Prophet regions for discounted, uniformly bounded random variables

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Abstract

Let $X_1, X_2, \ldots$ be any sequence of $[0,1]$-valued random variables. A complete comparison is made between the expected maximum $\mathrm{E}(\max_{j\leq n} Y_j)$ and the stop rule supremum $\sup_t \mathrm{E}\, Y_t$ for two types of discounted sequences: (i) $Y_j = b_j X_j$, where $\{b_j\}$ is a nonincreasing sequence of positive numbers with $b_1 = 1$; and (ii) $Y_j = B_1 \cdots B_{j-1} X_j$, where $B_1, B_2, \ldots$ are independent $[0,1]$-valued random variables that are independent of the X_j, having a common mean β. For instance, it is shown that the set of points $\{(x,y) : x = \sup_t \mathrm{E}\, Y_t$ and $y = \mathrm{E}(\max_{j\leq n} Y_j)$ for some sequence $X_1, \ldots, X_n$ and $Y_j = b_j X_j\}$ is precisely the convex closure of the union of the sets $\{(b_j x, b_j y) : (x,y) \in C_j\}$, $j = 1, \ldots, n$, where $C_j = \{(x,y) : 0 \leq x \leq 1, x \leq y \leq x[1 + (j-1)(1 - x^{1/(j-1)})]\}$ is the prophet region for undiscounted random variables given by Hill and Kertz (*Trans. Amer. Math. Soc.* **278**, 197-207 (1983)). As a special case, it is shown that the maximum possible difference $\mathrm{E}(\max_{j\leq n} \beta^{j-1} X_j) - \sup_t \mathrm{E}(\beta^{t-1} X_t)$ is attained by independent random variables when $\beta \leq 27/32$, but by a martingale-like sequence when $\beta > 27/32$. Prophet regions for infinite sequences are also given.

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1 Introduction

Let $X_1, X_2, \ldots$ be any sequence of $[0,1]$-valued random variables. This paper gives comparisons of the expected maximum and stop rule supremum of sequences $\{Y_j\}$ that are derived from the X_j by a form of discounting. Specifically, the sequences under consideration are

$$Y_j = b_j X_j, \qquad j = 1, 2, \ldots, \tag{1}$$

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where $\{b_j\}$ is a nonincreasing sequence of positive numbers with $b_1 = 1$; and

$$Y_j = B_1 \cdots B_{j-1} X_j, \qquad j = 1, 2, \ldots, \tag{2}$$

where $B_1, B_2, \ldots$ are independent $[0,1]$-valued random variables that are independent of the X_j, and have a common mean β with $0 < \beta < 1$. The models (1) and (2) will be referred to as *deterministic discounting* and *random discounting*, respectively.

The results of this paper extend earlier work by Hill and Kertz [8], whose results can be summarized as follows. Let $X_1, X_2, \ldots$ be any sequence of $[0,1]$-valued random variables, let $V(X_1, X_2, \ldots)$ denote the supremum of $\mathrm{E}X_t$ over all stop rules t, and for $n \in \mathbb{N}$, let $V(X_1, \ldots, X_n)$ denote the supremum of $\mathrm{E}X_t$ over all stop rules t such that $t \leq n$ a.s. Then

(a) For every $n \geq 2$, the set of all ordered pairs $\{(x,y) : x = V(X_1, \ldots, X_n)$ and $y = \mathrm{E}(\max_{1 \leq j \leq n} X_j)$ for some sequence $X_1, \ldots, X_n\}$ is precisely the set

$$C_n := \{(x,y) : 0 \leq x \leq 1,\ x \leq y \leq x[1 + (n-1)(1 - x^{1/(n-1)})]\}, \tag{3}$$

and

(b) the set of all ordered pairs $\{(x,y) : x = V(X_1, X_2, \ldots)$ and $y = \mathrm{E}(\sup_{n \geq 1} X_n)$ for some sequence $X_1, X_2, \ldots\}$ is precisely the set $C := \bigcup_{n=1}^{\infty} C_n = \{(x,y) : 0 < x < 1,\ x \leq y < x - x \ln x\} \cup \{(0,0),(1,1)\}$.

From these results, Hill and Kertz derived the following sharp ratio and difference inequalities comparing the optimal expected return and expected supremum: for every $n \in \mathbb{N}$,

$$\mathrm{E}\left(\max_{j \leq n} X_j\right) - V(X_1, \ldots, X_n) \leq (1 - 1/n)^n, \tag{4}$$

and

$$\mathrm{E}\left(\max_{j \leq n} X_j\right) < nV(X_1, \ldots, X_n) \qquad \text{if} \quad \mathrm{P}\left(\max_{j \leq n} X_j > 0\right) > 0; \tag{5}$$

and the analogous sharp inequality for infinite sequences is

$$\mathrm{E}\left(\sup_{n \geq 1} X_n\right) - V(X_1, X_2, \ldots) < e^{-1}. \tag{6}$$

Inequalities such as (4)-(6) have been called *prophet inequalities*, because they compare the optimal expected return of a prophet, or player with complete foresight, to that of a gambler, who has no foresight. Similarly, the sets of ordered pairs in (a) and (b) have been called *prophet regions*. A vast literature on prophet/gambler comparisons exists, with the article by Krengel and Sucheston [11] being considered the stem paper. The notion of a prophet region was introduced by Hill [7], who gave a complete comparison of $\mathrm{E}(\max_{j \leq n} X_j)$ and $\sup_t \mathrm{E}\, X_t$ for sequences of independent random variables $X_1, \ldots, X_n$ taking values in $[0,1]$. Boshuizen [3] generalized this result to independent random variables with a discount factor. Jones [10] and Samual-Cahn [14] gave prophet inequalities for independent, respectively i.i.d. random variables with a cost for observation. More recently, Saint-Mont [12, 13] has given further extensions for the independent case, involving simultaneous costs and discountings. A useful overview of the earlier prophet inequalities, which also describes commonly used proof techniques, is the survey paper by Hill and Kertz [9]. The more recent book by Harten *et al.* [6] contains proofs and extremal distributions for most of the known inequalities, as well as game-theoretic versions of prophet inequalities.

This paper extends the results of Hill and Kertz [8] to two cases where the random variables $X_1, X_2, \ldots$ are discounted. The first case is the deterministic model (1). It will be shown that for a finite sequence $X_1, \ldots, X_n$, the prophet region is the convex closure of the union of the sets $\{(b_j x, b_j y) : (x, y) \in C_j\}$, $j = 1, \ldots, n$. For $n = 2$ this set agrees with Boshuizen's prophet region for the independent case. From the prophet region, sharp difference and ratio inequalities generalizing (4) and (5) follow easily. The special case of geometric discounting, where $b_j = \beta^{j-1}$, is examined in detail. A surprising result is that the maximum possible difference $\mathrm{E}(\max_{j \le n} \beta^{j-1} X_j) - \sup_t \mathrm{E}(\beta^{t-1} X_t)$ is attained by independent random variables when $\beta \le 27/32$, but by a martingale-like sequence when $\beta > 27/32$.

For infinite sequences, the prophet region is the convex closure of the union of the sets $\{(b_j x, b_j y) : (x, y) \in C_j\}$, $j = 1, 2, \ldots$, provided that $\lim_{j \to \infty} b_j = 0$. Otherwise, the prophet region is that convex closure, with the possible exception of part of its upper boundary. It seems difficult in general to determine how much of the upper boundary is contained in the prophet region. On the other hand, it will be shown in Theorem 3.8 that the best-possible difference prophet inequality for infinite sequences is attained when the sequence $\{b_j (1 - 1/j)^j\}$ has a maximum, but holds with strict inequality otherwise.

The second type of discounting considered in this paper is the random discount model (2). In this case, the prophet region for finite sequences is best characterized by its upper boundary function, which matches the upper boundary function of C_n on $0 \le x \le \beta^{n-1}$, and is piecewise linear on $\beta^{n-1} \le x \le 1$. From this result, the prophet region for infinite sequences follows by taking limits as $n \to \infty$.

Prophet inequalities for a similar model with random discounting in which the random variables $X_1, X_2, \ldots$ are assumed to be independent are given in Allaart [2]. Indeed, the prophet regions given there are proper subsets of the regions in the present article. However, the methods of proof required to deal with the independent case are quite different from the supermartingale arguments used in this paper.

This paper is organized as follows. Section 2 introduces the necessary notation. Section 3 deals with the case of deterministic discount factors, and is divided into three subsections: the prophet region for the finite-horizon case is given in Subsection 3.1; the proof is given in Subsection 3.2; and Subsection 3.3 develops analogous results for infinite sequences. Section 4 contains an application to optimal stopping with a random time horizon: the optimal expected return of a player who knows the number of available observations ahead of time is compared to that of a player who does not. Finally, Section 5 gives prophet regions for the case of random discounting.

2 Notation and definitions

Throughout this paper, the following notation will be used. For subsets A of the underlying probability space, I_A denotes the indicator function of A. For real numbers x and y, $x \vee y$ denotes the maximum of x and y, and $x^+ := \max\{x, 0\}$. For a function $f : [0,1] \to \mathbb{R}$, $\hat{f}$ denotes the smallest concave function dominating f. For a collection of random variables $\{X_s : s \in S\}$, $\operatorname{ess\,sup}\{X_s : s \in S\}$ denotes the essential supremum of the collection. (See pages 8-9 of [4] for the definition and existence of essential supremum.)

