior of μ. Let

$$K_\alpha := \left\{ x \in \mathbf{R} : \lim_{r \downarrow 0} \frac{\log \mu\bigl(B(x,r)\bigr)}{\log r} = \alpha \right\}, \quad \alpha \geqq 0, \tag{6}$$

and define $\delta(\alpha) := \dim_H K_\alpha$. In Section 5, it will be shown that the function $\delta(\alpha)$ has the usual concave shape on some compact interval, and vanishes outside that interval. Fig. 1 shows the three possible types of behavior. The case $p = \frac{1}{2}$ is unique; the value $p = \frac{2}{3}$ is the first in a sequence of values of p for which $\delta(\alpha)$ has a jump discontinuity; and the case $p = 0.8$ illustrates "typical" behavior. (See Proposition 5.3 below and the discussion that follows it.)

In the next theorem, $\#\mathcal{M}$ denotes the cardinality of $\mathcal{M}$.

THEOREM 1.3 (The size of $\mathcal{M}$). (a) *If* $p < \frac{1}{2}$, *then* $\mathcal{M}$ *is finite almost surely, and* $\#\mathcal{M}$ *takes values in the set* $\left\{2^l(2^m - 1),\ l \in \mathbf{Z}_+,\ m \in \mathbf{N}\right\}$, *with*

$$\mathrm{P}\bigl[\#\mathcal{M} = 2^l(2^m-1)\bigr] = \begin{cases} p^{m-1}\left(1 - \dfrac{p}{q}\right), & l = 0,\ m \in \mathbf{N}, \\[2ex] p^m \left(\dfrac{p}{q}\right)^l \left(1 - \dfrac{p}{q}\right), & l \geqq 1,\ m \in \mathbf{N}. \end{cases} \tag{7}$$

(b) *If* $p \geqq \frac{1}{2}$, *then* $\mathcal{M}$ *is a topological Cantor set with probability one, and*

$$\dim_H \mathcal{M} = \frac{p-q}{2p} \quad a.s.$$

REMARK 1.4. (i) When $p < \frac{1}{2}$, (7) shows that $\#\mathcal{M} = 2^{\zeta}(2^{\xi} - 1)$, where ζ and ξ are independent random variables, the distribution of ζ is a mixture of point mass at 0 and a geometric distribution with parameter $1 - (p/q)$, and the distribution of ξ is geometric with parameter q.

(ii) From part (a) of Theorem 1.3 we deduce that $\mathrm{E}(\#\mathcal{M})^{\alpha} < \infty$ if and only if $\alpha < \log_2(q/p)$. In particular, $\mathrm{E}(\#\mathcal{M}) < \infty$ if and only if $p < \frac{1}{3}$. The first few moments of $\#\mathcal{M}$ (when they exist) can easily be calculated. For example, $\mathrm{E}(\#\mathcal{M}) = (1-2p)/(1-p)(1-3p)$ when $p < \frac{1}{3}$.

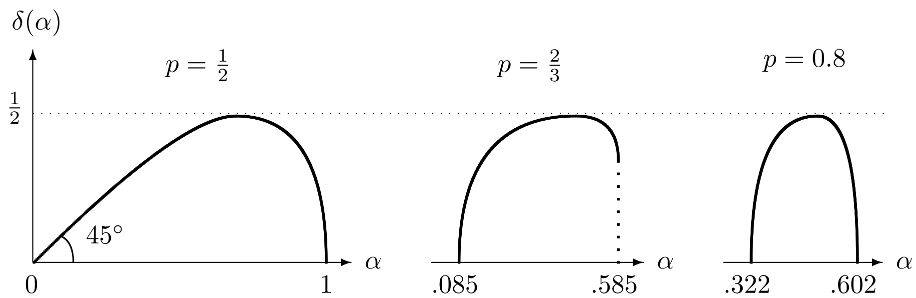

Fig. 1: The multifractal spectrum of μ: three essentially different cases

The key to proving the above results is to consider a random walk associated with the slopes of the piecewise linear finite-stage approximations of F. This random walk, which moves on the nonpositive integers and has a reflecting barrier at 0, is discussed in Section 2. In Section 3 we derive the first expression in Theorem 1.1, prove Theorem 1.2, and show that the conditional distribution of M given that $M \leqq \frac{1}{3}$ is a self-similar measure. This self-similarity is exploited in Section 4 to recursively compute the moments of M. For example, when $p = \frac{1}{2}$ we obtain that $\mathrm{E}M = 1/\sqrt{3}$. Section 5 gives the complete multifractal spectrum of μ. The proof of Theorem 1.3 is developed in Section 6. In Section 7 we derive the second expression in Theorem 1.1, and analyze the distribution of X_0. Finally, Section 8 extends some of the results to functions of the form $\sum a^{n-1}\varepsilon_n\phi(2^{n-1}x)$, where $0 < a < 1$.

2. The associated random walk

Let

$$F_0 \equiv 0, \qquad F_n(x) := \sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k-1} \varepsilon_k \phi(2^{k-1}x), \quad n \in \mathbf{N}.$$

Put $M_n := \max F_n(x)$, and define

$$\mathcal{M}_n := \left\{ x \in [0,1] : F_n(x) = M_n \right\}.$$

We can think of M_n as the temporary maximum at stage n, and of $\mathcal{M}_n$ as the set of points where the temporary maximum is attained.

Now define

$$R_n := \max_{x \in \mathcal{M}_n} \limsup_{h \to 0} \frac{F_n(x+h) - F_n(x)}{2|h|}, \quad n \in \mathbf{Z}_+. \tag{8}$$

Clearly, $R_n \leqq 0$. Furthermore, $R_0 = 0$, and the definition of F_n implies that

$$R_{n+1} = \begin{cases} -1 & \text{if } \ R_n = 0, \\ R_n + \varepsilon_{n+1} & \text{if } \ R_n < 0. \end{cases}$$

Thus, the process $\{R_n\}$ is a random walk on $\{\dots, -2, -1, 0\}$, with a reflecting barrier at 0. Note that $\{R_n\}$ moves parallel to $\{S_n\}$, except at those times when $R_n = 0$ and $\varepsilon_{n+1} = 1$.

What is the significance of the process $\{R_n\}$? Put $x_{n,j} := j/2^n$, $j = 0, 1, \dots, 2^n$. Observe that $F(x_{n,j}) = F_n(x_{n,j})$ for all n and j. Furthermore, the graph of F_n consists of line segments joined end-to-end at the points $x_{n,j}$. Some of these have an endpoint whose x-coordinate lies in $\mathcal{M}_n$. Among that more select group of line segments, the flattest have slope $\pm 2R_n$. Since $F - F_n$ is periodic with period 2^{-n}, it follows that the maximum of F can be attained on an interval $(x_{n,j-1}, x_{n,j})$ if and only if either $x_{n,j-1}$ or $x_{n,j}$ lies in $\mathcal{M}_n$, and the slope of the graph of F_n on $(x_{n,j-1}, x_{n,j})$ is equal to $\pm 2R_n$. When $R_n = 0$, the graph of F_n "levels off", and there is a possibility of upward growth at the next step, to be realized if $\varepsilon_{n+1} = 1$. Thus, intuitively, the magnitude of R_n measures how far the construction of the graph of F is, after n stages, from being able to make another push upward.

In order to record the *times* of upward growth, define $N_0 \equiv 0$, and recursively,

$$N_k := \inf \{ n > N_{k-1} : M_n > M_{n-1} \}, \quad k \in \mathbf{N}. \tag{9}$$

Thus, N_k is the kth time of increase of the sequence $\{M_n\}$. It is also the kth time the random walks $\{R_n\}$ and $\{S_n\}$ move in opposite directions. From this relationship between N_k, R_n and S_n, it can be seen that $N_k = \tau_{2k-1}$ for every k, while $\tau_{2k} = \inf\{n > N_k : R_n = 0\}$. Thus, the distribution of the first-passage times $\{\tau_j\}$ will be particularly useful. Observe that, given that τ_j is finite, $\tau_{j+1} - \tau_j$ is independent of τ_j, and has the same distribution as τ_1. From Feller [3, p. 255]:

$$\mathrm{P}\,(\tau_1 = 2j+1) = \frac{1}{j+1}\binom{2j}{j} p^{j+1} q^j = q p_j, \quad j \in \mathbf{Z}_+. \tag{10}$$

We will also need the distribution of τ_2. From (10) and the fact that $\mathrm{P}(\tau_1 = 2j+1) = q\,\mathrm{P}\,(\tau_2 = 2j)$ for $j \geqq 1$, we obtain that

$$\mathrm{P}\,(\tau_2 = 2j) = p_j, \quad j \in \mathbf{N}.$$

Note that τ_j is defective when $p < \frac{1}{2}$, with $\mathrm{P}(\tau_j < \infty) = (p/q)^j$, $j \in \mathbf{N}$.

3. The distribution of M

Since $N_k = \tau_{2k-1}$, we can write

$$M = \lim_{n\to\infty} M_n = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{N_k - 1} = 2\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{\tau_{2k-1}}, \tag{11}$$

which yields the first expression in Theorem 1.1.

We now investigate the distribution of M. Note first that if $\varepsilon_1 = 1$, then $M \in \left[1, \frac{4}{3}\right]$. If $\varepsilon_1 = -1$, we get the same graph shifted one unit down and half a unit to the left. Hence,

$$\mu(B) = (p/q)\mu(B-1), \quad B \subset \left[1, \frac{4}{3}\right]. \tag{12}$$

It suffices therefore to study the conditional distribution of M given that $\varepsilon_1 = -1$. Define the measure $\tilde{\mu}$ by

$$\tilde{\mu}(B) := \mathrm{P}\,(M \in B \mid \varepsilon_1 = -1) = \frac{\mu(B \cap I)}{\mu(I)}, \quad B \in \mathrm{Borels}\,(\mathbf{R}),$$

where $I := \left[0, \frac{1}{3}\right]$. Define the intervals

$$I_j := \left[4^{-j}, \left(\frac{4}{3}\right)4^{-j}\right], \quad j \in \mathbf{Z}_+,$$

and let $f_j(x) = 4^{-j}(x+1)$, $x \in \mathbf{R}$, $j \in \mathbf{N}$. Thus, f_j maps I bijectively to I_j.

