Optimal stopping rules for American and Russian options in a correlated random walk model

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Abstract

Optimal stopping rules are developed for the American call option and the Russian option under a correlated random walk model. The optimal rules are of twin threshold form: one threshold for stopping after an up-step, and another for stopping after a down-step. Depending on the choice of parameter values, one of the thresholds may be infinite. Precise expressions for the thresholds and optimal expected returns are given both in the positively and the negatively correlated case, for problems both with and without discounting. The optimal rules are illustrated by several numerical examples.

Key words and phrases: Correlated random walk; stopping rule; optimality principle; discount factor; directional reinforcement

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1 Introduction

This paper presents solutions to two optimal-stopping problems associated with a multiplicative correlated random walk. These problems may be interpreted as giving the optimal exercise time of a certain type of option when the stock price is subject to directional reinforcement, or momentum. The model to be used is as follows: For $n \in \mathbb{N}$, let $S_n = S_0 + X_1 + \cdots + X_n$,

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where S_0 is any integer, and $\{X_n\}_{n\geq 0}$ is the unique $\{-1,1\}$-valued Markov chain such that

$$\mathrm{P}(X_1=1|X_0=1)=p, \qquad \mathrm{P}(X_1=-1|X_0=-1)=q.$$

Here p and q are real numbers in $(0,1)$, not necessarily summing to 1. The process $\{S_n\}$ is called a *correlated random walk* (CRW); if $p+q=1$ it is a standard Bernoulli random walk. But if $p+q>1$, it exhibits directional reinforcement or momentum, in the sense that it is more likely to continue in the same direction than to turn around. This is called the *positively correlated case* below. Vice versa, if $p+q<1$, then the walk reverses its direction more often than a standard random walk. We will call this the *negatively correlated case.*

Let $a>1$, $0<\beta\leq 1$ and $K>0$ be real constants. The two optimal-stopping problems investigated in this paper are:

Problem 1. (*Perpetual American call option*) Maximize over all stopping rules τ the expectation

$$\mathrm{E}\left[\beta^\tau\left(a^{S_\tau}-K\right)^+\right].$$

Problem 2. (*Russian option*) Maximize over all stopping rules τ the expectation

$$\mathrm{E}\left[\beta^\tau a^{M_\tau}\right],$$

where $M_n:=\max\{m,S_0,S_1,\dots,S_n\}$, m being an arbitrary integer.

The exponentiated process $\{a^{S_n}\}$ can be called a *multiplicative correlated random walk.* In the case of independent steps it is simply a special case of the binomial model or Cox-Ross-Rubinstein model, which has been widely used as a discrete-time analog of the Black-Scholes model of geometric Brownian motion. Optimal stopping problems such as Problems 1 and 2 above have been solved for many types of stochastic processes, including the binomial model as well as more general partial sums of i.i.d. random variables (e.g. Dubins and Teicher [9], Darling et al. [7], Ferguson and McQueen [11], and Kramkov and Shiryaev [14]); geometric Brownian motion (Shepp and Shiryaev [17]); Lévy processes (Mordecki [15]); and general one-dimensional diffusions (e.g. Dayanik and Karatzas [8]). A general treatment of optimal stopping in discrete time can be found in Chapter II of Shiryaev [18].

What all the above processes lack, however, is a form of directional reinforcement. By contrast, the CRW model provides an extension of the binomial model that allows for dependence between steps, yet is sufficiently simple that explicit optimal stopping rules can be obtained. The model is motivated by the popular notion among stock market technical analysts that movements in security prices are not memoryless. Rather, price changes from one day to the next affect succeeding day changes. Various assertions have been given to justify this concept of momentum. Of course, it is far from agreed that the market consistently exhibits any behavior other than a random walk. Rather than taking a stand in the discussion, this paper aims to provide insight into how an investor might take advantage of momentum, if present, by examining a simple model of such processes. The work builds on earlier investigations in this direction by Allaart and Monticino [3, 4] and Allaart [1, 2].

Correlated random walks were introduced by Goldstein [12], and were subsequently used to model various kinds of physical and biological processes. See, for instance, Henderson and Renshaw [13], or Renshaw and Henderson [16]. The theoretical properties of CRW with and without barriers have been well developed, and much is known about these processes now. See Chen and Renshaw [5] for a comprehensive list of references.