For any filtration $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \dots)$, let $T_{\mathcal{F}}$ be the set of all stopping rules t such that $\{t = j\} \in \mathcal{F}_j$ for all $j \in \mathbb{N}$. If $\mathbf{Y} = (Y_1, Y_2, \dots)$ is a sequence of random variables and $\mathcal{F} = \{\mathcal{F}_j\}$ is a filtration, define the *value* $V_{\mathcal{F}}(\mathbf{Y})$ of $\mathbf{Y}$ with respect to $\mathcal{F}$ by $V_{\mathcal{F}}(\mathbf{Y}) = \sup\{\mathrm{E}\, Y_t : t \in T_{\mathcal{F}}\}$. Likewise, for $n \geq 1$, let $V_{\mathcal{F}}(Y_1, \dots, Y_n) = \sup\{\mathrm{E}\, Y_t : t \in T_{\mathcal{F}}, t \leq n\}$. The *conditional value* with respect to $\mathcal{F}$ of $Y_m, Y_{m+1}, \dots$ given $\mathcal{F}_j$ is $V_{\mathcal{F}}(Y_m, Y_{m+1}, \dots | \mathcal{F}_j) = \operatorname{ess\,sup}\{\mathrm{E}(Y_t|\mathcal{F}_j) : t \in T_{\mathcal{F}}, t \geq m\}$, and $V_{\mathcal{F}}(Y_m, Y_{m+1}, \dots, Y_n|\mathcal{F}_j) = \operatorname{ess\,sup}\{\mathrm{E}(Y_t|\mathcal{F}_j) : t \in T_{\mathcal{F}}, m \leq t \leq n\}$. Where the subscript $\mathcal{F}$ is omitted, $\mathcal{F}$ is understood to be the natural filtration of the sequence under consideration.

3 Deterministic discounting

In this section, let $b_1 = 1 \geq b_2 \geq b_3 \geq \dots$ be fixed positive numbers, and let $Y_j = b_j X_j$ for $j \in \mathbb{N}$. Define the filtration $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \dots)$ by $\mathcal{F}_j = \sigma(X_1, \dots, X_j)$, $j \in \mathbb{N}$.

3.1 The finite-horizon case

For $n \geq 1$, define the function

$$\Psi_n(x) = x + \max_{1 \leq j \leq n-1} jx \left(1 - (x/b_{j+1})^{1/j}\right)^+, \qquad 0 \leq x \leq 1.$$

Theorem 3.1 *The set of all points* $\{(x,y) : x = V_{\mathcal{F}}(Y_1, \dots, Y_n)$ and $y = \mathrm{E}(Y_1 \vee \cdots \vee Y_n)$ where $Y_j = b_j X_j$ for all j and some sequence $X_1, \dots, X_n \in [0,1]\}$ *is precisely the set*

$$\Gamma_n = \{(x,y) : 0 \leq x \leq 1,\, x \leq y \leq \widehat{\Psi}_n(x)\}.$$

Stated differently, Theorem 3.1 says that the prophet region is exactly the convex closure of the union of the sets $\{(b_j x, b_j y) : (x,y) \in C_j\}$, $j = 1, 2, \dots, n$, where $C_1 = \{(x,x) : 0 \leq x \leq 1\}$, and for $n \geq 2$, C_n is the set (3). Thus, the theorem can be viewed as a generalization of the prophet region of Hill and Kertz [8].

For comparison, Boshuizen's results for the independent case are given below.

Boshuizen's results [3, Theorem 2.5]. Let $n \geq 2$ and $0 < \beta \leq 1$.

(i) Let $P_n = \{(x,y) : x = V(X_1, \beta X_2, \dots, \beta^{n-1} X_n), y = \mathrm{E}(\max_{1 \leq i \leq n} \beta^{i-1} X_i)$, *for some* sequence of independent $[0,1]$-valued r.v.'s $X_1, \dots, X_n\}$. *Then* $P_n = \{(x,y) : 0 \leq x \leq 1, x \leq y \leq \Phi(x)\}$, *where*

$$\Phi(x) = \begin{cases} 2x - x^2/\beta, & \text{if } \; 0 \leq x \leq 1 - \sqrt{1-\beta} \\ 1 - g(\beta)(1-x), & \text{if } \; 1 - \sqrt{1-\beta} \leq x \leq 1, \end{cases} \tag{7}$$

where $g(\beta) = 2(\sqrt{1-\beta} - (1-\beta))/\beta$.

(ii) For every sequence $X_1, \dots, X_n$ *of* $[0,1]$*-valued r.v.'s,*

$$\mathrm{E}\left(\max_{1 \le i \le n} \beta^{i-1} X_i\right) - V(X_1, \beta X_2, \dots, \beta^{n-1} X_n) \le \beta/4, \tag{8}$$

and this bound is attained.

As Boshuizen points out at the end of his article, the factors $\beta, \beta^2, \dots$ may be replaced by any nonincreasing sequence $(1 \ge) b_2 \ge b_3 \ge \dots$. The results of (i) and (ii) above then hold with b_2 in place of β. As will be seen shortly, the prophet region for the dependent case coincides with the region given in (i) when $n = 2$, but is strictly larger when $n > 2$.

Remark 3.2 Clearly, the relationship (1) determines a one-to-one correspondence between sequences $\{X_j\}$ taking values in $[0,1]$ and sequences $\{Y_j\}$ such that for each j, Y_j takes values in $[0, b_j]$. To keep the presentation simpler, all further results and proofs of this section will be given directly in terms of the sequence $\{Y_j\}$.

In general, it is not possible to give explicit expressions for $\widehat{\Psi}_n(x)$. An exception is the case $n = 2$.

Example 3.3 Let $n = 2$, $b_1 = 1$ and $b_2 = \beta \le 1$. Then $\Psi_2(x) = \max\{x, x(2 - x/\beta)\}$. To find $\widehat{\Psi}_2(x)$, we must find the line through the point $(1,1)$ that is tangent to the curve $y = x(2 - x/\beta)$, as well as the point of intersection. Routine calculus yields that $\widehat{\Psi}_2(x) = \Phi(x)$, where Φ is the function defined in (7). Thus, the prophet region coincides with Boshuizen's region for the independent case. Indeed, in the extremal case Y_1 and Y_2 can be chosen to be independent: if $x \le 1 - \sqrt{1-\beta}$ let $Y_1 \equiv x$, and $Y_2 \in \{0, \beta\}$ with mean x; otherwise, let $Y_1 \in \{1 - \sqrt{1-\beta}, 1\}$ with mean x, and $Y_2 \in \{0, \beta\}$ with mean $1 - \sqrt{1-\beta}$. A simple calculation shows that $V(Y_1, Y_2) = x$, and $\mathrm{E}(Y_1 \vee Y_2) = \widehat{\Psi}_2(x)$.

The following inequalities follow immediately from Theorem 3.1.

Corollary 3.4 *For each* $n \ge 2$,

$$\mathrm{E}(Y_1 \vee \dots \vee Y_n) - V(Y_1, \dots, Y_n) \le \max_{2 \le j \le n} b_j (1 - 1/j)^j, \tag{9}$$

and

$$\mathrm{E}(Y_1 \vee \dots \vee Y_n) < nV(Y_1, \dots, Y_n) \qquad \text{if} \quad \mathrm{P}(\max_{j \le n} X_j > 0) > 0.$$

Both bounds are sharp for every sequence $\{b_j\}$, *and* (9) *is attained.*

Of particular interest is the case of geometric discounting, when $b_j = \beta^{j-1}$ for all $j \in \mathbb{N}$. For this case, Corollary 3.4 can be stated as

Corollary 3.5 *Let* $0 < \beta < 1$. *Then*

$$\mathrm{E}(X_1 \vee \beta X_2 \vee \dots \vee \beta^{n-1} X_n) - V(X_1, \beta X_2, \dots, \beta^{n-1} X_n) \le \max_{2 \le j \le n} \beta^{j-1} (1 - 1/j)^j, \tag{10}$$

and this bound is attained.

β	.8	.9	.95	.99	.999	.9999	$\to 1$
j^*	2	3	4	8	23	71	$\to \infty$
$\beta^{j^*-1}(1-1/j^*)^{j^*}$	.200	.240	.271	.320	.352	.363	$\to 1/e$

Table 1: The critical horizon j^* and the prophet's maximum advantage $\beta^{j^*-1}(1-1/j^*)^{j^*}$, for various choices of β.

In fact, it can be shown (by replacing j with a continuous variable x, and using routine calculus), that $\alpha_j := \beta^{j-1}(1-1/j)^j$ is unimodal in j. Hence α_j is maximized at the index

$$j^* := \min\{j : \alpha_{j+1} < \alpha_j\} = \min\{j : \gamma_j > \beta\}, \tag{11}$$

where $\gamma_j = (1-1/j)^j/[1-1/(j+1)]^{j+1}$. Thus, if $j(n) := \min\{n, j^*\}$, (10) is equivalent to

$$\mathrm{E}(X_1 \vee \beta X_2 \vee \cdots \vee \beta^{n-1}X_n) - V(X_1, \beta X_2, \ldots, \beta^{n-1}X_n) \leq \beta^{j(n)-1}(1-1/j(n))^{j(n)}.$$

The interpretation of j^* is that, once $n \geq j^*$, the prophet's advantage (in the extremal case) does not increase further when additional random variables are observed.