LEMMA 3.1. (a) $\tilde{\mu}(0) = \max\{0, 1-(p/q)^2\}$.
(b) *For any Borel set* B *with* $0 \notin B$,

$$\tilde{\mu}(B) = \sum_{j=1}^{\infty} p_j \tilde{\mu}\big(f_j^{-1}(B)\big). \tag{13}$$

PROOF. Let $\widetilde{M}$ be a random variable having $\tilde{\mu}$ as its probability distribution. Since $(\tau_1 \mid \varepsilon_1 = -1) \overset{d}{=} 1 + \tau_2$, it follows from (11) that

$$\widetilde{M} \overset{d}{=} \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{\tau^{(1)}+\dots+\tau^{(k)}},$$

where $\tau^{(1)}, \tau^{(2)}, \dots$ are i.i.d. random variables having the same distribution as τ_2. Thus (enlarging the probability space if necessary), $\widetilde{M}$ has the representation

$$\widetilde{M} = \left(\frac{1}{2}\right)^{\tau} \big(1 + \widehat{M}\big),$$

where $\tau \overset{d}{=} \tau_2$, $\widehat{M} \overset{d}{=} \widetilde{M}$, and τ and $\widehat{M}$ are independent. It follows that

$$\tilde{\mu}(0) = \mathrm{P}\,(\tau_1 = \infty \mid \varepsilon_1 = -1) = \mathrm{P}\,(\tau_2 = \infty) = \max\{0, 1-(p/q)^2\},$$

and for any Borel set B with $0 \notin B$,

$$\tilde{\mu}(B) = \sum_{j=1}^{\infty} \mathrm{P}\,(\tau = 2j)\, \mathrm{P}\,\big(\widetilde{M} \in B \mid \tau = 2j\big)$$

$$= \sum_{j=1}^{\infty} p_j\, \mathrm{P}\,\big(f_j(\widehat{M}) \in B\big) = \sum_{j=1}^{\infty} p_j \tilde{\mu}\big(f_j^{-1}(B)\big). \quad \square$$

PROOF OF THEOREM 1.2. Part (a) follows immediately from (11), since τ_{2k-1} is odd for every k. Suppose $p < \frac{1}{2}$. Then with probability 1 there is an index k_0 such that $N_k = \infty$ for every $k \geqq k_0$. Hence by (11), μ gives all its mass to the dyadic rational points, so μ is purely atomic. By Lemma 3.1, $\tilde{\mu}(0) = 1 - (p/q)^2$. Now let $y = \sum_{i=1}^{k} 4^{-n_i}$, where $1 \leqq n_1 < n_2 < \dots < n_k$. Then $y \in I_{n_1}$, so that by (13),

$$\tilde{\mu}(y) = \sum_{j=1}^{\infty} p_j \tilde{\mu}\big(f_j^{-1}(y)\big) = p_{n_1} \tilde{\mu}\big(f_{n_1}^{-1}(y)\big).$$

Note that $f_{n_1}^{-1}(y) = \sum_{i=2}^{k} 4^{-(n_i-n_1)}$ if $k \geqq 2$, and $f_{n_1}^{-1}(y) = 0$ if $k = 1$. Thus, replacing y with $y' = \sum_{i=2}^{k} 4^{-(n_i-n_1)}$, k with $k' = k-1$ and n_i with $n_i' = n_{i+1} - n_1$, and iterating, we eventually obtain

$$\tilde{\mu}(y) = p_{n_1} p_{n_2-n_1} \cdots p_{n_k-n_{k-1}} \left(1 - (p/q)^2\right).$$

Since $\mu(y) = q\tilde{\mu}(y)$ and $q\left[1 - (p/q)^2\right] = 1 - (p/q)$, (4) follows under the restriction that $n_1 \geqq 1$. By (12), it holds when $n_1 = 0$ as well. Likewise, (3) holds. This proves part (b).

Next, suppose $p \geqq \frac{1}{2}$. In this case $\mathrm{P}(N_k < \infty \ \forall k) = 1$, so $\mu(y) = 0$ for any point y of the form $y = \sum_{i=1}^{k} 4^{-n_i}$. This leaves points of the form $y = \sum_{i=1}^{\infty} 4^{-n_i}$. But for each such y there are only countably many possible sequences $\{\varepsilon_n\}_{n\in\mathbf{N}}$ which yield $M = y$. Since each such sequence occurs with probability zero, it follows that μ is nonatomic. Moreover, Lemma 3.1 shows that $\tilde{\mu}$ is self-similar, since (13) now holds even if $0 \in B$. The dimension of $\tilde{\mu}$ (and hence of μ) now follows from the standard dimension formula

$$\dim_H \tilde{\mu} = \frac{\sum_{j=1}^{\infty} p_j \log p_j}{\sum_{j=1}^{\infty} p_j \log c_j}, \tag{14}$$

where

$$c_j := \left|f_j(I)\right| / |I| = |I_j|/|I| = 4^{-j}, \quad j \in \mathbf{N}.$$

(The full details are given in Section 5, as part of the multifractal analysis carried out there.) This completes the proof of (c), and of the theorem. $\square$

4. Moments of M

Using the selfsimilarity of the measure $\tilde{\mu}$, the moments of M can be calculated recursively. Let $m_k := \mathrm{E}M^k$, $k \in \mathbf{N}$.

THEOREM 4.1. *Define* $\rho_k := \sqrt{1 - 4^{1-k}pq}$, $k \in \mathbf{N}$. *Then*

$$m_k = \frac{4^k(p - q + \rho_k)}{2(4^k - 1)} \left[1 + \sum_{i=1}^{k-1} \binom{k}{i} \frac{1 - \rho_i}{2q} m_i\right], \quad k = 1, 2, \ldots$$

(the empty sum being zero). In particular,

$$\mathrm{E}M = \frac{2}{3}\left(p - q + \sqrt{1 - pq}\,\right).$$

PROOF. Let

$$\tilde{m}_k := \int_{[0,\frac{1}{3}]} x^k d\tilde{\mu}(x), \tag{15}$$

$$a_k := \sum_{j=0}^{\infty} 4^{-jk}\mu(I_j), \qquad b_k := \sum_{j=1}^{\infty} 4^{-jk}\tilde{\mu}(I_j).$$

(In evaluating $\tilde{m}_0$, we define $0^0 \equiv 1$, so that $\tilde{m}_0 = 1$.) Then, for $k \geqq 1$,

$$m_k = \sum_{j=0}^{\infty} \int_{I_j} x^k d\mu(x) = \sum_{j=0}^{\infty} 4^{-jk} \sum_{i=0}^{k} \binom{k}{i} \int_{I_j} \left(4^j x - 1\right)^i d\mu(x)$$

$$= \sum_{j=0}^{\infty} 4^{-jk}\mu(I_j) \sum_{i=0}^{k} \binom{k}{i} \int_{[0,\frac{1}{3}]} y^i d\tilde{\mu}(y) = a_k \sum_{i=0}^{k} \binom{k}{i} \tilde{m}_i.$$

Similarly, $\tilde{m}_k = b_k \sum_{i=0}^{k} \binom{k}{i}\tilde{m}_i = (b_k/a_k)m_k$, for $k \geqq 1$. Since $\tilde{m}_0 = 1$, we obtain the recursive relationship

$$m_k = a_k\left[1 + \sum_{i=1}^{k} \binom{k}{i} \frac{b_i}{a_i} m_i\right].$$

Solving explicitly for m_k gives

$$m_k = \frac{a_k}{1 - b_k}\left[1 + \sum_{i=1}^{k-1} \binom{k}{i} \frac{b_i}{a_i} m_i\right], \quad k \geqq 1. \tag{16}$$

It remains to calculate a_k and b_k. Recall that the generating function of τ_1 is given by

$$G(s) := \sum_{i=1}^{\infty} \mathrm{P}\,(\tau_1 = i)s^i = \frac{1 - \sqrt{1 - 4pqs^2}}{2qs}.$$

(See Feller [3, p. 255].) Thus,

$$a_k = \sum_{j=0}^{\infty} 4^{-jk}\,\mathrm{P}\,(\tau_1 = 2j+1) = 2^k G(2^{-k}) = \frac{4^k(1 - \rho_k)}{2q}.$$

By (15), $b_k = q^{-1}(a_k - p)$. After some algebra, we obtain

$$\frac{a_k}{1 - b_k} = \frac{qa_k}{1 - a_k} = \frac{q}{a_k^{-1} - 1} = \frac{4^k(p - q + \rho_k)}{2(4^k - 1)}$$

(where the last step uses the definition of ρ_k), and

$$\frac{b_i}{a_i} = \frac{a_i - p}{qa_i} = \frac{1 - pa_i^{-1}}{q} = \frac{1 - \rho_i}{2q}.$$

Substituting these expressions into (16) completes the proof. $\square$

5. The multifractal spectrum of μ

In this section, assume $p \geqq \frac{1}{2}$. We are interested in the local power law behavior of the measure μ. Rather than considering the sets K_α defined by (6), it is slightly more convenient to consider the sets

$$\tilde{K}_\alpha := \left\{ x \in \mathbf{R} : \lim_{r \downarrow 0} \frac{\log \tilde{\mu}\big(B(x,r)\big)}{\log r} = \alpha \right\}, \quad \alpha \geqq 0. \tag{17}$$

Of course, $\dim_H K_\alpha = \dim_H \tilde{K}_\alpha$.

Since $\tilde{\mu}$ is self-similar, we expect it to satisfy the multifractal formalism. But this is not quite clear, because the iterated function system (IFS) $\{f_1, f_2, \dots\}$ is infinite. The multifractal formalism is known to hold in the infinite case if instead of considering open balls $B(x,r)$ and letting $r \downarrow 0$, one considers the basic intervals of level k generated by the IFS and lets $k \to \infty$. (See, for instance, Riedi and Mandelbrot [8].) However, it is much more natural from a geometric point of view to define $\tilde{K}_\alpha$ by (17). This may lead to insurmountable complications in more general settings, but here we are fortunate enough that the support of $\tilde{\mu}$ is a Cantor set, and hence can be viewed alternatively as the attractor of a *finite* IFS. Since this finite IFS satisfies a separation condition that the infinite IFS lacks, this representation will make it possible to replace open balls by the basic intervals generated by the IFS. The price to pay for this benefit is that some extra care is needed to estimate the $\tilde{\mu}$ measures of these basic intervals, but this will turn out to be a good trade-off.

We first develop the necessary notation and state the main results of this section. Recall from Section 3 that $c_j = 4^{-j}$ is the contraction ratio of the map f_j. Let D denote the set of all $\lambda \in \mathbf{R}$ for which the equation

$$\sum_{j=1}^{\infty} p_j^\lambda c_j^\beta = 1 \tag{18}$$

has a solution β. For $\lambda \in D$, the solution to (18) is plainly unique; denote it by $\beta(\lambda)$. Observe that $\beta(0) = \dim_H \operatorname{suppt}(\tilde{\mu}) = \frac{1}{2}$, and $\beta(1) = 0$.