This paper is organized as follows. Section 2 summarizes the main results and illustrates the optimal rules with a few concrete examples. Section 3 introduces notation and reviews a fundamental principle of infinite-horizon optimal stopping. Subsequently, Problem 1 is solved in Section 4, and Problem 2 in Section 5.

2 Summary and numerical examples

The call option problem considered here (Problem 1 above) is similar in nature to a problem studied by the author (Allaart [1]), where the objective function $\beta^n S_n$ was used. What makes the present model more interesting from an applied perspective is its more direct financial interpretation, since the process a^{S_n} cannot become negative. Mathematically, the problem considered here is more subtle because the value of the option need not be finite, and if it is, an optimal stopping rule need not exist. More specifically, given p, q and β, there is a critical value a_0 for the parameter a such that:

- if $a > a_0$, then the value of the option is infinite;
- if $a = a_0$, then the value is finite but no optimal rule exists;
- if $a < a_0$, then an optimal rule exists and is of the form

$$\tau_{r,l} := \inf\{n : S_n \geq r \text{ and } X_n = 1, \text{ or } S_n \geq l \text{ and } X_n = -1\},$$

where one (but not both) of the thresholds r and l may be infinite.

Example 2.1. Let $a = 1.05$, $p = q = 0.55$, $\beta = 0.99$, and $K = 2$. By Theorem 4.8 below, the optimal rule is $\tau_{34,23}$. To two decimal places, $a^{34} \doteq 5.25$, and $a^{23} \doteq 3.07$. So, if the price of the stock is \$1.00 at the initial time and the strike price is \$2.00, then it is optimal to proceed as follows: First, wait for the stock price to exceed \$3.07. Subsequently, stop as soon as the price either hits \$5.25, or takes a down-step, whichever comes first.

Suppose instead that the initial stock price is \$4.00. Now the optimal rule depends on the last step prior to the purchase of the option. If this step was down, the option should be exercised immediately. But if it was up, then it is optimal to wait until the price either hits \$5.25, or takes a down-step, whichever comes first.

Example 2.2 (Negative correlation)**.** Let a, β and K be as in Example 2.1, but now take $p = 0.3$ and $q = 0.5$. By Theorem 4.12 below, the optimal rule is $\tau_{16,19}$. To two decimal places, $a^{16} \doteq 2.18$ and $a^{19} \doteq 2.53$. Thus, if the initial stock price is \$1.00, it is optimal to stop when the price hits \$2.18. On the other hand, if the initial price is $a^{17} = \$2.29$, the optimal rule depends on the last step prior to the purchase of the option. If that step was up, the option should be exercised instantly. If it was down, then it is optimal to stop as soon as the price goes up, unless before that the price falls below \$2.18, in which case one should wait until the price gets back up to \$2.18 and then stop. Finally, if the initial price is above \$2.53, the option should be exercised immediately.

For the case of the Russian option (Problem 2 above) we adapt the method of Allaart [2], where a linear version of the problem with objective function $M_n - cn$ was considered. Here too the value of the option is finite only if a is not too large. When an optimal rule exists, it calls for stopping the first time that the difference $M_n - S_n$ gets above one of two thresholds (again depending on the direction of the most recent step).

Example 2.3. Let $a = 1.05$, $p = 0.6$, $q = 0.5$, and $\beta = 0.99$. Assume $S_0 = m$. By Theorems 5.2 and 5.3 below, it is optimal to stop the first time n at which $M_n - S_n \geq 4$. This comes down to stopping the first time that the price drops below 82% of its running maximum, since $a^{-4} \doteq 0.82$.

3 Preliminaries

Throughout this paper, let $\mathbf{Z}_+$ denote the set of nonnegative integers. Assume that the random variables S_0 and $\{X_n\}_{n \in \mathbf{Z}_+}$ are defined on a suitable probability space $(\Omega, \mathcal{F}, \mathrm{P})$, and define a filtration $\{\mathcal{F}_n\}_{n\in\mathbf{Z}_+}$ by $\mathcal{F}_n := \sigma(\{(S_0, X_0), \ldots, (S_n, X_n)\})$, $n \in \mathbf{Z}_+$. By a *stopping rule* we mean a random variable τ taking values in $\mathbf{Z}_+ \cup \{\infty\}$ such that $\{\tau \le n\} \in \mathcal{F}_n$ for all $n \in \mathbf{Z}_+$. We denote the set of all stopping rules by $\mathcal{T}$.