Table 1 gives values of j^* and the corresponding upper bound $\beta^{j^*-1}(1-1/j^*)^{j^*}$ for various choices of β. It is interesting to note that if $\beta \leq 27/32 \approx .8438$, then $j^* = 2$ and $\beta^{j^*-1}(1-1/j^*)^{j^*} = \beta/4$. Comparing this value with the right hand side of (8) yields the somewhat surprising conclusion that when $\beta \leq 27/32$, the best-possible difference prophet inequality in the dependent case is the same as in the independent case, for all $n \geq 2$. In this case, dependence does not benefit the prophet.

If β is near one, j^* may be estimated as follows. Using the Taylor approximation $\log(1-x) \approx -x - x^2/2$, we have

$$\log \gamma_j = j \log\left(1 - \frac{1}{j}\right) - (j+1)\log\left(1 - \frac{1}{j+1}\right) \approx \frac{1}{2}\left(\frac{1}{j+1} - \frac{1}{j}\right).$$

Equating this to $\log \beta$ yields a quadratic equation, whose positive root equals

$$j = \frac{1}{2}\left(\sqrt{1 - \frac{2}{\log \beta}} - 1\right).$$

This value appears to systematically underestimate the true value j^*; a better and simpler approximation is

$$j^* \approx \frac{1}{2}\sqrt{-\frac{2}{\log \beta}} = [2\log(1/\beta)]^{-1/2},$$

rounding up to the nearest integer in view of (11). This produces correct estimates of j^* for all values of β in Table 1.

A further simplification is possible by considering the Taylor expansion of $\log(1/\beta)$: if β is very close to one, say $\beta = 1/(1+r)$ for a very small positive number r, then $\log(1/\beta)$ will be very well approximated by r. Hence $j^* \approx (2r)^{-1/2}$.

3.2 Proof of Theorem 3.1

The proof of Theorem 3.1 is analogous to that of Theorem 3.2 of Hill and Kertz [8], but some modifications have to be made since the function Ψ_n is the maximum of several concave functions, but is not itself concave. The following lemma is essentially an amalgamation of Lemma 2.3 and Proposition 2.6 of [8].

Lemma 3.6 *Given any process $Y_1, \ldots, Y_n$ with $Y_j \in [0, b_j]$ for $j = 1, \ldots, n$, there exists a process $\hat{Y}_1, \ldots, \hat{Y}_n$ with $\hat{Y}_j \in [0, b_j]$ for $j = 1, \ldots, n$, satisfying the following properties:*

(i) $\hat{Y}_1, \ldots, \hat{Y}_n$ *is a supermartingale;*

(ii) $\hat{Y}_{m+1} \in \{0\} \cup [\hat{Y}_m, b_{m+1}]$ *a.s. for all* $m = 1, \ldots, n-1$*;*

(iii) $V(\hat{Y}_1, \ldots, \hat{Y}_n) = V(Y_1, \ldots, Y_n) = \mathrm{E}\,\hat{Y}_1$*; and*

(iv) $\mathrm{E}(\max_{j \le n} \hat{Y}_j) \ge \mathrm{E}(\max_{j \le n} Y_j)$.

Proof. We may assume (by replacing each Y_m with $Y_m \vee V(Y_{m+1}, \ldots, Y_n | \mathcal{F}_m)$, if necessary) that $Y_m \ge V(Y_{m+1}, \ldots, Y_n | \mathcal{F}_m)$ for all $m = 1, \ldots, n-1$. In particular, $Y_m \ge \mathrm{E}(Y_{m+1} | \mathcal{F}_m)$, and $Y_1, \ldots, Y_n$ is a supermartingale. Before constructing the process $\hat{Y}_1, \ldots, \hat{Y}_n$, first define a process $Y_1', Y_2', \ldots$ by $Y_m' = Y_m$ for $m \le n$, and $Y_m' \equiv 0$ for $m > n$. Put $b_m = 0$ for all $m > n$.

Next, define $t_1 = 1$, and recursively for $m = 1, 2, \ldots$, define $t_{m+1} = \min\{k > t_m : Y_k = 0$ or $Y_k \ge Y_{t_m}\}$. Define $\hat{Y}_m = Y_{t_m}$ for $m = 1, 2, \ldots, n$. Then the process $\hat{Y}_1, \ldots, \hat{Y}_n$ is a supermartingale (since the differences $t_{m+1} - t_m$ are uniformly bounded) with $\hat{Y}_{m+1} = 0$ or $\hat{Y}_{m+1} \ge \hat{Y}_m$ for all m, and satisfying (iii) and (iv). Finally, since the sequence $\{b_m\}$ is nonincreasing, $\hat{Y}_m \le b_{t_m} \le b_m$ for all m. $\square$

Proof of Theorem 3.1. First it will be shown that if $x = V(Y_1, \ldots, Y_n)$, then $\mathrm{E}(Y_1 \vee \cdots \vee Y_n) \le \widehat{\Psi}_n(x)$. By Lemma 3.6 and Jensen's inequality, it suffices to show that if $Y_1, \ldots, Y_k$ is a supermartingale with $Y_{m+1} \in \{0\} \cup [Y_m, b_{m+1}]$ a.s. for all $m = 1, 2, \ldots, k-1$, then

$$\mathrm{E}(Y_1 \vee \cdots \vee Y_k | Y_1) \le \Psi_k(Y_1) \quad \text{a.s.} \tag{12}$$

For then, $\mathrm{E}(Y_1 \vee \cdots \vee Y_n) \le \mathrm{E}\,\Psi_n(Y_1) \le \widehat{\Psi}_n(\mathrm{E}Y_1) = \widehat{\Psi}_n(x)$.

The proof of (12) will be by induction on k. Note first that, by the supermartingale property, $Y_j = 0$ a.s. on $\{Y_1 = 0\}$ for all $j \ge 2$. Observe also that (12) is trivial if $Y_1 \ge b_2$. Thus, it remains only to show (12) holds a.s. on $\{0 < Y_1 < b_2\}$.

For $k = 2$, we have

$$\begin{aligned}
\mathrm{E}(Y_1 \vee Y_2 | Y_1) &= Y_1 + \mathrm{E}[(Y_2 - Y_1) I_{\{Y_1 \le Y_2\}} | Y_1] \\
&\le Y_1 + \mathrm{E}[Y_2 (1 - Y_1/b_2) | Y_1] \\
&\le Y_1 + Y_1 (1 - Y_1/b_2)^+ \quad \text{a.s. on } \{0 < Y_1 < b_2\}.
\end{aligned}$$

Next, assume (12) is true for $k = m$, and show it is true for $k = m + 1$ by calculating

$$
\begin{aligned}
\mathrm{E}(Y_1 \vee \cdots \vee Y_{m+1} | Y_1) &= \mathrm{E}\left[Y_1 I(Y_2 = 0) + (Y_2 \vee \cdots \vee Y_{m+1}) I_{\{Y_1 \le Y_2\}} | Y_1\right] \\
&= Y_1 \mathrm{P}(Y_2 = 0 | Y_1) + \mathrm{E}\left[\mathrm{E}(Y_2 \vee \cdots \vee Y_{m+1} | Y_2) I_{\{Y_1 \le Y_2\}} | Y_1\right] \\
&\le Y_1 \mathrm{P}(Y_2 = 0 | Y_1) + \mathrm{E}\left[\left(Y_2 + \max_{1 \le j \le m-1} j Y_2 [1 - (Y_2/b_{j+2})^{1/j}]^+\right) I_{\{Y_1 \le Y_2\}} | Y_1\right] \\
&= Y_1 + \mathrm{E}\left[\max_{1 \le j \le m-1} f_j(Y_2) I_{\{Y_1 \le Y_2\}} Y_2 | Y_1\right], \qquad (13)
\end{aligned}
$$

where $f_j(x) = 1 + j(1 - (x/b_{j+2})^{1/j})^+ - Y_1/x$ for $j = 1, \ldots, m-1$, and the inequality follows from the induction hypothesis.

Note that for $b_{j+2} \le x \le b_2$, $f_j(x) = 1 - Y_1/x \le 1 - Y_1/b_2$, while for $0 \le x \le b_{j+2}$,

$$
f_j(x) \le (j+1)\left(1 - (Y_1/b_{j+2})^{1/(j+1)}\right),
$$

as can be seen by taking the derivative of $1 + j[1 - (x/b_{j+2})^{1/j}] - Y_1/x$. It follows that

$$
\begin{aligned}
\max_{1 \le j \le m-1} f_j(x) &\le \max_{1 \le j \le m-1} \max\left\{1 - Y_1/b_2, (j+1)\left(1 - (Y_1/b_{j+2})^{1/(j+1)}\right)\right\} \\
&= \max_{1 \le j \le m} j\left(1 - (Y_1/b_{j+1})^{1/j}\right).
\end{aligned}
$$

The development now continues

$$
\begin{aligned}
(13) &\le Y_1 + \mathrm{E}\left[\max_{1 \le j \le m} j\left(1 - (Y_1/b_{j+1})^{1/j}\right)^+ I_{\{Y_1 \le Y_2\}} Y_2 | Y_1\right] \\
&= Y_1 + \max_{1 \le j \le m} j\left(1 - (Y_1/b_{j+1})^{1/j}\right)^+ \mathrm{E}(Y_2 | Y_1) \\
&\le Y_1 + \max_{1 \le j \le m} j Y_1\left(1 - (Y_1/b_{j+1})^{1/j}\right)^+ = \Psi_{m+1}(Y_1),
\end{aligned}
$$

a.s. on $\{0 < Y_1 < b_2\}$. This establishes (12), and thus the set of ordered pairs $\{(x, y) : x = V(Y_1, \ldots, Y_n)$ and $y = \mathrm{E}(Y_1 \vee \cdots \vee Y_n)$ for some $Y_1, \ldots, Y_n$ with $Y_i \le b_i$ for $1 \le i \le n\}$ is a subset of Γ_n. The converse inclusion follows from the following proposition. □