PROPOSITION 5.1. (a) *$D=(-\infty,1]$ if $p=\frac12$; $D=\mathbf{R}$ if $p>\frac12$.*

(b) *The function $\beta(\lambda)$ is strictly decreasing and strictly convex on D, and is real-analytic on the interior of D.*

As is usual in these problems, we define the Legendre transform $\hat\beta$ of β by

$$\hat\beta(\alpha):=\inf_{\lambda\in D}\left\{\beta(\lambda)+\alpha\lambda\right\},\quad \alpha\geqq 0.$$

Let

$$u_j:=\frac{\log p_j}{\log c_j}=\frac{\log p_j}{-j\log 4},\quad j\in\mathbf{N},$$

and define

$$\alpha_{\min}:=\inf_{j\in\mathbf{N}}u_j,\qquad \alpha_{\max}:=\sup_{j\in\mathbf{N}}u_j.$$

THEOREM 5.2. *The multifractal formalism holds, i.e.*

$$\dim_H\tilde K_\alpha=\begin{cases}\hat\beta(\alpha), & \text{if } \alpha\in[\alpha_{\min},\alpha_{\max}],\\ 0, & \text{otherwise.}\end{cases}$$

For $j\in\mathbf{N}$, let p_j^* denote the value of p for which $u_j=u_{j+1}$. (Thus, for example, $p_1^*=\frac23$, $p_2^*=\frac{25}{33}$, and generally, $p_j^*\uparrow 1$ as $j\to\infty$.) Put $p_0^*=\frac12$.

PROPOSITION 5.3 (evaluation of $\hat\beta$). (a) $\alpha_{\min}=\min\left\{u_1,\lim_{j\to\infty}u_j\right\}$ $=\min\left\{-\log p/\log 2,-\log(4pq)/\log 4\right\}$.

(b) $\alpha_{\max}=u_j$, *where j is such that* $p_{j-1}^*\leqq p<p_j^*$.

(c) *For* $\alpha\in(\alpha_{\min},\alpha_{\max})$,

$$\hat\beta(\alpha)=\beta\big(\lambda(\alpha)\big)+\lambda(\alpha)\alpha,$$

where $\lambda(\alpha)$ is the unique value of λ where $d\beta/d\lambda=-\alpha$.

(d) $\hat\beta(\alpha_{\min})=0$.

(e) *If $p\notin\{p_j^*\}_{j\in\mathbf{N}}$, then $\hat\beta(\alpha_{\max})=0$. If $p=p_j^*$ for $j\geqq 1$, then $\hat\beta(\alpha_{\max})$ is equal to the unique number s such that $(4^{-s})^j+(4^{-s})^{j+1}=1$.*

Observe that part (c) implies in the usual way that $d\hat\beta/d\alpha=\lambda(\alpha)$ and $d^2\hat\beta/d\alpha^2<0$ for $\alpha_{\min}<\alpha<\alpha_{\max}$. Thus, $\hat\beta$ is strictly concave on $[\alpha_{\min},\alpha_{\max}]$, and has a maximum value of $\beta(0)=\frac12$ at the point α where $\lambda(\alpha)=0$.

Notice further that if $p>\frac12$, the boundary points $\alpha_{\min}$ and $\alpha_{\max}$ correspond to $\lambda\to\infty$ and $\lambda\to-\infty$, respectively. Thus, the graph of $\hat\beta$ has vertical tangents at $\alpha_{\min}$ and $\alpha_{\max}$. On the other hand, if $p=\frac{1}{2}$, the point $\alpha_{\min}=0$ corresponds to $\lambda=1$, so the graph of $\hat{\beta}$ departs from the origin under a $45°$ angle. Finally, one checks easily that in the latter case, $\alpha_{\max}=1$. Fig. 1 illustrates the possible shapes of the graph of $\hat{\beta}$.

We now turn to the proofs. First, define the region $V\subset\mathbf{R}^2$ by

$$V:=\bigl\{(\lambda,\beta):\ \lambda\log 4pq-\beta\log 4<0\bigr\}.$$

LEMMA 5.4. *For all λ in the interior of D, we have $\bigl(\lambda,\beta(\lambda)\bigr)\in V$.*

PROOF. By Stirling's formula,

$$\binom{2j}{j}\sim\frac{4^j}{\sqrt{\pi j}},\quad \text{as}\ \ j\to\infty. \tag{19}$$

Hence, from (2),

$$p_j\asymp j^{-3/2}(4pq)^j, \tag{20}$$

where $a_j\asymp b_j$ means that $K^{-1}<a_j/b_j<K$ for all sufficiently large j and some positive constant K. It follows that

$$\alpha_{\min}\leqq\lim_{j\to\infty}u_j=\frac{-\log 4pq}{\log 4}. \tag{21}$$

If there exists j_0 such that $\alpha_{\min}=u_{j_0}$, then

$$1=\sum_{j=1}^{\infty}p_j^{\lambda}c_j^{\beta(\lambda)}>p_{j_0}^{\lambda}c_{j_0}^{\beta(\lambda)}=c_{j_0}^{\beta(\lambda)+\lambda\alpha_{\min}},$$

and consequently,

$$\beta(\lambda)+\lambda\alpha_{\min}>0. \tag{22}$$

On the other hand, if no such j_0 exists, then $\alpha_{\min}=\lim_{j\to\infty}u_j$, so there exists $j_1\in\mathbf{N}$ such that $1>p_jc_j^{-\alpha_{\min}}>\frac{1}{2}$ for all $j\geqq j_1$. But then

$$\begin{aligned}1\geqq\sum_{j=j_1}^{\infty}p_j^{\lambda}c_j^{\beta(\lambda)}&=\sum_{j=j_1}^{\infty}p_j^{\lambda}c_j^{-\lambda\alpha_{\min}}c_j^{\beta(\lambda)+\lambda\alpha_{\min}}\\&>\sum_{j=j_1}^{\infty}\min\left\{\left(\frac{1}{2}\right)^{\lambda},1\right\}c_j^{\beta(\lambda)+\lambda\alpha_{\min}},\end{aligned}$$

which again implies (22). Together, (21) and (22) yield the lemma. $\square$

PROOF OF PROPOSITION 5.1. Part (a) follows from (20) and $c_j = 4^{-j}$. To prove (b), note that the function $\Phi(\lambda, \beta) := \sum_{j=1}^{\infty} p_j^{\lambda} c_j^{\beta}$ is finite and real-analytic on V. Thus, by Lemma 5.4 and the Implicit Function Theorem, $\beta(\lambda)$ is real-analytic on the interior of D. It follows that, at all λ in the interior of D, equation (18) can be differentiated implicitly twice with respect to λ, and the differentiation and summation may be interchanged. Thus, the usual formulas for $d\beta/d\lambda$ and $d^2\beta/d\lambda^2$ apply (e.g. Falconer [2, pp. 287–288]), and the remaining assertions of part (b) of the proposition follow. $\square$

In particular, we have the "inverse relation"

$$\alpha = -\frac{d\beta}{d\lambda} = \frac{\sum_{j=1}^{\infty} p_j^{\lambda} c_j^{\beta} \log p_j}{\sum_{j=1}^{\infty} p_j^{\lambda} c_j^{\beta} \log c_j}, \tag{23}$$

where $\lambda = \lambda(\alpha)$. This relationship will be of crucial importance later.

PROOF OF PROPOSITION 5.3. Put $C := \log(p/q)$, and let

$$v_j := (\log 4) j(j+1)(u_{j+1} - u_j), \quad j \in \mathbf{N}.$$

Writing

$$u_j = (j \log 4)^{-1} \left[\log(j+1) - \log \binom{2j}{j} - j \log pq - C \right]$$

and using that

$$\log \binom{2j+2}{j+1} - \log \binom{2j}{j} = \log 2 + \log(2j+1) - \log(j+1), \tag{24}$$

we obtain, after some simplifications,

$$v_j = \log \binom{2j}{j} - \log(j+1) + C - j \big\{ \log 2 + \log(2j+1) - \log(j+2) \big\}. \tag{25}$$

Hence, using (24) once more and cancelling common terms, we arrive at

$$\begin{aligned} v_{j+1} - v_j &= (j+1) \big\{ \log(2j+1) + \log(j+3) - \log(2j+3) - \log(j+2) \big\} \\ &= (j+1) \log \left(\frac{2j^2 + 7j + 3}{2j^2 + 7j + 6} \right) < 0, \end{aligned}$$

so v_j is strictly decreasing in j. Further, from (25) and (19) it can be seen that $\lim_{j\to\infty} v_j / \log j = -\frac{3}{2}$. Thus, v_j is eventually negative. It follows that the sequence $\{u_j\}$ is either unimodal or strictly decreasing. This yields parts (a) and (b) of the proposition. Part (c) follows from the differentiability of $\beta(\lambda)$. Noting that the infimum of u_j is attained by at most one j (i.e. $j=1$), and the maximum of u_j is attained jointly by j and $j+1$ when $p=p_j^*$, parts (d) and (e) follow essentially as in Cawley and Mauldin [1]. (The slight modifications needed to deal with the infinite sums are straightforward.) $\square$

For $n \in \mathbf{N}$ and $\mathbf{i}=(i_1,\dots,i_n) \in \{1,2\}^n$, let

$$J_{\mathbf{i}} = J_{i_1,\dots,i_n} := \varphi_{i_1} \circ \varphi_{i_2} \circ \dots \circ \varphi_{i_n}(J),$$

where $J = \left[0, \frac{1}{3}\right]$, $\varphi_1(x) = x/4$ and $\varphi_2(x) = (1+x)/4$ (so φ_1 and φ_2 are the orientation preserving affine maps which map J onto the subintervals $\left[0, \frac{1}{12}\right]$ and $\left[\frac{1}{4}, \frac{1}{3}\right]$ respectively). Thus, the $J_{\mathbf{i}}$ for $\mathbf{i} \in \{1,2\}^n$ are the basic intervals at stage n of the construction of the Cantor set $\tilde{\mathcal{C}} := \mathcal{C} \cap \left[0, \frac{1}{3}\right]$. Now, for $x \in \tilde{\mathcal{C}}$ and $n \in \mathbf{N}$, let $J_n(x)$ denote the stage-n interval $J_{i_1,\dots,i_n}$ that contains x. Since there is a gap between J_1 and J_2, the following useful lemma holds (see Falconer [2, Lemma 17.5]).

LEMMA 5.5. *For any (positive and finite) Borel measure m on $\mathbf{R}$ and all $x \in \tilde{\mathcal{C}}$,*

$$\lim_{r \downarrow 0} \frac{\log m\big(B(x,r)\big)}{\log r} = \lim_{n \to \infty} \frac{\log m\big(J_n(x)\big)}{\log \big|J_n(x)\big|}, \tag{26}$$

with either both limits existing or neither.

Note that it is not at all clear whether (26) holds when the intervals $J_n(x)$ are replaced with the nth stage "basic intervals" of the infinite IFS $\{f_1, f_2, \dots\}$. The reason is that there is no minimum gap between the intervals $I_i = f_i(J)$, $i \in \mathbf{N}$. Hence, we are forced to work with the basic intervals of the Cantor set $\tilde{\mathcal{C}}$ instead.