Proposition 3.7 *For every point $(x, y) \in \Gamma_n$, there exists a sequence of random variables $Y_1, \ldots, Y_n$ with $Y_j \in [0, b_j]$ for all j which is both Markov and a supermartingale, such that $V(Y_1, \ldots, Y_n) = x$ and $\mathrm{E}(Y_1 \vee \cdots \vee Y_n) = y$.*

Proof. Let $S_n = \{(x, y) : 0 \le x \le 1, y \in \{x, \Psi_n(x)\}\}$, and note that Γ_n is the convex closure of S_n. The statement of the proposition is proved first for $(x, y) \in S_n$. If $y = x$, the statement follows by taking $Y_1 \equiv x$, and $Y_2 = \cdots = Y_n \equiv 0$. So assume $y = \Psi_n(x) > x$, and choose k, $1 \le k \le n-1$ such that $x + kx(1 - (x/b_{k+1})^{1/k}) = \Psi_n(x)$. Note that $\Psi_n(x) > x$ implies that $x \le b_{k+1}$. Define a sequence $Y_1', \ldots, Y_{k+1}'$ by $Y_1' \equiv x$; and for $2 \le m \le k+1$,

$$
\mathrm{P}(Y_m' = 0 | \mathcal{F}_{m-1}') = 1 \quad \text{on } \{Y_{m-1}' = 0\},
$$

and

$$\mathrm{P}(Y'_m = x(b_{k+1}/x)^{(m-1)/k} | \mathcal{F}'_{m-1}) = (x/b_{k+1})^{1/k} = 1 - \mathrm{P}(Y'_m = 0 | \mathcal{F}'_{m-1}) \quad \text{on } \{Y'_{m-1} \neq 0\}.$$

Then $Y'_1, \dots, Y'_{k+1}$ is both Markov and a martingale, $0 \leq Y'_m \leq b_{k+1} \leq b_m$ for all $m = 1, \dots, k+1$, and $\mathrm{E}(Y'_1 \vee \cdots \vee Y'_{k+1}) = \Psi_n(x)$. (This process is a modified version of the extremal process given by both Hill and Kertz [8] and Dubins and Pitman [5].) To obtain a sequence of length n, define $Y'_j \equiv 0$ for $j > k+1$. Then the process $Y'_1, \dots, Y'_n$ is both Markov and a supermartingale, and the statement of the proposition holds for each point (x, y) in S_n. To see that it holds for all members of Γ_n, note that since S_n is connected, each point (x, y) in Γ_n can be written as a convex combination of two points in S_n, say $(x, y) = p_1(x_1, y_1) + p_2(x_2, y_2)$, where $(x_j, y_j) \in S_n$ for $j = 1, 2$, and $p_1 + p_2 = 1$. Let $\mathbf{Y}^{(1)}, \mathbf{Y}^{(2)}$ be sequences with $(V(\mathbf{Y}^{(j)}), \mathrm{E}(\sup \mathbf{Y}^{(j)})) = (x_j, y_j)$ (each both Markov and a supermartingale), and let $\mathbf{Y}$ be the mixture defined by $\mathrm{P}(\mathbf{Y} = \mathbf{Y}^{(j)}) = p_j$, $j = 1, 2$. Then $\mathbf{Y}$ is both Markov and a supermartingale, $V(\mathbf{Y}) = x$, and $\mathrm{E}(\sup \mathbf{Y}) = y$. $\Box$

3.3 The infinite-horizon case

Let Γ denote the set of all ordered pairs $\{(x, y) : x = V(Y_1, Y_2, \dots)$ and $y = \mathrm{E}(\sup_n Y_n)$ for some sequence $Y_1, Y_2, \dots$ with $Y_j \leq b_j$ for all $j \in \mathbb{N}\}$. Since it is always possible to append an infinite string of zeros to any finite sequence $Y_1, \dots, Y_n$, it is clear that Γ contains each Γ_n as a subset. On the other hand, each (x, y) in Γ is the limit point of a sequence $\{(x_n, y_n)\}$, where $(x_n, y_n) \in \Gamma_n$. Thus

$$\bigcup_{n=1}^{\infty} \Gamma_n \subseteq \Gamma \subseteq \mathrm{cl}\left(\bigcup_{n=1}^{\infty} \Gamma_n\right),$$

where $\mathrm{cl}(A)$ denotes the topological closure of A. In general, it seems difficult to determine whether Γ contains its upper boundary. But if $\lim_{n\to\infty} b_n = 0$, the answer is affirmative. To see this, define the function

$$\Psi(x) := \lim_{n\to\infty} \Psi_n(x) = x + \sup_{j\geq 1} jx \left(1 - (x/b_{j+1})^{1/j}\right)^+, \tag{14}$$

and note that the upper boundary function of Γ is $\widehat{\Psi}$. Let $b := \lim_{n\to\infty} b_n$. If $b = 0$, then the supremum in (14) is attained for every x, and so $\Gamma = \{(x, y) : 0 \leq x \leq 1, x \leq y \leq \widehat{\Psi}(x)\}$.

The case when $b > 0$ is rather more complicated, due to the fact that the supremum in (14) is attained for $x > b$, but may be strict for $x \leq b$. This suggests that the prophet region could perhaps contain part, but not all of its upper boundary. However, the author does not know of a specific example where this happens.

The following is an infinite-horizon analogue of the prophet inequality (9).

Theorem 3.8 *For every sequence* $Y_1, Y_2, \dots$ *with* $0 \leq Y_j \leq b_j$ $(j \in \mathbb{N})$,

$$\mathrm{E}\left(\sup_{n\geq 1} Y_n\right) - V(Y_1, Y_2, \dots) \leq \sup_{j\geq 2} b_j (1 - 1/j)^j.$$

This bound is sharp, and holds with strict inequality if the sequence $b_j(1-1/j)^j$ does not have a largest term.

Note that the weak inequality statement follows immediately from Corollary 3.4 by taking limits as $n \to \infty$. The proof of strict inequality below uses several ideas from the proof of Theorem 4.2 of Hill and Kertz [8], and the following elementary limit result.

Lemma 3.9 *If $\{c_j\}$ is a sequence of positive numbers and $c_j \to c > 0$, then for all $x > 0$,*

$$\lim_{j\to\infty} j\left(1-(x/c_j)^{1/j}\right) = -\ln(x/c). \tag{15}$$

Proof. Let $f_j(u) = j(1-u^{1/j})$, $f(u) = -\ln u$, and $\alpha_j = x/c_j$. Then f_j converges to f uniformly on every interval (u_1, u_2) with $u_2 > u_1 > 0$, and $\alpha_j \to x/c > 0$. Hence $f_j(\alpha_j) \to f(x/c)$, and (15) holds. □

Proof of Theorem 3.8 *(Strict inequality).* Suppose the sequence $b_j(1-1/j)^j$ does not have a largest term, and let $\gamma := \sup_{j\geq 1} b_j(1-1/j)^j = \lim_{j\to\infty} b_j(1-1/j)^j$, and $b := \lim_{j\to\infty} b_j$. Then γ and b are both strictly positive, and $\gamma = b/e$. As in the finite-dimensional case, we may assume that $Y_1, Y_2, \ldots$ is a supermartingale. Define the function

$$\phi(x) := \sup_{j\geq 1} jx\left(1-(x/b_{j+1})^{1/j}\right).$$

Then $\phi(x)$ has a unique maximum value of γ at $x = \gamma$. To see this, note that for each $j \geq 1$,

$$\max_{x>0} jx\left(1-(x/b_{j+1})^{1/j}\right) = b_{j+1}[1-1/(j+1)]^{j+1} < \gamma,$$

while by Lemma 3.9, $\lim_{j\to\infty} jx[1-(x/b_{j+1})^{1/j}] = -x\ln(x/b) = x(1-\ln(x/\gamma))$, which has a unique maximum at $x = \gamma$.