In what follows, fix $\alpha \in (\alpha_{\min}, \alpha_{\max})$, let $\lambda = \lambda(\alpha)$, and let $\beta = \beta(\lambda)$. Define a probability measure ν on $\tilde{\mathcal{C}}$ by the equation

$$\nu = \sum_{j=1}^{\infty} \bar{p}_j (\nu \circ f_j^{-1}),$$

where

$$\bar{p}_j := p_j^{\lambda} c_j^{\beta}, \quad j \in \mathbf{N}. \tag{27}$$

For $x \in \tilde{\mathcal{C}}$, write

$$x = \sum_{i=1}^{\infty} \omega_i 4^{-i}, \quad \omega_i \in \{0,1\} \quad \text{for} \quad i \in \mathbf{N}.$$

For $k \in \mathbf{N}$ and x as above, let $n(k)$ denote the number of 1's among $\omega_1, \dots, \omega_k$. For $n \in \mathbf{N}$, put $s_n := \inf\left\{k : n(k) = n\right\}$. Let $t_n := s_n - s_{n-1}$, where $s_0 \equiv 0$. Then $\{t_n\}$ is a sequence of independent, identically distributed random variables under the measure ν, with $\nu(t_1 = j) = \bar{p}_j$, $j \in \mathbf{N}$. Hence, by the strong law of large numbers (SLLN),

$$\lim_{n\to\infty} \frac{s_n}{n} = \mathrm{E}^{(\nu)}\, s_1 = \sum_{j=1}^{\infty} j\bar{p}_j = -(\log 4)^{-1} \sum_{j=1}^{\infty} \bar{p}_j \log c_j, \quad \nu\text{-a.s.} \tag{28}$$

Furthermore, the relationship between $\left\{n(k)\right\}$ and $\{s_n\}$ implies that

$$\frac{n(k)}{k} \to \frac{1}{\mathrm{E}^{(\nu)}\, s_1} \qquad \text{and} \qquad \frac{s_{n(k)}}{k} \to 1, \quad \nu\text{-a.s.} \tag{29}$$

Next, define

$$r_k := \sum_{j=k+1}^{\infty} p_j, \qquad \bar{r}_k := \sum_{j=k+1}^{\infty} \bar{p}_j,$$

and let $l(k) := s_{n(k)}$. Observe that

$$\log \tilde{\mu}\big(J_k(x)\big) = \sum_{i=1}^{n(k)} \log p_{t_i} + \log r_{k-l(k)}, \tag{30}$$

$$\log \nu\big(J_k(x)\big) = \sum_{i=1}^{n(k)} \log \bar{p}_{t_i} + \log \bar{r}_{k-l(k)}. \tag{31}$$

LEMMA 5.6. *$\lim_{k\to\infty} \frac{\log r_{k-l(k)}}{k} = \lim_{k\to\infty} \frac{\log \bar{r}_{k-l(k)}}{k} = 0$, ν-a.s.*

PROOF. If $p > \frac{1}{2}$, we have $r_k \asymp p_k \asymp k^{-3/2}(4pq)^k$, by (20). If $p = \frac{1}{2}$, then $r_k \asymp k^{-1/2}$ (by comparison with the integral $\int_k^\infty x^{-3/2}dx$). Similarly, $\bar{r}_k \asymp \bar{p}_k \asymp k^{-3\lambda/2}(4pq)^{\lambda k} 4^{-\beta k}$. The lemma now follows easily from the fact that $\big(k - l(k)\big)/k = 1 - s_{n(k)}/k \to 0$, ν-a.s. $\square$

All the preparatory work has been done. Theorem 5.2 will now follow from Proposition 5.8 below and the following lemma, which has been called "Volume Lemma", "Frostman's Lemma" or "Billingsley's Lemma"; see Przytycki and Urbański [7, Section 7.6].

LEMMA 5.7. *Let m be a positive finite Borel measure on* $\mathbf{R}$.

(a) *If $K \subset \mathbf{R}$ is a Borel set satisfying $m(K) > 0$ and*

$$\liminf_{r \downarrow 0} \frac{\log m\big(B(x,r)\big)}{\log r} = \delta \quad \textit{for all } x \in K, \tag{32}$$

then $\dim_H K = \delta$.

(b) *If the lower limit in* (32) *is equal to δ for m-a.e. x, then* $\dim_H m = \delta$.

PROPOSITION 5.8. (a) $\nu(\tilde{K}_\alpha) = 1$.

(b) *For every $x \in \tilde{K}_\alpha$,* $\lim_{r \downarrow 0} \log \nu\big(B(x,r)\big) / \log r = \hat{\beta}(\alpha)$.

PROOF. Note that $\big|J_k(x)\big| = \frac{1}{3} \cdot 4^{-k}$, so $\log\big|J_k(x)\big| \sim -k \log 4$ as $k \to \infty$. Thus, by (30) and Lemma 5.6, we have for ν-almost every x,

$$\lim_{k \to \infty} \frac{\log \tilde{\mu}\big(J_k(x)\big)}{\log\big|J_k(x)\big|} = \lim_{k \to \infty} \frac{\sum_{i=1}^{n(k)} \log p_{t_i} + \log r_{k-l(k)}}{-k \log 4}$$

$$= -(\log 4)^{-1} \lim_{k \to \infty} \frac{n(k)}{k} \left(\frac{1}{n(k)} \sum_{i=1}^{n(k)} \log p_{t_i} \right) = \frac{\sum_{j=1}^{\infty} \bar{p}_j \log p_j}{\sum_{j=1}^{\infty} \bar{p}_j \log c_j} = \alpha.$$

The third equality follows from (28), (29), and SLLN applied to the random variables $\{\log p_{t_i}\}_{i \in \mathbf{N}}$, which are i.i.d. under ν. The last equality follows from (23) and (27). Thus, part (a) follows using Lemma 5.5.

To prove part (b), start by writing

$$\begin{aligned} &\log \nu\big(J_k(x)\big) - \lambda \log \tilde{\mu}\big(J_k(x)\big) \\ &= \log \bar{r}_{k-l(k)} - \lambda \log r_{k-l(k)} - (\beta \log 4) l(k), \end{aligned} \tag{33}$$

where we have used (27), (30), (31) and the fact that

$$\sum_{i=1}^{n(k)} \log c_{t_i}^{\beta} = -\beta \sum_{i=1}^{n(k)} t_i \log 4 = -(\beta \log 4) l(k).$$

Assume $p > \frac{1}{2}$; the case $p = \frac{1}{2}$ is very similar. Choose a number $K > 1$ such that for each $j \in \mathbf{N}$,

$$K^{-1} < r_j \cdot j^{3/2} (4pq)^{-j} < K \tag{34}$$

and

$$K^{-1} < \bar{r}_j \cdot j^{3\lambda/2}(4pq)^{-\lambda j} 4^{\beta j} < K \tag{35}$$

(see the proof of Lemma 5.6). Put $C := \big(|\lambda| + 1\big) \log K$. Taking logarithms in (34) and (35), we obtain

$$|\log \bar{r}_j - \lambda \log r_j + \beta j \log 4| < C, \quad \text{for all } j \in \mathbf{N}.$$

Now substitute $j = k - l(k)$ in this last inequality and conclude that

$$\lim_{k\to\infty} \frac{\log \bar{r}_{k-l(k)} - \lambda \log r_{k-l(k)} - (\beta \log 4) l(k)}{-k \log 4} = \beta.$$

Thus, (33) and the definition of $\tilde{K}_\alpha$ imply

$$\lim_{k\to\infty} \frac{\log \nu\big(J_k(x)\big)}{\log \big|J_k(x)\big|} = \lambda\alpha + \beta = \hat{\beta}(\alpha), \quad \text{for all } x \in \tilde{K}_\alpha.$$

Another application of Lemma 5.5 completes the proof. □

Proof of Theorem 5.2. For $\alpha \in (\alpha_{\min}, \alpha_{\max})$, the dimension of $\tilde{K}_\alpha$ follows from Proposition 5.8 and Lemma 5.7. For all other values of α the dimension of $\tilde{K}_\alpha$ follows exactly as in the finite case; see Cawley and Mauldin [1]. □

Finally, Proposition 5.8 and Lemma 5.7 imply that $\dim_H \nu = \hat{\beta}(\alpha)$. Taking $\lambda = 1$ we obtain $\dim_H \tilde{\mu} = \hat{\beta}(\alpha) = \alpha$. Thus, (14) follows from (23).

6. The size of $\mathcal{M}$

In this section we prove Theorem 1.3. Suppose first that $p < \frac{1}{2}$.

Proof of Theorem 1.3, Part (a). Recall the definitions and interpretation of the random walks $\{S_n\}$ and $\{R_n\}$ from Sections 1 and 2, respectively. For $m \in \mathbf{Z}_+$, let A_m be the event that the walk $\{R_n\}$ returns to zero exactly m times, and (if $m \geqq 1$) each of the returns is followed by a down-step of the walk $\{S_n\}$ on the next step. For $k \in \mathbf{N}$, let B_k be the event that $\{R_n\}$ returns to zero at least k times, and the kth return is the first that is followed immediately by an up-step of $\{S_n\}$. Note that after each visit to 0 (including the one at time 0), the random walk $\{R_n\}$ automatically falls back to -1, and the probability that it will return to 0 again is $p/q < 1$. Thus,

$$\mathrm{P}(A_m) = \left(\frac{p}{q} \cdot q\right)^m \left(1 - \frac{p}{q}\right) = p^m \left(1 - \frac{p}{q}\right), \quad m \in \mathbf{Z}_+,$$

and

$$\mathrm{P}(B_k) = \left(\frac{p}{q}\cdot q\right)^{k-1}\cdot\frac{p}{q}\cdot p = \frac{p^{k+1}}{q}, \quad k \in \mathbf{N}.$$

Taken together, the A_m's and B_k's form a collection of disjoint sets whose union is $\{R_n = 0$ *finitely often*$\}$, a set of probability 1.

Let $m \in \mathbf{Z}_+$, and suppose that the event A_m occurs. We may assume without loss of generality that $\varepsilon_1 = 1$. (If $\varepsilon_1 = -1$, we get the same graph shifted one unit down and half a unit to the left, and by the periodicity of F this does not affect the *size* of $\mathcal{M}$.) If the mth and last return to 0 happens at time n, then $\mathcal{M}_n$ consists of 2^m dyadic closed intervals of length 2^{-n}, two of which are joined together at the point $x = \frac{1}{2}$. (Fig. 2 illustrates this for the case $m = 2$, $n = 4$.) Since $\varepsilon_{n+1} = -1$ and the walk $\{R_n\}$ does not return to 0 again, the maximum points of F are precisely the endpoints of these intervals. Hence $\#\mathcal{M} = 2^{m+1} - 1$.

Next, let $k \in \mathbf{N}$, and suppose that B_k occurs. If the kth return to zero happens at time n, then, as above, $\mathcal{M}_n$ consists of 2^k dyadic intervals of length 2^{-n}. But now, since $\varepsilon_{n+1} = 1$, the graph of F_{n+1} is topped by 2^k "tents" over these 2^k intervals. Thus $\#\mathcal{M} = 2^k N$, where N is a random variable having the same distribution as $\#\mathcal{M}$. In particular, $\#\mathcal{M}$ is even.