By letting n tend to ∞ in (12), and using the fact that $Y_1, Y_2, \ldots$ is a uniformly bounded supermartingale, it follows that

$$\begin{aligned} \mathrm{E}(\sup Y_n) - V(Y_1, Y_2, \ldots) &= \mathrm{E}(\sup Y_n) - \mathrm{E}\, Y_1 = \mathrm{E}[\mathrm{E}(\sup Y_n - Y_1 | Y_1)] \\ &\leq \mathrm{E}\left(\lim_{n\to\infty} \Psi_n(Y_1) - Y_1\right) = \mathrm{E}\,\phi(Y_1) \leq \gamma. \end{aligned} \tag{16}$$

Suppose, by way of contradiction, that $\mathrm{E}(\sup Y_n) - V(Y_1, Y_2, \ldots) = \gamma$. Then equality must hold throughout in (16), and hence $Y_1 = \gamma$ a.s. By an argument similar to that in the proof of Lemma 3.6, we may assume that $\mathrm{P}(Y_2 = 0) + \mathrm{P}(Y_2 \geq \gamma) = 1$. Let $\alpha = \mathrm{P}(Y_2 \geq \gamma)$, and $x_2 = \mathrm{E}(Y_2 | Y_2 \geq \gamma)$. Without loss of generality, we may assume that $\mathrm{P}(Y_2 > Y_1) > 0$. (Otherwise simply delete Y_2 and consider the sequence $Y_1, Y_3, Y_4, \ldots$.) Hence $\alpha > 0$ and

$x_2 > \gamma$. Now

$$
\begin{aligned}
\mathrm{E}\left(\sup_{n\ge 1} Y_n\right) - V(Y_1, Y_2, \dots) &= \gamma(1-\alpha) + \int_{Y_2\ge\gamma} \mathrm{E}\left(\sup_{n\ge 2} Y_n | Y_2\right) d\,\mathrm{P} - \gamma \\
&\le \int_{Y_2\ge\gamma} \left[Y_2 + \sup_{j\ge 1} jY_2 \left(1 - (Y_2/b_{j+2})^{1/j}\right)^+ \right] d\,\mathrm{P} - \gamma\alpha \\
&\le \gamma(1-\alpha) + \int_{Y_2\ge\gamma} \phi_2(Y_2) d\,\mathrm{P},
\end{aligned}
$$

where $\phi_2(x) = \sup_{j\ge 1} jx[1-(x/b_{j+2})^{1/j}]^+$, and the last inequality follows since $\int_{Y_2\ge\gamma} Y_2 = \mathrm{E}\, Y_2 \le \gamma$ by the supermartingale property of $Y_1, Y_2, \dots$. Observe that ϕ_2, like ϕ, has a unique maximum value of γ at $x = \gamma$. The same is then true for $\hat{\phi}_2$, the smallest concave function dominating ϕ_2. Thus, Jensen's inequality implies that

$$
\begin{aligned}
\mathrm{E}\left(\sup_{n\ge 1} Y_n\right) - V(Y_1, Y_2, \dots) &\le \gamma(1-\alpha) + \int_{Y_2\ge\gamma} \hat{\phi}_2(Y_2) d\,\mathrm{P} \\
&\le \gamma(1-\alpha) + \alpha\hat{\phi}_2\left(\int_{Y_2\ge\gamma} Y_2 d(\mathrm{P}/\alpha)\right) = \gamma(1-\alpha) + \alpha\hat{\phi}_2(x_2) < \gamma,
\end{aligned}
$$

since $x_2 \ne \gamma$ and $\alpha > 0$. This contradiction completes the proof. □

4 Application to optimal stopping with a random horizon

The results of Section 3.1 can be applied to obtain prophet-like inequalities for a situation with a constant discount factor and a random time horizon. Specifically, let $X_1, X_2, \dots$ be any sequence of $[0,1]$-valued random variables, and let N be a random variable taking values in the positive integers, independent of $X_1, X_2, \dots$. Fix $0 < \beta < 1$, and assume the reward for stopping at time t is $\beta^{t-1} X_t I_{\{t\le N\}}$. Consider one player, henceforth to be called the *informed gambler*, who knows the value of N before observing the first random variable; and another, to be called the *uninformed gambler*, who knows only the distribution of N, but not its value. How widely can the players' optimal expected returns diverge?

To formalize this problem, introduce the sigma algebras $\mathcal{F}_j = \sigma(X_1, \dots, X_j, N)$, and $\mathcal{G}_j = \sigma(X_1 I_{\{N\ge 1\}}, \dots, X_j I_{\{N\ge j\}})$, $j \in \mathbb{N}$. (Alternatively, we could use the sigma algebra $\mathcal{G}'_j = \sigma(X_1, I_{\{N\ge 1\}}, \dots, X_j, I_{\{N\ge j\}})$. It is not hard to see that the uninformed gambler's optimal value is the same in both cases.) Let $p_j = \mathrm{P}(N = j)$ for $j = 1, 2, \dots$, and define $Z_j = X_j I_{\{N\ge j\}}$. Then the informed gambler's and the uninformed gambler's optimal values will be $V_{\mathcal{F}}(Z_1, \beta Z_2, \beta^2 Z_3, \dots)$ and $V_{\mathcal{G}}(Z_1, \beta Z_2, \beta^2 Z_3, \dots)$, respectively. The following consequence of Corollary 3.5 gives a sharp inequality relating these two values.

Theorem 4.1 *(i) For every probability distribution* $\{p_j\}$,

$$
V_{\mathcal{F}}(Z_1, \beta Z_2, \beta^2 Z_3, \dots) - V_{\mathcal{G}}(Z_1, \beta Z_2, \beta^2 Z_3, \dots) \le \beta^{j^*-1}\left(1 - \frac{1}{j^*}\right)^{j^*}, \tag{17}
$$

where j^* is defined as in (11), and this bound is attained.

(ii) If furthermore, $N \leq n$ a.s., then

$$V_{\mathcal{F}}(Z_1, \beta Z_2, \beta^2 Z_3, \dots) - V_{\mathcal{G}}(Z_1, \beta Z_2, \beta^2 Z_3, \dots) \leq \beta^{j(n)-1} \left(1 - \frac{1}{j(n)}\right)^{j(n)}, \tag{18}$$

where $j(n) = \min\{j^*, n\}$. This bound is attained.

Proof. Note that

$$\begin{aligned} V_{\mathcal{F}}(Z_1, \beta Z_2, \beta^2 Z_3, \dots) &= \sum_{j=1}^{\infty} p_j V(X_1, \beta X_2, \dots, \beta^{j-1} X_j) \qquad (19) \\ &\leq \sum_{j=1}^{\infty} p_j \,\mathrm{E}(X_1 \vee \beta X_2 \vee \cdots \vee \beta^{j-1} X_j) \\ &= \mathrm{E}(Z_1 \vee \beta Z_2 \vee \cdots \vee \beta^{N-1} Z_N) = \mathrm{E}\left(\sup_{j \geq 1} \beta^{j-1} Z_j\right). \end{aligned}$$

Applying Corollary 3.5 to the sequence $\{Z_j\}$ gives the inequalities of Theorem 4.1. The bounds are attained as follows. For (i), let $j_0 = j^*$; for (ii), let $j_0 = j(n)$. Now define $X_i \equiv \beta^{j_0-i}(1-1/j_0)^{j_0-i}$ for $i = 1, \dots, j_0$, and $X_i \equiv 0$ for $i > j_0$. Let $p_i = (1/j_0)(1-1/j_0)^{i-1}$ for $i = 1, \dots, j_0$, $p_{j_0} = (1-1/j_0)^{j_0-1}$, and $p_i = 0$ for all $i > j_0$. Then

$$V(X_1, \beta X_2, \dots, \beta^{j-1} X_j) = \beta^{j_0-1} \,\mathrm{E}\, X_j = \beta^{j_0-1} \left(1 - \frac{1}{j_0}\right)^{j_0-j}, \qquad j = 1, \dots, j_0,$$

and

$$V_{\mathcal{G}}(Z_1, \beta Z_2, \beta^2 Z_3, \dots) = \mathrm{E}\, Z_1 = \mathrm{E}\, X_1 = \beta^{j_0-1} \left(1 - \frac{1}{j_0}\right)^{j_0-1},$$

since the process $Z_1, \beta Z_2, \dots, \beta^{j_0-1} Z_{j_0}$ is a supermartingale with respect to the filtration $\mathcal{G}$. A routine calculation using (19) shows that

$$V_{\mathcal{F}}(Z_1, \beta Z_2, \beta^2 Z_3, \dots) - V_{\mathcal{G}}(Z_1, \beta Z_2, \beta^2 Z_3, \dots) = \beta^{j_0-1} \left(1 - \frac{1}{j_0}\right)^{j_0},$$

attaining equality in both (17) and (18). □

The bounds in Theorem 4.1 can be interpreted as the maximum amount a player should be willing to pay for the privilege of being disclosed the number of available observations ahead of time. For instance, if $\beta = .9$, one should never pay more than .240 for this privilege. (See Table 1.)

Note that the inequalities do not change if the informed gambler is given the additional power of knowing the values of $X_1, X_2, \dots$ ahead of time. Also, the informed gambler's advantage is not reduced if the X_i are assumed to be independent. This is in clear contrast to the classical comparison of prophet and gambler.

Other comparisons of the optimal expected returns of an informed gambler and an uninformed gambler are given in Allaart [1], from which the terms "informed gambler" and "uninformed gambler" were taken.

5 Random discounting

In this section, let $X_1, X_2, \ldots$ be a general sequence of $[0,1]$-valued random variables, and let $B_1, B_2, \ldots$ be independent $[0,1]$-valued random variables, independent of $X_1, X_2, \ldots$, having a common mean $0 < \beta \leq 1$. Define the discounted sequence $\{Y_j\}$ by

$$Y_j = B_1 \cdots B_{j-1} X_j, \qquad j = 1, 2, \ldots. \tag{20}$$

Define the filtration $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \ldots)$ by $\mathcal{F}_j = \sigma(X_1, \ldots, X_j, B_1, \ldots, B_{j-1})$, $j \in \mathbb{N}$. Note that this definition expresses the position that the gambler can observe the X's and the B's individually. The goal of this section is to establish a complete comparison between $V_{\mathcal{F}}(Y_1, \ldots, Y_n)$ and $\mathrm{E}(Y_1 \vee \cdots \vee Y_n)$ for $n \geq 1$. Since the case $\beta = 1$ is simply the nondiscounted case treated by Hill and Kertz [8], only the case $\beta < 1$ will be considered here.