From the considerations in the last two paragraphs, it follows that $\#\mathcal{M}$ takes values in the set $\left\{2^l(2^m - 1) : l \in \mathbf{Z}_+,\, m \in \mathbf{N}\right\}$, with

$$\mathrm{P}(\#\mathcal{M} = 2^m - 1) = \mathrm{P}\left(A_{m-1}\right) = p^{m-1}\left(1 - \frac{p}{q}\right), \tag{36}$$

and for $l \geqq 1$,

$$\begin{aligned} \mathrm{P}\left[\#\mathcal{M} = 2^l(2^m - 1)\right] &= \sum_{k=1}^{l} \mathrm{P}\left(B_k\right) \mathrm{P}\left[\#\mathcal{M} = 2^l(2^m - 1) \mid B_k\right] \\ &= \sum_{k=1}^{l} q^{-1}p^{k+1}\, \mathrm{P}\left[\#\mathcal{M} = 2^{l-k}(2^m - 1)\right]. \end{aligned} \tag{37}$$

Fix m, and let $a_j := \mathrm{P}\left[\#\mathcal{M} = 2^j(2^m - 1)\right]$, $j \in \mathbf{Z}_+$. Then (37) gives

$$a_l = \frac{p}{q}\sum_{k=1}^{l} p^k a_{l-k} = \frac{p}{q}\sum_{j=0}^{l-1} p^{l-j} a_j.$$

Taking $l = 1$ gives $a_1 = q^{-1}p^2 a_0$, while for $l \geqq 2$, comparison of the above summation with the analogous one for a_{l-1} gives

$$a_l = p\left(\frac{p}{q}\sum_{j=0}^{l-2} p^{l-j-1} a_j + \frac{p}{q}a_{l-1}\right) = p\left(1 + \frac{p}{q}\right)a_{l-1} = \frac{p}{q}a_{l-1}.$$

Combining these results with (36) yields the formulas in (7). $\square$

Fig. 2 illustrates the above ideas. In the figure, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $\varepsilon_3 = -1$, and $\varepsilon_4 = 1$. The graph of F_3 shows that it is possible to have $2^2 - 1 = 3$ maximum points. The graph of F_4 has four "plateaus" over the intervals $\left[\frac{1}{4}, \frac{5}{16}\right]$, $\left[\frac{7}{16}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{9}{16}\right]$ and $\left[\frac{11}{16}, \frac{3}{4}\right]$. There are now two possibilities for the next stage: if $\varepsilon_5 = 1$, the plateaus are replaced with small "tents" and $\#\mathcal{M}_5 = 4$ (after which the construction "starts over" at a smaller scale). But if $\varepsilon_5 = -1$, the plateaus are replaced by "dents" and $\#\mathcal{M}_5 = 7 = 2^3 - 1$.

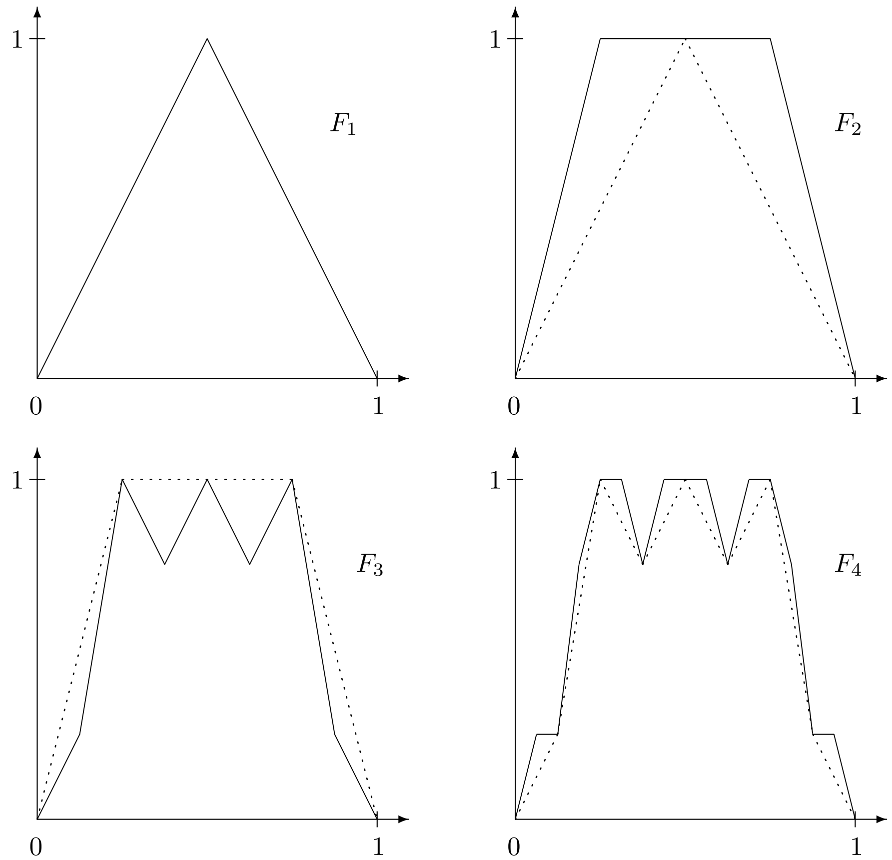

Fig. 2. The first four stages in the construction of F, for a realization of $\{\varepsilon_n\}$ with $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $\varepsilon_3 = -1$, and $\varepsilon_4 = 1$

We now proceed to the case $p \geqq \frac{1}{2}$. Let $d := (p - q)/2p$; we aim to show that $\dim_H \mathcal{M} = d$ almost surely.

For $k \in \mathbf{N}$, let T_k be the kth time n for which the graph of F_n has one or more "plateaus". More precisely, define $T_0 \equiv 0$, and recursively,

$$\begin{aligned} T_k &:= \inf\{n > T_{k-1} : \mathcal{M}_n \text{ contains an interval}\} \\ &= \inf\{n > T_{k-1} : R_n = 0\}, \qquad k \geqq 1. \end{aligned}$$

Recall the definition of generalized Hausdorff measure: for an increasing function $h : (0, \varepsilon) \to (0, \infty)$, a Borel set $E \subset \mathbf{R}$ and a number δ with $0 < \delta < \varepsilon$,

$$\mathcal{H}_\delta^h(E) := \inf \sum_{i=1}^{\infty} h\big(|U_i|\big),$$

where the infimum is over all covers $\{U_i\}_{i=1}^{\infty}$ of E by open intervals of length $|U_i| < \delta$ (such covers are called δ-*covers*); and $\mathcal{H}^h(E) := \lim_{\delta \downarrow 0} \mathcal{H}_\delta^h(E)$.

LEMMA 6.1. *For any increasing function $h : (0, \varepsilon) \to (0, \infty)$,*

$$\frac{1}{4} \liminf_{k\to\infty} 2^k h\big(2^{-T_k}\big) \leqq \mathcal{H}^h(\mathcal{M}) \leqq \liminf_{k\to\infty} 2^k h\big(2^{-T_k}\big) \quad a.s.$$

PROOF. Consider a realization of $\{\varepsilon_n\}_{n\in\mathbf{N}}$ such that $\tau_j < \infty$ for every $j \in \mathbf{N}$. This event has probability 1 when $p \geqq \frac{1}{2}$. It implies that $T_k < \infty$ for all k, and $M > M_n$ for every n. Of course $T_k \uparrow \infty$.

For $k \in \mathbf{N}$, let $E_k := \mathcal{M}_{T_k}$, and notice that E_k is the union of 2^k nonoverlapping dyadic intervals of length 2^{-T_k}. Call these the *intervals of level* k, and label them, from left to right, by $I_{k,j}$ $(j = 1, \dots, 2^k)$. Clearly, the eventual maximum of F can occur only on one of these (and then on all) 2^k intervals. Essentially, at time T_k the construction of F starts over on a smaller scale, with 2^k identical copies side by side constructed over the subintervals $I_{k,j}$. It follows that $\mathcal{M} = \bigcap_{k=1}^{\infty} E_k$, $E_{k+1} \subset E_k$, and $E_{k+1} \cap I_{k,j} = I_{k+1,2j-1} \cup I_{k+1,2j}$. (In fact, the intervals $I_{k+1,2j-1}$ and $I_{k+1,2j}$ lie adjacent to the center of $I_{k,j}$ if $\varepsilon_{T_k+1} = 1$, and are adjacent to the left and right endpoints of $I_{k,j}$ if $\varepsilon_{T_k+1} = -1$.)

(i) *The upper bound.* Given $\delta > 0$, there exists $k \in \mathbf{N}$ such that $2^{-T_k} < \delta$, and the 2^k intervals $I_{k,j}$ $(j = 1, \dots, 2^k)$ cover $\mathcal{M}$. Thus,

$$\mathcal{H}^h(\mathcal{M}) \leqq \liminf_{k\to\infty} 2^k h\big(2^{-T_k}\big).$$

(ii) *The lower bound.* Fix $0 < \delta < \varepsilon$, and let $\mathcal{U}$ be a δ-cover of $\mathcal{M}$ by open intervals. Since $\mathcal{M}$ is compact, we may assume that $\mathcal{U}$ is finite. Next, for any interval U there is a smallest integer n such that U contains a closed dyadic interval of length 2^{-n}; moreover, U intersects at most four dyadic intervals of this length. Since h is increasing, it follows that there is a finite δ-cover $\mathcal{J}$ of $\mathcal{M}$ by closed dyadic intervals such that

$$\sum_{U\in\mathcal{U}} h\big(|U|\big) \geqq \frac{1}{4}\sum_{J\in\mathcal{J}} h\big(|J|\big). \tag{38}$$

By removing intervals in the collection $\mathcal{J}$ that do not intersect $\mathcal{M}$ and by replacing each interval J in $\mathcal{J}$ with the smallest dyadic closed subinterval I of J such that $(J\backslash I)\cap\mathcal{M}=\emptyset$, we may assume that $\mathcal{J}$ is a subcollection of the intervals $I_{k,j}$ ($k\in\mathbf{N}$, $j=1,\dots,2^k$), with no intervals being repeated and no interval in $\mathcal{J}$ being a subset of another.

Let l be the largest integer such that $\mathcal{J}$ includes an interval of level l, and let k be the next largest, so $k<l$. If $I_{l,j}\in\mathcal{J}$, then $\mathcal{J}$ cannot include the kth level interval, say $I_{k,i}$, of which $I_{l,j}$ is a subset; hence all 2^{l-k} of the lth level intervals contained in $I_{k,i}$ are in $\mathcal{J}$. It follows that if $2^k h\big(2^{-T_k}\big)\leqq 2^l h\big(2^{-T_l}\big)$, then upon replacing all intervals of level l in $\mathcal{J}$ with the intervals of level k that contain them, the sum $\sum_{J\in\mathcal{J}} h\big(|J|\big)$ will not increase. If on the other hand, $2^k h\big(2^{-T_k}\big) > 2^l h\big(2^{-T_l}\big)$, then upon replacing each kth level interval in $\mathcal{J}$ with the collection of its descendants at level l, the sum $\sum_{J\in\mathcal{J}} h\big(|J|\big)$ will decrease. Either way, we can replace $\mathcal{J}$ with a collection $\mathcal{J}'$ that still covers $\mathcal{M}$, but that includes intervals of strictly fewer levels than $\mathcal{J}$, while $\sum_{J\in\mathcal{J}'} h\big(|J|\big)\leqq\sum_{J\in\mathcal{J}} h\big(|J|\big)$. After finitely many iterations of this procedure, we will be left with a collection $\mathcal{J}^*$ whose intervals cover $\mathcal{M}$ and are all of the same level, say l. But then $\mathcal{J}^*=\{I_{l,j}: j=1,\dots,2^l\}$, and therefore,

$$\sum_{J\in\mathcal{J}} h\big(|J|\big) \geqq \sum_{J\in\mathcal{J}^*} h\big(|J|\big) = 2^l h\big(2^{-T_l}\big) \geqq \inf_{k:\,2^{-T_k}<\delta} 2^k h\big(2^{-T_k}\big). \tag{39}$$

It follows from (38) and (39) that

$$\sum_{i\in\mathcal{I}} h\big(|U_i|\big) \geqq \frac{1}{4}\inf_{k:\,2^{-T_k}<\delta} 2^k h\big(2^{-T_k}\big)$$

and hence,

$$\mathcal{H}^h(\mathcal{M}) \geqq \frac{1}{4}\lim_{\delta\downarrow 0}\inf_{k:\,2^{-T_k}<\delta} 2^k h\big(2^{-T_k}\big) = \frac{1}{4}\liminf_{k\to\infty} 2^k h\big(2^{-T_k}\big). \qquad \square$$

Proof of Theorem 1.3, Part (b). Note that T_k permits the natural decomposition $T_k=\sigma_1+\dots+\sigma_k$, where $\sigma_1,\sigma_2,\dots$ are i.i.d. with the same distribution as T_1. Clearly, $\sigma_1 \overset{d}{=} 1+\tau_1$. Hence $\mathrm{E}\sigma_1 = 1+\mathrm{E}\,\tau_1 = 1+1/(p-q)$ $= 2p/(p-q) = 1/d$ if $p > \frac{1}{2}$, and $\mathrm{E}\sigma_1 = \infty$ if $p = \frac{1}{2}$. By the SLLN it follows that almost surely,

$$\lim_{k\to\infty} k^{-1}T_k = \begin{cases} d^{-1}, & \text{if } \ p > \dfrac{1}{2} \\ \infty, & \text{if } \ p = \dfrac{1}{2}. \end{cases} \tag{40}$$

Consider the case $p > \frac{1}{2}$. If $s < d$, choose $\eta > 0$ such that $s + \eta < d$. Then, using (40),

$$1 - sT_k/k > 1 - (d-\eta)T_k/k \to \eta/d,$$

so

$$\liminf_{k\to\infty} 2^k\big(2^{-T_k}\big)^s = \liminf_{k\to\infty}\big(2^{1-sT_k/k}\big)^k \geqq \liminf_{k\to\infty}\big(2^{\eta/2d}\big)^k = \infty.$$

Similarly, if $s > d$, we can choose $\eta > 0$ such that $s - \eta > d$. Then

$$1 - sT_k/k < 1 - (d+\eta)T_k/k \to -\eta/d,$$

so

$$\liminf_{k\to\infty} 2^k\big(2^{-T_k}\big)^s = \liminf_{k\to\infty}\big(2^{1-sT_k/k}\big)^k \leqq \liminf_{k\to\infty}\big(2^{-\eta/2d}\big)^k = 0.$$

Hence, by Lemma 6.1 applied to $h(t) = t^s$, $\dim_H \mathcal{M} = d$ almost surely.

Finally, if $p = \frac{1}{2}$, then $T_k/k \to \infty$ a.s., and so $\liminf_{k\to\infty}\big(2^{1-sT_k/k}\big)^k = 0$ for any positive s. Thus, $\dim_H \mathcal{M} = 0$ almost surely. $\square$

Remark 6.2. For the case $p > \frac{1}{2}$, the law of the iterated logarithm (LIL) makes a more precise statement possible. Let $A := \mathrm{E}\,\sigma_1 = 2p/(p-q)$, and $B := \operatorname{Var}\sigma_1$. It is easily verified that $B = 4pq/(p-q)^3$. By the LIL,

$$\limsup_{n\to\infty} \frac{T_n - An}{\sqrt{2Bn\log\log Bn}} = 1 \quad \text{a.s.} \tag{41}$$

Since $T_n/n \to d^{-1}$ a.s., this suggests defining the gauge functions

$$h_\alpha(t) := t^d \exp\big\{\alpha p^{-1}(q\log 2)^{1/2}\big(\log(1/t)\log\log\log(1/t)\big)^{1/2}\big\},$$

$0 < t < e^{-e}$. After some careful analysis, it follows from (40), (41) and Lemma 6.1 that almost surely,

$$\mathcal{H}^{h_\alpha}(\mathcal{M}) = \begin{cases} 0, & \text{if } \ \alpha < 1 \\ \infty, & \text{if } \ \alpha > 1. \end{cases}$$

(The somewhat messy details are omitted.) In particular, taking $\alpha = 0$ shows that $\mathcal{H}^d(\mathcal{M}) = 0$ almost surely.

7. The leftmost point of $\mathcal{M}$

We consider now the random variable $X_0 = \min \mathcal{M}$. Our first goal is to verify the second expression in Theorem 1.1. To this end, define

$$X_0^{(n)} := \min \mathcal{M}_n, \quad n \in \mathbf{N}.$$

Thus, $X_0^{(n)}$ is the leftmost maximum point of F_n in $[0,1]$. It is clear that $X_0 = \lim_{n\to\infty} X_0^{(n)}$.

Recall that $\tau_1 = \inf\{n : M_n > 0\}$. The portion of the graph of F_{τ_1} above the x-axis consists of one or more "tents" of equal height. The leftmost such "tent" spans the interval $[0, 2^{-\tau_1+1}]$ of the x-axis; more precisely,

$$F_{\tau_1}(x) = \left(\frac{1}{2}\right)^{\tau_1 - 1} \phi(2^{\tau_1-1}x), \quad 0 \leqq x \leqq 2^{-\tau_1+1}.$$

It follows that $X_0^{(\tau_1)} = 2^{-\tau_1}$. (Recall the convention $2^{-\infty} = 0$.)

Next, observe that the first time $m > \tau_1$ for which $R_m = 0$ is τ_2, and the graph of F_{τ_2} has a plateau above the interval $\left[2^{-\tau_1} - 2^{-\tau_2}, 2^{-\tau_1}\right]$. Hence

$$X_0^{(\tau_2)} = 2^{-\tau_1} - 2^{-\tau_2}.$$

After time τ_2, the construction starts anew (probabilistically speaking), with the interval $\left[2^{-\tau_1} - 2^{-\tau_2}, 2^{-\tau_1}\right]$ taking the place of $[0,1]$. We therefore obtain

$$X_0^{(\tau_{2k})} = \sum_{j=1}^{k} \left[\left(\frac{1}{2}\right)^{\tau_{2j-1}} - \left(\frac{1}{2}\right)^{\tau_{2j}}\right], \quad \text{for } k \in \mathbf{N},$$

and letting $k \to \infty$ yields the second expression in Theorem 1.1.

Let ν denote the probability distribution of X_0. Define the intervals

$$J := \left[0, \frac{1}{2}\right], \qquad J_k := [4^{-k}, 2 \cdot 4^{-k}], \quad k \in \mathbf{N},$$

and let Ψ_k be the orientation *reversing* affine map of J onto J_k; that is, $\Psi_k(x) = 2 \cdot 4^{-k}(1-x)$, $k \in \mathbf{N}$. Let $\pi_k := \mathrm{P}\ (\tau_1 = 2k-1)$, $k \in \mathbf{N}$.

THEOREM 7.1. (a) *The measure ν satisfies*

$$\nu = \left(1 - \frac{p}{q}\right)^{+} \delta_{\{0\}} + \sum_{k=1}^{\infty} \pi_k (\nu \circ \Psi_k^{-1}). \tag{42}$$

(b) $\dim_H \operatorname{suppt}(\nu) = 1 - \log\left(\sqrt{5}-1\right)/\log 2 \approx .6942$.

(c) *If* $p < \frac{1}{2}$, *then* ν *is purely atomic. The atoms of* ν *are* 0 *and the points* $x_{i_1\cdots i_r} := \Psi_{i_1} \circ \ldots \circ \Psi_{i_r}(0)$, *where* $r \in \mathbf{N}$ *and* $i_1, \ldots, i_r \in \mathbf{N}$. *Moreover,* $\nu(0) = 1 - (p/q)$, *and* $\nu(x_{i_1\cdots i_r}) = \pi_{i_1} \cdots \pi_{i_r}\bigl(1-(p/q)\bigr)$.

(d) *If* $p \geqq \frac{1}{2}$, *then* ν *is singular continuous, and*

$$\dim_H \nu = 2q \dim_H \mu + (p-q)\left(\frac{-p\log p - q\log q}{\log 2}\right). \tag{43}$$

Notice that $2q + (p-q) = 1$. Hence (43) expresses the dimension of ν as a weighted average of the dimension of μ and the "entropy dimension" $(-p\log p - q\log q)/\log 2$. The appearance of $\dim_H \mu$ in this formula is not surprising if one compares the expressions in (1). However, the author does not know a simple intuitive explanation for the exact form of (43).

Proof of Theorem 7.1. In view of (1) and since $\tau_1, \tau_2-\tau_1, \tau_3-\tau_2, \ldots$ are i.i.d., we can write

$$X_0 = \left(\frac{1}{2}\right)^{\tau_1}(1-\widehat{X}_0),$$

where $\widehat{X}_0 \stackrel{d}{=} X_0$, and τ_1 and $\widehat{X}_0$ are independent. It follows that

$$\nu(0) = \mathrm{P}(X_0 = 0) = \mathrm{P}(\tau_1 = \infty) = (1-p/q)^+,$$

and for any Borel set B with $0 \notin B$,

$$\nu(B) = \sum_{k=1}^{\infty} \mathrm{P}(\tau_1 = 2k-1)\,\mathrm{P}(X_0 \in B \mid \tau_1 = 2k-1)$$

$$= \sum_{k=1}^{\infty} \pi_k\,\mathrm{P}\left(\Psi_k(\widehat{X}_0) \in B\right) = \sum_{k=1}^{\infty} \pi_k \nu\bigl(\Psi_k^{-1}(B)\bigr).$$

This proves part (a).