Definition 5.1 *For $n \geq 1$ and $0 < \beta < 1$, $\Omega_{n,\beta} : [0,1] \to \mathbb{R}$ is the function*

$$\Omega_{n,\beta}(x) = \begin{cases} x + (n-1)x\left(1 - x^{1/(n-1)}\right), & x \leq \beta^{n-1} \\ j(1-\beta)x + \beta^j, & \beta^j < x \leq \beta^{j-1}, \quad j = 1, \ldots, n-1. \end{cases}$$

Note that $\Omega_{n,\beta}$ is concave, as can be seen by examining the behavior of its derivative.

The next theorem, whose proof is developed following Remark 5.5 below, is the main result of this section.

Theorem 5.2 *The set* $\{(x,y) : x = V_{\mathcal{F}}(Y_1, \ldots, Y_n)$ and $y = \mathrm{E}(Y_1 \vee \cdots \vee Y_n)$ for some sequence $Y_1, \ldots, Y_n$ of the form (20)$\}$ *is precisely the set* $\{(x,y) : 0 \leq x \leq 1, x \leq y \leq \Omega_{n,\beta}(x)\}$.

Corollary 5.3 *For fixed $n \geq 1$ and $0 < \beta < 1$,*

$$\mathrm{E}(Y_1 \vee \cdots \vee Y_n) - V_{\mathcal{F}}(Y_1, \ldots, Y_n) \leq \begin{cases} (1 - 1/n)^n, & \beta \geq 1 - 1/n \\ \\ (1-\beta)\lfloor 1/(1-\beta) \rfloor \beta^{\lfloor 1/(1-\beta) \rfloor}, & \beta < 1 - 1/n, \end{cases}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. This bound is attained for all n and all β.

Proof. The function $x \to (n-1)x\left(1 - x^{1/(n-1)}\right)$ attains its maximum at the point $x = (1 - 1/n)^{n-1}$, which falls inside the range $[0, \beta^{n-1}]$ if and only if $\beta \geq 1 - 1/n$. If, on the other hand, $\beta < 1 - 1/n$, then the maximum value of $\Omega_{n,\beta}(x) - x$ occurs at the point β^{j^*}, where j^* is the largest integer j such that $j(1-\beta) - 1 \leq 0$. $\square$

The following result for infinite sequences follows easily from Theorem 5.2.

Corollary 5.4 *The set of points* $\{(x,y) : x = V_{\mathcal{F}}(Y_1, Y_2, \dots)$ and $y = \mathrm{E}(\sup_{n\geq 1} Y_n)$ for some sequence $Y_1, Y_2, \dots$ of the form (20)$\}$ *is precisely the set* $\{(x,y) : 0 \leq x \leq 1, x \leq y \leq \Omega_\beta(x)\}$, *where* $\Omega_\beta(0) = 0$, *and*

$$\Omega_\beta(x) = j(1-\beta)x + \beta^j, \qquad \beta^j < x \leq \beta^{j-1}, \quad j = 1, 2, \dots.$$

As a consequence,

$$\mathrm{E}\left(\sup_{n\geq 1} Y_n\right) - V_{\mathcal{F}}(Y_1, Y_2, \dots) \leq (1-\beta)\lfloor 1/(1-\beta)\rfloor \beta^{\lfloor 1/(1-\beta)\rfloor},$$

and this bound is attained for all $0 < \beta < 1$.

Observe that the case of geometric discounting ($Y_j = \beta^{j-1} X_j$) is a degenerate case of the random discounting scheme of this section. It is interesting to investigate the additional advantage the prophet can gain over the gambler if the discount factors are made random variables with mean β. To this end, graphs of the bounds $y = \max_{j\geq 2} \beta^{j-1}(1-1/j)^j$ and $y = (1-\beta)\lfloor 1/(1-\beta)\rfloor \beta^{\lfloor 1/(1-\beta)\rfloor}$ are given in Figure 1.

Remark 5.5 For sequences of *independent* random variables $X_1, \dots, X_n$, the prophet region is slightly smaller, as was shown in Allaart [2]. The two main results of that paper are:

(i) The set of points

$\{(x,y) : x = V(Y_1, \dots, Y_n)$ and $y = \mathrm{E}(Y_1 \vee \cdots \vee Y_n)$ for some sequence $Y_1, \dots, Y_n$ of the form (20) with $X_1, \dots, X_n$ independent$\}$

is precisely the set $\{(x,y) : 0 \leq x \leq 1, x \leq y \leq \Psi_{n,\beta}(x)\}$, where

$$\Psi_{n,\beta}(x) = \begin{cases} [(1-\beta)n + 2\beta]x - \beta^{-(n-2)}x^2, & \text{if } x \leq \beta^{n-1}, \\ (1-\beta)jx + \beta^j, & \text{if } \beta^j \leq x \leq \beta^{j-1},\ 1 \leq j \leq n-1. \end{cases}$$

(ii) If $X_1, \dots, X_n$ are independent, then

$$\mathrm{E}(Y_1 \vee \cdots \vee Y_n) - V(Y_1, \dots, Y_n) \leq \begin{cases} \beta^{n-2}\{(1-\beta)(n-1) + \beta\}^2/4, & \beta \geq 1 - 1/n, \\ \\ (1-\beta)\lfloor 1/(1-\beta)\rfloor \beta^{\lfloor 1/(1-\beta)\rfloor}, & \beta < 1 - 1/n, \end{cases}$$

and this bound is attained for all n and all β.

Notice that the upper boundary of the prophet region for the independent case coincides with that for the arbitrarily dependent case to the right of the point $x = \beta^{n-1}$, but is different to the left of this point. In fact, it will be shown later that the extremal sequence $(X_1, \dots, X_n)$ attaining the upper bound $\Omega_{n,\beta}(x)$ is a degenerate (and hence independent) sequence when $x \geq \beta^{n-1}$, but has a martingale-like structure when $x < \beta^{n-1}$. Similarly, the best-possible difference inequality for the independent case is the same as that for the arbitrarily dependent case when $\beta < 1 - 1/n$, but is different when $\beta \geq 1 - 1/n$.

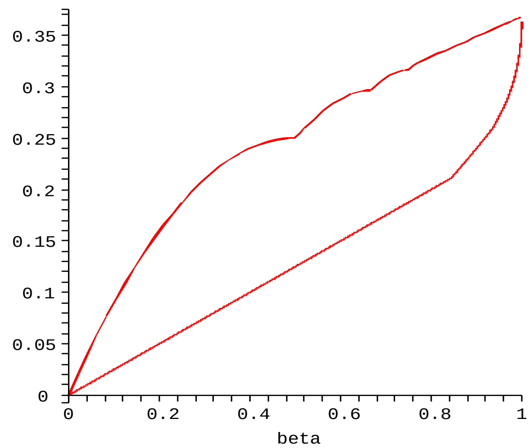

Figure 1: The bounds of Corollaries 3.5 and 5.4, as functions of β. Bottom: $y = \max_{j\ge 2}\beta^{j-1}(1-1/j)^j$. Top: $y = (1-\beta)\lfloor 1/(1-\beta)\rfloor \beta^{\lfloor 1/(1-\beta)\rfloor}$.

The remainder of the section is devoted to proving Theorem 5.2. First it will be shown that, for the purpose of bounding the value $\mathrm{E}(Y_1\vee\cdots\vee Y_n)$ for given $V_{\mathcal{F}}(Y_1,\ldots,Y_n)$, we may assume that (i) $B_j\in\{0,1\}$ for all j, and (ii) the sequence $Y_1,\ldots,Y_n$ is a supermartingale.

The first lemma shows that $V_{\mathcal{F}}(Y_1,\ldots,Y_n)$ depends on $B_1,\ldots,B_{n-1}$ only through their mean β.

Lemma 5.6

$$V_{\mathcal{F}}(Y_1,\ldots,Y_n) = V(X_1,\beta X_2,\ldots,\beta^{n-1}X_n).$$

Proof. Follows easily by backward induction (see [4]) using the recursive relation

$$V_{\mathcal{F}}(Y_j,\ldots,Y_n|\mathcal{F}_{j-1}) = \mathrm{E}[Y_j\vee V_{\mathcal{F}}(Y_{j+1},\ldots,Y_n|\mathcal{F}_j)|\mathcal{F}_{j-1}], \qquad 1\le j\le n,$$

and the fact that $B_1,B_2,\ldots$ are independent of each other and of the sequence $\{X_j\}$. $\square$

An important consequence of Lemma 5.6 is that, by an argument similar to that in Lemma 3.6, we may assume that the sequence $X_1,\beta X_2,\ldots,\beta^{n-1}X_n$ is a supermartingale. From this, it follows easily that the sequence $Y_1,\ldots,Y_n$ is a supermartingale as well.