To obtain the dimension of the support of ν, observe that $\operatorname{suppt}(\nu)$ is the attractor of the IFS $\{\Phi_1, \Phi_2\}$, where

$$\Phi_1(x) := x/4, \qquad \Phi_2(x) := (1-x)/2.$$

(Φ_1 and Φ_2 map J onto the subintervals $\left[0, \frac{1}{8}\right]$ and $\left[\frac{1}{4}, \frac{1}{2}\right]$ respectively, with Φ_2 reversing the orientation.) To understand this, it is sufficient to notice that $\left[\frac{1}{4}, \frac{1}{2}\right] = J_1$, $\left[0, \frac{1}{8}\right]$ is the smallest closed interval containing the union

$E := \bigcup_{k=2}^{\infty} J_k \cup \{0\}$, and $4E = E \cup J_1$. Since $\{\Phi_1, \Phi_2\}$ satisfies the open set condition, $\dim_H \operatorname{suppt}(\nu)$ is the number s such that $\left(\frac{1}{4}\right)^s + \left(\frac{1}{2}\right)^s = 1$. Solving for s yields statement (b).

Suppose now that $p < \frac{1}{2}$. Then with probability one there exists $j \in \mathbf{N}$ such that $\tau_j = \infty$, so X_0 is a dyadic rational point. Hence ν is purely atomic. The probabilities $\nu(x_{i_1 \cdots i_r})$ follow easily from (42).

Finally, suppose $p \geqq \frac{1}{2}$. Then, by a reasoning analogous to that in the proof of Theorem 1.2, ν is continuous. By part (a), ν is self-similar; and since the intervals J_k are pairwise disjoint, Theorem 4.1 of Hanus et al. [4] yields

$$\dim_H \nu = \frac{\sum_{k=1}^{\infty} \pi_k \log \pi_k}{\sum_{k=1}^{\infty} \pi_k \log t_k}, \tag{44}$$

where $t_k := |J_k|/|J| = 2(4^{-k})$, and the expression is interpreted as 0 when the denominator is $-\infty$. Note that $\pi_k = q p_{k-1}$ for $k \geqq 2$, and $\pi_1 = p$. Thus,

$$\begin{aligned} \sum_{k=1}^{\infty} \pi_k \log \pi_k &= p \log p + \sum_{k=2}^{\infty} (q p_{k-1}) \log (q p_{k-1}) \\ &= p \log p + q \log q + q \sum_{j=1}^{\infty} p_j \log p_j. \end{aligned} \tag{45}$$

Finally, since $\log t_k = -(\log 2)(2k-1)$, we have

$$\sum_{k=1}^{\infty} \pi_k \log t_k = -(\log 2)\,\mathrm{E}\,\tau_1 = \begin{cases} -\log 2/(p-q) & \text{if } \ p > \dfrac{1}{2} \\ -\infty & \text{if } \ p = \dfrac{1}{2}. \end{cases} \tag{46}$$

The dimension formula (43) now follows from (5), (44), (45) and (46). $\square$

8. The functions $\sum a^{n-1} \varepsilon_n \phi(2^{n-1}x)$

It seems natural to try and generalize the work of the preceding sections to the random functions

$$F(x) := \sum_{n=1}^{\infty} a^{n-1} \varepsilon_n \phi(2^{n-1}x),$$

where a is a constant with $0 < a < 1$. Note that by Theorem 2 of Kono [6], F is absolutely continuous if $a < \frac{1}{2}$, and F is nowhere differentiable if $a > \frac{1}{2}$.

Put $M := \max F(x)$, let $F_n(x) := \sum_{k=1}^n a^{k-1}\varepsilon_k \phi(2^{k-1}x)$, and redefine $\mathcal{M}$, μ, M_n, and $\mathcal{M}_n$ accordingly. While this case is in general more difficult to analyze, some interesting things can nonetheless be said. We first state the main results, and then develop the proofs. In what follows, let $\lambda := 2a$.

THEOREM 8.1. *Suppose $a < \frac{1}{2}$. Then μ is purely atomic, and $\mathcal{M}$ is finite almost surely. If in fact $a \leqq \frac{1}{4}$, then* $\mathrm{P}(M = 1) = p = 1 - \mathrm{P}\ (M = 0)$, *and* $\#\mathcal{M} = 1$.

For some particular values of a in $\left(\frac{1}{4}, \frac{1}{2}\right)$, the distributions of M and $\#\mathcal{M}$ can be obtained completely.

THEOREM 8.2. *Suppose λ satisfies the equation $\sum_{i=1}^r \lambda^i = 1$, where $r \geqq 2$. (Note that this automatically implies $\frac{1}{4} < a < \frac{1}{2}$.) Write $z := a^{r+1}$. Then*

(a) *The support of μ is the Cantor set*

$$\mathcal{C}_r := \left\{ \sum_{n=0}^{\infty} \omega_n z^n : \omega_n \in \{0,1\} \text{ for all } n \right\},$$

and $\dim_H \mathcal{C}_r = -\log 2/\log z$.

(b) $\mathrm{E}M = p/(1 - p^r z)$.

(c) *The measure μ is specified completely by*

$$\mu(0) = q(1 - p^r)/(1 - p^r q),$$

and

$$\mu\left(\sum_{i=1}^{k} z^{n_i}\right) = (p^r q)^{n_k} \left(\frac{p}{q}\right)^k \mu(0)$$

for any choice of $k \in \mathbf{N}$ and integers $0 \leqq n_1 < n_2 < \cdots < n_k$.

(d) *The distribution of $\#\mathcal{M}$ is given by*

$$\mathrm{P}\left[\#\mathcal{M} = 2^l(2^m - 1)\right] = \begin{cases} (p^r q)^{m-1}(1 - p^r), & l = 0,\ m \in \mathbf{N}, \\ p^{rl+1}(p^r q)^{m-1}(1 - p^r), & l \geqq 1,\ m \in \mathbf{N}. \end{cases}$$

(e) $\mathrm{E}(\#\mathcal{M})^\alpha < \infty$ *if and only if $p < 2^{-\alpha/r}$. In particular, if $p < 2^{-1/r}$, then*

$$\mathrm{E}(\#\mathcal{M}) = \frac{1 - p^r}{(1 - 2p^r)(1 - p^r q)}.$$

Observe that λ in Theorem 8.2 is a root of a polynomial with coefficients ± 1 only. In general, let $\mathcal{P}(-1,1)$ denote the set of all polynomials $f : \mathbf{R} \to \mathbf{R}$ of the form $f(x) = \sum_{i=0}^m \eta_i x^i$, where $m \in \mathbf{Z}_+$ and $\eta_0, \ldots, \eta_m \in \{-1, 1\}$.

THEOREM 8.3. *Suppose $a > \frac{1}{2}$. Then:*
(a) *With probability one, $M_n < M$ for every n.*
(b) *If $f(\lambda) \neq 0$ for every $f \in \mathcal{P}(-1,1)$, then $\#\mathcal{M} = 2$ almost surely.*
(c) *If $1 + \lambda + \cdots + \lambda^{r-1} = \lambda^r$ for some $r \geqq 2$, then, with $\rho := q^{r-1}p$,*

$$\mathrm{P}(\#\mathcal{M} = 2^l) = \rho^{l-1}(1-\rho), \quad l \in \mathbf{N}.$$

Part (a) seems to suggest that μ should be continuous when $a > \frac{1}{2}$. However, it seems difficult to prove that μ does not give positive mass to possible proper limit points of the sequence $\{M_n\}$. If this can be ruled out, then a natural next question is for which values of a in $\left(\frac{1}{2}, 1\right)$ the measure μ is in fact *absolutely* continuous. Some useful techniques for attacking this question might come from the study of Bernoulli convolutions, where absolute continuity is a central problem, and polynomials with coefficients ± 1 play an important role. (See Solomyak [9].)

We now turn to the proofs. Define R_n as in (8). Then $R_0 = 0$ and $R_1 = -1$ as before, but for $n \geqq 1$ we have:

$$\varepsilon_{n+1} = 1 \Rightarrow \begin{cases} R_{n+1} = -|R_n + \lambda^n|, \\ M_{n+1} = M_n + 2^{-n}(R_n + \lambda^n)^+, \end{cases} \tag{47}$$

$$\varepsilon_{n+1} = -1 \Rightarrow \begin{cases} R_{n+1} = R_n - \lambda^n, \\ M_{n+1} = M_n. \end{cases} \tag{48}$$

It follows that for every n,

$$R_n \in \left\{ f(\lambda) : f \in \mathcal{P}(-1,1),\ \deg(f) = n-1 \right\}. \tag{49}$$

Call $n \in \mathbf{N}$ a *critical moment* if $R_n > -\lambda^n$, and a *renewal moment* if $R_n = 0$. Observe from (47) and (48) that $M_{n+1} > M_n$ if and only if n is a critical moment and $\varepsilon_{n+1} = 1$. Every renewal moment is plainly a critical moment.

LEMMA 8.4. *The conclusions in the following statements hold almost surely.*
(a) *If $a \leqq \frac{1}{4}$, no critical moments occur.*
(b) *If $\frac{1}{4} < a < \frac{1}{2}$, at most finitely many critical moments occur.*
(c) *If $a > \frac{1}{2}$, infinitely many critical moments occur.*

PROOF. (a) If $a \leqq \frac{1}{4}$, then $\lambda \leqq \frac{1}{2}$, and so for each $n \in \mathbf{N}$,

$$R_n \leqq -1 + \lambda + \lambda^2 + \cdots + \lambda^{n-1} \leqq -\lambda^n.$$

(b) Suppose $\frac{1}{4} < a < \frac{1}{2}$, so $\frac{1}{2} < \lambda < 1$. Let n_0 be an integer such that $\lambda^{n_0} < \frac{1}{2}$. Note that

$$-\big(1 + \lambda + \dots + \lambda^{n_0-1}\big) + \big(\lambda^{n_0} + \lambda^{n_0+1} + \dots\big) = \frac{2\lambda^{n_0} - 1}{1 - \lambda} < 0.$$

Thus, if for some k, $\varepsilon_{k+1} = \varepsilon_{k+2} = \dots = \varepsilon_{k+n_0} = -1$, then we have for all $n \geqq k + n_0$,

$$R_n + \lambda^n \leqq R_k + \lambda^k\big[-\big(1 + \lambda + \dots + \lambda^{n_0-1}\big) + \lambda^{n_0} + \dots + \lambda^{n-k}\big] < 0,$$

since $R_k \leqq 0$. Hence no critical moments occur after time $k + n_0$. Since $p < 1$, such a k exists with probability one.