Lemma 5.7 *Let $j\in\{1,\ldots,n-1\}$ be fixed, and define $\tilde{B}_1,\ldots,\tilde{B}_{n-1}$ by $\tilde{B}_i=B_i$ if $i\ne j$, and $\tilde{B}_j=$ the $\{0,1\}$-valued r.v., independent of $X_1,\ldots,X_n$ and $B_1,\ldots,B_{n-1}$, with $\mathrm{E}\tilde{B}_j=\beta$. Let $\tilde{Y}_i=\tilde{B}_1\cdots\tilde{B}_{i-1}X_i$ for all i. Then $\mathrm{E}(\tilde{Y}_1\vee\cdots\vee\tilde{Y}_n)\ge \mathrm{E}(Y_1\vee\cdots\vee Y_n)$.*

Proof. Observe that $Y_1\vee\cdots\vee Y_n = W\vee B_jZ$, where

$$W = Y_1\vee\cdots\vee Y_j,$$
$$Z = B_1\cdots B_{j-1}(X_{j+1}\vee B_{j+1}X_{j+2}\vee\cdots\vee B_{j+1}\cdots B_{n-1}X_n).$$

It is well known that the random variable $\tilde{B}_j$ convexly dominates B_j. Since for fixed nonnegative numbers w and z the function $x \to w \vee zx$ is convex, and W and Z are independent of B_j, the lemma follows. □

In view of Lemma 5.7, we can and will assume from here on that $\mathrm{P}(B_j = 1) = \beta = \mathrm{P}(B_j = 0)$ for all j. With this assumption, the expected maximum can be written as

$$\mathrm{E}(Y_1 \vee \cdots \vee Y_n) = \mathrm{E}[f_n(X_1, \ldots, X_n)],$$

where for real numbers $x_1, x_2, \ldots,$ we define

$$\begin{aligned} f_1(x_1) &= x_1 \\ f_k(x_1, \ldots, x_k) &= \sum_{j=1}^{k-1} \beta^{j-1}(1-\beta)(x_1 \vee \cdots \vee x_j) + \beta^{k-1}(x_1 \vee \cdots \vee x_k), \qquad k \geq 2. \end{aligned}$$

Proposition 5.8 *For every $k \geq 2$, for every sequence of r.v.'s $X_1, \ldots, X_k$ such that $(\beta^{j-1} X_j)_{j=1}^k$ is a supermartingale, and for every x in $[0,1]$ the following statements hold almost surely on the set $\{X_1 \leq x\}$.*

(i) If $x \leq \beta^{k-1}$, then

$$\mathrm{E}[f_k(x, X_2, \ldots, X_k)|X_1] \leq x + (k-1)\left(1 - x^{1/(k-1)}\right) X_1.$$

(ii) If $\beta^j < x \leq \beta^{j-1}$ for $j = 1, \ldots, k-1$, then

$$\mathrm{E}[f_k(x, X_2, \ldots, X_k)|X_1] \leq \begin{cases} x + \left[(j-1)(1-\beta) + 1 - x\beta^{-(j-1)}\right] X_1, & X_1 \leq \beta^j \\ (j-1)(1-\beta)X_1 + \beta^j + (1-\beta)x, & X_1 > \beta^j. \end{cases}$$

The proof of Proposition 5.8 is somewhat technical, and is deferred until the end of the section. First, the statement of the proposition is used to complete the proof of Theorem 5.2.

Proof of Theorem 5.2. We may assume that the sequence $Y_1, \ldots, Y_n$ is a supermartingale, so that $V_{\mathcal{F}}(Y_1, \ldots, Y_n) = \mathrm{E}\, Y_1 = \mathrm{E}\, X_1$. Since

$$\mathrm{E}(Y_1 \vee \cdots \vee Y_n | X_1) = \mathrm{E}[f_n(X_1, \ldots, X_n)|X_1],$$

Proposition 5.8 implies that on $\{X_1 \leq \beta^{n-1}\}$,

$$\mathrm{E}(Y_1 \vee \cdots \vee Y_n | X_1) \leq X_1 + (n-1)X_1 \left(1 - X_1^{1/(n-1)}\right), \tag{21}$$

while on $\{\beta^j < X_1 \leq \beta^{j-1}\}$ $(j = 1, \ldots, n-1)$,

$$\begin{aligned} \mathrm{E}(Y_1 \vee \cdots \vee Y_n | X_1) &\leq X_1 + X_1(j-1)(1-\beta) + \beta^j - \beta X_1 \\ &= j(1-\beta)X_1 + \beta^j. \end{aligned} \tag{22}$$

Taking expectations on both sides of (21) and (22) yields

$$\mathrm{E}(Y_1 \vee \cdots \vee Y_n) \leq \mathrm{E}(\Omega_{n,\beta}(X_1)) \leq \Omega_{n,\beta}(\mathrm{E}\, X_1) = \Omega_{n,\beta}(V_{\mathcal{F}}(Y_1, \ldots, Y_n)),$$

where the second inequality follows from the concavity of $\Omega_{n,\beta}$.

Conversely, let (x, y) be a point in the set $\{(x, y) : x \leq y \leq \Omega_{n,\beta}(x)\}$. For a sequence $\mathbf{X} = (X_1, \ldots, X_n)$, the sequence $\mathbf{Y} = (Y_1, \ldots, Y_n)$ defined by (20), where $\mathrm{P}(B_j = 1) = \beta = 1 - \mathrm{P}(B_j = 0)$ for all j, will be called the *sequence associated to* $\mathbf{X}$. We first construct a sequence $\mathbf{X}' = (X'_1, \ldots, X'_n)$ such that the sequence $\mathbf{Y}' = (Y'_1, \ldots, Y'_n)$ associated to $\mathbf{X}'$ satisfies $V_{\mathcal{F}'}(\mathbf{Y}') = x$, and $\mathrm{E}(Y'_1 \vee \cdots \vee Y'_n) = \Omega_{n,\beta}(x)$. There are two cases to consider.

Case 1. $x \leq \beta^{n-1}$. Let $\mathbf{X}' = (X'_1, \ldots, X'_n)$ be given by $X'_1 \equiv x$, and for $m = 2, \ldots, n$,

$$\mathrm{P}(X'_m = 0 | \mathcal{F}'_{m-1}) = 1 \qquad \text{on } \{X'_{m-1} = 0\},$$

and

$$\mathrm{P}(X'_m = x^{1-(m-1)/(n-1)} | \mathcal{F}_{m-1}) = x^{1/(n-1)}/\beta = 1 - \mathrm{P}(X'_m = 0 | \mathcal{F}_{m-1}) \quad \text{on } \{X'_{m-1} \neq 0\}.$$

Then the sequence $\mathbf{Y}' = (Y'_1, \ldots, Y'_n)$ associated to $\mathbf{X}'$ is a martingale satisfying

$$\mathrm{P}(Y'_m = 0 | \mathcal{F}'_{m-1}) = 1 \qquad \text{on } \{Y'_{m-1} = 0\},$$

and

$$\mathrm{P}(Y'_m = x^{1-(m-1)/(n-1)} | \mathcal{F}'_{m-1}) = x^{1/(n-1)} = 1 - \mathrm{P}(Y'_m = 0 | \mathcal{F}'_{m-1}) \quad \text{on } \{Y'_{m-1} \neq 0\},$$

for $m = 2, \ldots, n$. This is a special case of the martingale in the proof of Proposition 3.7. Hence $V_{\mathcal{F}'}(\mathbf{Y}') = \mathrm{E}\, Y'_1 = x$, and $\mathrm{E}(Y'_1 \vee \cdots \vee Y'_n) = x + (n-1)x\left(1 - x^{1/(n-1)}\right)$.

Case 2. $\beta^j < x \leq \beta^{j-1}$ for some $j \in \{1, \ldots, n-1\}$. In this case, let $X'_m \equiv \min\{x\beta^{-(m-1)}, 1\}$ for $m = 1, \ldots, n$. Then the associated sequence $\mathbf{Y}' = (Y'_1 \ldots, Y'_n)$ is a supermartingale, and so $V_{\mathcal{F}'}(\mathbf{Y}') = \mathrm{E}\, Y'_1 = x$. Moreover,

$$\mathrm{E}(Y'_1 \vee \cdots \vee Y'_n) = \sum_{m=1}^{j} \beta^{m-1}(1-\beta)\, \mathrm{E}\, X'_m + \beta^j\, \mathrm{E}\, X'_{j+1} = j(1-\beta)x + \beta^j.$$

In both cases, it follows that $\mathrm{E}(Y'_1 \vee \cdots \vee Y'_n) = \Omega_{n,\beta}(x)$.

Next, let $\tilde{\mathbf{X}} = (\tilde{X}_1, \ldots, \tilde{X}_n)$ be the sequence $\tilde{X}_m \equiv x$ for $m = 1, \ldots, n$, and define the mixture $\mathbf{X} = (X_1, \ldots, X_n)$ by $\mathrm{P}(\mathbf{X} = \mathbf{X}') = (y - x)/(\Omega_{n,\beta}(x) - x) = 1 - \mathrm{P}(\mathbf{X} = \tilde{\mathbf{X}})$. Then the sequence $\mathbf{Y} = (Y_1, \ldots, Y_n)$ associated to $\mathbf{X}$ is a supermartingale, $V_{\mathcal{F}}(\mathbf{Y}) = x$, and $\mathrm{E}(Y_1 \vee \cdots \vee Y_n) = y$. □

In order to prove Proposition 5.8, the following functions are needed.

Definition 5.9 *(i) For $k \geq 1$ and $0 \leq x \leq \beta^k$, $u_{k,x} : [0,1] \to \mathbb{R}$ is the function*

$$u_{k,x}(y) = \begin{cases} ky(1 - x^{1/k}), & y \leq x \\ y - x + ky(1 - y^{1/k}), & x < y \leq \beta^k \\ j(1-\beta)y + \beta^j - x, & \beta^j < y \leq \beta^{j-1}, \quad j = 1, \ldots, k. \end{cases}$$

(ii) For $k \geq 1$ and $\beta^k \leq x \leq \beta^{k-1}$, $w_{k,x} : [0,1] \to \mathbb{R}$ is the function

$$w_{k,x}(y) = \begin{cases} \left[(k-1)(1-\beta) + 1 - x\beta^{-(k-1)}\right] y, & y \leq \beta^k \\ (k-1)(1-\beta)y + \beta^k - \beta x, & \beta^k < y \leq x \\ j(1-\beta)y + \beta^j - x, & \max(x, \beta^j) < y \leq \beta^{j-1}, \quad j = 1, \ldots, k. \end{cases}$$

Lemma 5.10 *(i) If $x \leq \beta^{k+1}$, then*

$$\hat{u}_{k,x}(y) = \begin{cases} (k+1)\left(1 - x^{1/(k+1)}\right) y, & y \leq x^{k/(k+1)} \\ u_{k,x}(y), & y > x^{k/(k+1)}. \end{cases}$$

(ii) If $\beta^{k+1} \leq x \leq \beta^k$, then

$$\hat{u}_{k,x}(y) = \begin{cases} \left[k(1-\beta) + 1 - x\beta^{-k}\right] y, & y \leq \beta^k \\ u_{k,x}(y), & y > \beta^k. \end{cases}$$

Proof. An examination of the slope of $u_{k,x}$ reveals that $u_{k,x}$ is linear on $[0,x]$ and concave on $[x,1]$, with an increase in slope at $y = x$. Let

$$M = \max_{0<y\leq 1} \frac{u_{k,x}(y)}{y}.$$

The slope of $u_{k,x}$ on $[\beta^k, 1]$ is less than the slope on $[0,x]$, and $u_{k,x}$ is strictly concave on $[x, \beta^k]$. Thus, there is a unique y^* in $[x, \beta^k]$ such that $My^* = u_{k,x}(y^*)$. The function ϕ defined by $\phi(y) = My$ if $y \leq y^*$, and $= u_{k,x}(y)$ if $y > y^*$, is clearly concave and dominates $u_{k,x}$ on $[0,1]$. Since any smaller function dominating $u_{k,x}$ would fail to be concave on $[0, y^*]$, we conclude that $\phi = \hat{u}_{k,x}$.

It remains to calculate M and y^*. Routine differentiation of the function $h(y) := 1 - x/y + k(1 - y^{1/k})$ for $x < y \leq \beta^k$ shows that $y^* = x^{k/(k+1)}$, $M = (k+1)(1 - x^{1/(k+1)})$ if $x \leq \beta^{k+1}$; and $y^* = \beta^k$, $M = k(1-\beta) + 1 - x\beta^{-k}$ otherwise. The lemma follows. □

Lemma 5.11

$$\hat{w}_{k,x}(y) = \begin{cases} \left[(k-1)(1-\beta) + 1 - x\beta^{-(k-1)}\right] y, & y \leq \beta^{k-1} \\ j(1-\beta)y + \beta^j - x, & \beta^j < y \leq \beta^{j-1}, \quad j = 1, \ldots, k-1. \end{cases}$$

Proof. It is easily checked that the function $w_{k,x}$ is continuous and piecewise linear on $[0,1]$, and is concave on $[\beta^{k-1},1]$. Moreover, the slope of $w_{k,x}$ on $[0,\beta^k]$ is both greater than the slope on $[\beta^k,x]$, and greater than the slope anywhere on $[\beta^{k-1},1]$. Define $\phi(y) = \left[(k-1)(1-\beta)+1-x\beta^{-(k-1)}\right] y$ for $y \geq 0$. Then ϕ is linear, and a simple calculation shows that $\phi(0) = w_{k,x}(0)$, and $\phi(\beta^{k-1}) = w_{k,x}(\beta^{k-1})$. Hence $\hat{w}_{k,x}(y) = \phi(y)$ for $y \leq \beta^{k-1}$, and $= w_{k,x}(y)$ for $y > \beta^{k-1}$. $\Box$

Proof of Proposition 5.8. The proof will be by induction on k. Note first that

$$\begin{aligned}\mathrm{E}[f_2(x,X_2)|X_1] &= x + \beta\,\mathrm{E}[(X_2-x)^+|X_1] \\ &\leq x + \beta\,\mathrm{E}[X_2|X_1](1-x) = x + \max\{X_1,\beta\}(1-x),\end{aligned}$$

from which the statement of the proposition follows for $k=2$.

Next, let $l \geq 2$, and suppose statements (i) and (ii) of Proposition 5.8 are both true for $k=l$. Write

$$\begin{aligned}\mathrm{E}[f_{l+1}(x,X_2,\ldots,X_{l+1})|X_1] = (1-\beta)x &+ \beta\,\mathrm{E}\left[\mathrm{E}[f_l(x,X_3,\ldots,X_{l+1})|X_2]I_{\{X_2\leq x\}}|X_1\right] \\ &+ \beta\,\mathrm{E}\left[\mathrm{E}[f_l(X_2,X_3,\ldots,X_{l+1})|X_2]I_{\{X_2>x\}}|X_1\right].\end{aligned}$$

Now if $x \leq \beta^{l-1}$, the induction hypothesis yields

$$\begin{aligned}&\mathrm{E}[f_{l+1}(x,X_2,\ldots,X_{l+1})|X_1] \\ &\quad\leq (1-\beta)x + \beta\,\mathrm{E}\left[x + (l-1)X_2\left(1-x^{1/(l-1)}\right)I_{\{X_2\leq x\}}|X_1\right] \\ &\qquad + \beta\,\mathrm{E}\left[\left\{X_2 + (l-1)X_2\left(1-X_2^{1/(l-1)}\right)\right\}I_{\{x<X_2\leq\beta^l\}}|X_1\right] \\ &\qquad + \beta\sum_{j=1}^{l-1}\mathrm{E}\left[\{j(1-\beta)X_2+\beta^j\}I_{\{\beta^j<X_2\leq\beta^{j-1}\}}|X_1\right] \\ &\quad= x + \beta\,\mathrm{E}[u_{l-1,x}(X_2)|X_1] \leq x + \beta\hat{u}_{l-1,x}(\mathrm{E}[X_2|X_1]) \leq x + \beta\hat{u}_{l-1,x}(X_1/\beta),\end{aligned} \tag{23}$$

where the second inequality follows from Jensen's inequality, and the last inequality follows since $\hat{u}_{l-1,x}(y)$ is increasing in y, and by the supermartingale property of $X_1, \beta X_2, \ldots$. Statement (i) now follows for $k=l+1$ since if $x \leq \beta^l$, then $X_1/\beta \leq x/\beta \leq x^{(l-1)/l}$, and so Lemma 5.10 (i) gives

$$\beta\hat{u}_{l-1,x}(X_1/\beta) = l\left(1-x^{1/l}\right)X_1.$$

For statement (ii), we first show that if $\beta^j < x \leq \beta^{j-1}$ for $j=1,2,\ldots,l$, then

$$\mathrm{E}[f_{l+1}(x,X_2,\ldots,X_{l+1})|X_1] \leq x + \beta\hat{w}_{j,x}(\mathrm{E}[X_2|X_1]).$$

For $j = l$, this follows from the development (23) since $x \geq \beta^l$ implies that $\hat{u}_{l-1,x} = \hat{w}_{l,x}$. If $j < l$, the induction hypothesis implies

$$\begin{aligned}
&\mathrm{E}[f_{l+1}(x,X_2,\ldots,X_{l+1})|X_1] \\
&\quad \leq (1-\beta)x + \beta\,\mathrm{E}\left[\left\{x + \left((j-1)(1-\beta) + 1 - x\beta^{-(j-1)}\right)X_2\right\} I_{\{X_2 \leq \beta^j\}}|X_1\right] \\
&\qquad + \beta\,\mathrm{E}\left[\{(j-1)(1-\beta)X_2 + \beta^j + (1-\beta)x\} I_{\{\beta^j < X_2 \leq x\}}|X_1\right] \\
&\qquad + \beta \sum_{r=1}^{j} \mathrm{E}\left[\{r(1-\beta)X_2 + \beta^r\} I_{\{\max(x,\beta^r) < X_2 \leq \beta^{r-1}\}}|X_1\right] \\
&\quad = x + \beta\,\mathrm{E}[w_{j,x}(X_2)|X_1] \leq x + \beta\hat{w}_{j,x}(\mathrm{E}[X_2|X_1]).
\end{aligned}$$

There are now three cases to be considered (the second case being empty if $j = 1$):

Case 1. $X_1 \leq \beta^j$. Then $X_1/\beta \leq \beta^{j-1}$, and by Lemma 5.11,

$$\beta\hat{w}_{j,x}(\mathrm{E}[X_2|X_1]) \leq \beta\hat{w}_{j,x}(X_1/\beta) = \left[(j-1)(1-\beta) + 1 - x\beta^{-(j-1)}\right] X_1.$$

Case 2. $\beta^j < X_1 \leq \beta$. Then $\beta^{j-1} < X_1/\beta \leq \beta^{j-2}$ (since $X_1 \leq x \leq \beta^{j-1}$), and so

$$\beta\hat{w}_{j,x}(\mathrm{E}[X_2|X_1]) \leq \beta\hat{w}_{j,x}(X_1/\beta) = (j-1)(1-\beta)X_1 + \beta^j - \beta x.$$

Case 3. $\beta < X_1$. Then certainly $x > \beta$ so that $j = 1$, and hence

$$\beta\hat{w}_{j,x}(\mathrm{E}[X_2|X_1]) \leq \beta\hat{w}_{1,x}(1) = \beta(1-x).$$

In each case, statement (ii) follows for $k = l + 1$. This completes the proof. □

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