(c) Finally, let $a > \frac{1}{2}$, so that $\lambda > 1$. Let n_0 be an integer such that $\lambda^{n_0} \geqq 2$. Then for every k,

$$\begin{aligned} &R_k + \lambda^k\big(1 + \lambda + \dots + \lambda^{n_0-1}\big) \\ &\geqq -\big(1 + \lambda + \dots + \lambda^{k-1}\big) + \lambda^k\big(1 + \lambda + \dots + \lambda^{n_0-1}\big) \\ &= (\lambda - 1)^{-1}\big[\lambda^k(\lambda^{n_0} - 1) - (\lambda^k - 1)\big] = (\lambda - 1)^{-1}\big[\lambda^k(\lambda^{n_0} - 2) + 1\big] > 0. \end{aligned}$$

So if $\varepsilon_{k+1} = \varepsilon_{k+2} = \dots = \varepsilon_{k+n_0} = 1$, there will be an index $j < n_0$ such that

$$R_{k+j} + \lambda^{k+j} = R_k + \lambda^k(1 + \lambda + \dots + \lambda^j) > 0,$$

and then $k + j$ is a critical moment. But such "positive runs" of length n_0 happen infinitely often with probability one. $\square$

Proof of Theorem 8.1. Suppose $a < \frac{1}{2}$. By Lemma 8.4, at most finitely many critical moments occur. Hence with probability one, $M = M_n$ for some $n \in \mathbf{N}$. Since M_n has only finitely many possible values, it follows that μ gives all its mass to a countable set of points. Thus, μ is purely atomic. That $\mathcal{M}$ is finite a.s. follows since in particular, $\mathrm{P}(R_n = 0$ infinitely often$) = 0$.

If $a \leqq \frac{1}{4}$, then no critical moments occur, so $M = M_1 \in \{0, 1\}$, and F has a unique maximum point in $[0, 1)$. $\square$

Lemma 8.5. *Let λ be the unique positive root of the polynomial $f(x) = 1 - \sum_{i=1}^r x^i$, where $r \geqq 2$. Then for any polynomial $g \in \mathcal{P}(-1, 1)$ with constant term 1, we have either $g(\lambda) > 0$, or else $g - f$ is divisible by x^{r+1}. In particular, any polynomial $g \in \mathcal{P}(-1, 1)$ having λ as a root must have f as a factor.*

Proof. Let n denote the degree of g. If $n \leqq r$, then

$$g(\lambda) \geqq 1 - \lambda - \lambda^2 - \cdots - \lambda^n \geqq 1 - \lambda - \lambda^2 - \cdots - \lambda^r = 0,$$

and $g(\lambda) = 0$ if and only if $g = f$. If $n > r$, then either $g(x) = f(x) + x^{r+1}h(x)$ for some $h \in \mathcal{P}(-1,1)$, or else g and f differ in their coefficient of x^j for some $j \leqq r$. But in the latter case,

$$g(\lambda) \geqq f(\lambda) + 2\lambda^j - \sum_{i=r+1}^{\infty} \lambda^i = 2\lambda^j - \frac{\lambda^{r+1}}{1-\lambda}$$

$$\geqq 2\lambda^r - \frac{\lambda^{r+1}}{1-\lambda} = \lambda^r \left(\frac{2-3\lambda}{1-\lambda} \right) > 0,$$

since $f(\lambda) = 0$ implies that $\lambda < \frac{2}{3}$. $\square$

Proof of Theorem 8.2. Let λ satisfy the given equation. We claim that the only critical moments are renewal moments, and these are necessarily integer multiples of $r+1$. Clearly, $r+1$ *can* be a renewal moment, since if $\varepsilon_2 = \cdots = \varepsilon_{r+1} = 1$, then $R_{r+1} = -1 + \lambda + \lambda^2 + \cdots + \lambda^r = 0$.

On the other hand, if $\varepsilon_{k+1} = -1$ for some $1 \leqq k \leqq r$, then for each $n \in \mathbf{N}$, $R_n + \lambda^n = -g(\lambda)$ for a polynomial g satisfying the hypothesis of Lemma 8.5. Moreover, with f defined as in Lemma 8.5, g differs from f in its coefficient of x^k, so that $g - f$ is not divisible by x^{r+1}. Hence $R_n + \lambda^n = -g(\lambda) < 0$.

It follows that the first critical moment, if any, must occur at time $n = r + 1$, and this will be a renewal moment. After this, the process $\{R_n\}$ essentially starts over, albeit it with an extra factor λ^{r+1}. The claim follows.

Observe next that if $k(r+1)$ is a renewal moment and $\varepsilon_{k(r+1)+1} = 1$, then the increment in the temporary maximum is $a^{k(r+1)} = z^k$. Thus, the set of possible values of M is the set $\mathcal{C}_r$ given in part (a). The calculation of $\dim_H \mathcal{C}_r$ is routine.

For part (b), write $M = \sum_{k=0}^{\infty} \omega_k z^k$, where $\omega_k \in \{0,1\}$ for all k. Note that for $k \geqq 1$, $\omega_k = 1$ if and only if $\varepsilon_i = 1$ for all $j(r+1) + 2 \leqq i \leqq (j+1)(r+1)$, $j = 0, \ldots, k-1$, and $\varepsilon_{k(r+1)+1} = 1$. (The values of $\varepsilon_{j(r+1)+1}$ for $j = 0, \ldots, k-1$ do not affect ω_k.) Clearly, $\omega_0 = 1$ if and only if $\varepsilon_1 = 1$. Thus,

$$\mathrm{P}(\omega_k = 1) = p^{kr+1}, \quad k = 0, 1, 2, \ldots,$$

and

$$\mathrm{E}M = \sum_{k=0}^{\infty} \mathrm{E}(\omega_k) z^k = \sum_{k=0}^{\infty} p^{kr+1} z^k = \frac{p}{1 - p^r z}.$$

To prove part (c), define N_k ($k \in \mathbf{N}$) as in (9). Since $N_1 - 1$ (if finite) is a renewal moment, N_1 takes values in $\{k(r+1)+1,\, k \in \mathbf{Z}_+\} \cup \{\infty\}$. Observe that $N_1 = k(r+1)+1$ exactly when the conditions for $\omega_k = 1$ above are met, and in addition, $\varepsilon_{j(r+1)+1} = -1$ for $j = 0, 1, \dots, k-1$. Thus,

$$\mathrm{P}\left[N_1 = k(r+1)+1\right] = (qp^r)^k p, \quad k \in \mathbf{Z}_+,$$

and it follows that

$$\mu(0) = \mathrm{P}\,(N_1 = \infty) = 1 - \sum_{k=0}^{\infty} (qp^r)^k p = 1 - \frac{p}{1-p^r q} = \frac{q(1-p^r)}{1-p^r q}.$$

Observe next that, given that N_k is finite, $N_{k+1} - N_k$ is independent of N_k, and

$$\begin{aligned}\mathrm{P}\left[N_{k+1} - N_k = j(r+1) \mid N_k < \infty\right] &= (p^r q)^{j-1} p^{r+1} \\ &= q^{-1}\,\mathrm{P}\left[N_1 = j(r+1)+1\right], \quad j \in \mathbf{N}.\end{aligned}$$

Thus, if $y = \sum_{i=1}^k z^{n_i}$ where $k \in \mathbf{N}$ and $0 \leqq n_1 < n_2 < \dots < n_k$, we obtain

$$\mu(y) = \mathrm{P}\left[\bigcap_{i=1}^{k} \{N_i = n_i(r+1)+1\} \cap \{N_{k+1} = \infty\}\right]$$

$$= \mathrm{P}\left[N_1 = n_1(r+1)+1, N_2 - N_1 = (n_2 - n_1)(r+1), \dots,\right.$$

$$\left. N_k - N_{k-1} = (n_k - n_{k-1})(r+1), N_{k+1} - N_k = \infty\right]$$

$$= (qp^r)^{n_1} p \left[\prod_{i=1}^{k-1} \left\{ (p^r q)^{n_{i+1} - n_i - 1} p^{r+1} \right\}\right] q^{-1}\mu(0) = (p^r q)^{n_k} \left(\frac{p}{q}\right)^k \mu(0).$$

The proof of part (d) is similar to the proof of Theorem 1.3 (a). The only significant difference is that now each return of R_n from -1 to 0 happens with probability p^r (instead of p/q), since r consecutive up-steps are needed.

Finally, part (e) follows from (d) by routine calculation. □

Proof of Theorem 8.3. Suppose $a > \frac{1}{2}$. By part (c) of Lemma 8.4, there are infinitely many critical moments. But given that n is a critical moment, $M_{n+1} > M_n$ with the fixed probability $p > 0$. Hence $M_{n+1} > M_n$ infinitely often with probability one, proving statement (a).

For (b), note first that by the symmetry of F on $[0, 1]$ there will be *at least* two maximum points as soon as two increments in M_n occur; and this happens with probability 1 by the foregoing argument. In order for there to be *more* than two maximum points, there must be some $n \geqq 1$ for which $R_n = 0$. By (49) this can happen only if λ is a root of a polynomial in $\mathcal{P}(-1,1)$.

The proof of (c) is based on Lemma 8.5, and uses reciprocal polynomials. For a polynomial f of degree n, define a polynomial $\hat{f}$ by $\hat{f}(x) = x^n f(1/x)$. Note that the mapping $f \mapsto \hat{f}$ is multiplicative, maps $\mathcal{P}(-1,1)$ bijectively into itself, and is its own inverse; that is, $\hat{\hat{f}} = f$.

Now let λ satisfy the hypothesis of part (c). Then $f(\lambda) = 0$, where $f(x) = 1 + x + \dots + x^{r-1} - x^r$. Since $\hat{f}(x) = x^r + x^{r-1} + \dots + x - 1$, it follows that $\hat{\lambda} := 1/\lambda$ satisfies the hypothesis of Lemma 8.5; that is, $\sum_{i=1}^r \hat{\lambda}^i = 1$.

Suppose g is another polynomial in $\mathcal{P}(-1,1)$ having λ as a root. Then $\hat{g}$ has $\hat{\lambda}$ as a root, and repeated application of Lemma 8.5 yields that $\hat{g}$ must be of the form

$$\hat{g}(x) = \hat{f}(x)\big(\pm 1 \pm x^{r+1} \pm \dots \pm x^{m(r+1)}\big) =: \hat{f}(x)h(x),$$

for some $m \in \mathbf{Z}_+$. But then $g(x) = f(x)\hat{h}(x)$, so that g must be of the form

$$g(x) = f(x)\big(\pm 1 \pm x^{r+1} \pm \dots \pm x^{m(r+1)}\big).$$

Thus, $\deg(g)$ is one less than a multiple of $r+1$, and (49) implies that the renewal moments can only be multiples of $r+1$. By part (a), $\#\mathcal{M} = 2^l$ if and only if there are exactly $l-1$ renewal moments. In order for this to happen, the sequence $\varepsilon_1 =$ arbitrary, $\varepsilon_2 = -1, \dots, \varepsilon_r = -1$, $\varepsilon_{r+1} = 1$ must be repeated exactly $l-1$ times. Since this sequence has probability $\rho = q^{r-1}p$, part (c) of the theorem follows. $\square$

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