The process $\{(S_n, X_n)\}_{n\in\mathbf{Z}_+}$ is a bivariate Markov chain. Define the conditional probability measures $\mathrm{P}_{s,+}$ and $\mathrm{P}_{s,-}$ by

$$\mathrm{P}_{s,+}(\cdot) := \mathrm{P}(\cdot|S_0 = s, X_0 = 1), \qquad \mathrm{P}_{s,-}(\cdot) := \mathrm{P}(\cdot|S_0 = s, X_0 = -1),$$

and let $\mathrm{E}_{s,+}$ and $\mathrm{E}_{s,-}$ denote the corresponding expectation operators.

Since all our objective functions are nonnegative, we set the reward for never stopping equal to zero. Thus, throughout this paper, an expectation involving a stopping time τ is always to be read as an integral over the set $\{\tau < \infty\}$ with respect to the relevant probability measure. For example, $\mathrm{E}_{s,+}[\beta^\tau (a^{S_\tau} - K)^+]$ is to be read as $\int_{\{\tau<\infty\}} \beta^\tau (a^{S_\tau} - K)^+ \, d\,\mathrm{P}_{s,+}$.

This section ends with a brief review of the theory of infinite-horizon optimal stopping. Suppose random variables $\{Y_n\}_{n\in\mathbf{Z}_+\cup\{\infty\}}$ are given, and a stopping rule τ is sought that will maximize $\mathrm{E}Y_\tau$. Call a stopping rule τ^* *optimal* if $\mathrm{E}Y_{\tau^*} = \sup_{\tau\in\mathcal{T}} \mathrm{E}\, Y_\tau$. The following conditions are basic:

A1: $\mathrm{E}(\sup_n Y_n) < \infty$;
A2: $\limsup_{n\to\infty} Y_n \le Y_\infty$ almost surely.

It is classical knowledge (e.g. Theorem 4.5′ of Chow et al. [6]) that under conditions A1 and A2, an optimal stopping rule exists and is given by the *principle of optimality*: stop the first time that the current return is at least as large as the optimal expected return if you continue at least one more time period. More precisely, the rule

$$\tau^* := \inf \left\{ n \ge 0 : Y_n \ge \operatorname*{ess\,sup}_{\tau>n} \mathrm{E}(Y_\tau | Y_0, \ldots, Y_n) \right\}$$

is optimal, and it is the smallest (fastest) among all optimal rules. In both problems considered in this paper we take $Y_\infty \equiv 0$, so that A2 becomes

A2′: $\limsup_{n\to\infty} Y_n = 0$ almost surely.

4 Optimal rules for the American call option

In this section we solve Problem 1. For $\tau \in \mathcal{T}$ and $k \in \mathbf{Z}$, let

$$W_{k,\cdot}(\tau) := \mathrm{E}_{k,\cdot} \left[\beta^\tau \left(a^{S_\tau} - K \right)^+ \right],$$

where "$\cdot$" may be $+$ or $-$. Let

$$V(k,\cdot) := \sup_{\tau\in\mathcal{T}} W_{k,\cdot}(\tau), \qquad k \in \mathbf{Z}. \tag{4.1}$$

We aim to compute $V(k,\cdot)$ and determine an optimal stopping rule τ^*, if one exists. The optimal rule τ^* can be thought of as the optimal exercise time of a perpetual American call option with strike price K when the stock price follows a multiplicative correlated random walk on $\{a^k : k \in \mathbf{Z}\}$.

In the analysis of this problem, the first-entrance times

$$\tau_m := \inf\{n \geq 0 : S_n = m\} \qquad (m \in \mathbf{Z}).$$

will play an important role. Define

$$\varrho_+ := \mathrm{E}_{0,+}\,\beta^{\tau_1}, \qquad \text{and} \qquad \varrho_- := \mathrm{E}_{0,-}\,\beta^{\tau_1}. \tag{4.2}$$

If $\beta = 1$, we have $\varrho_\pm = \mathrm{P}_{0,\pm}(\tau_1 < \infty)$ by the convention of integrating only over the set $\{\tau_1 < \infty\}$. If $m > k$, it follows easily from the Markov property and spatial homogeneity of the process (S_n, X_n) that

$$\mathrm{E}_{k,+}\beta^{\tau_m} = \varrho_+^{m-k}, \qquad \mathrm{E}_{k,-}\beta^{\tau_m} = \varrho_-\varrho_+^{m-k-1}. \tag{4.3}$$

Furthermore, ϱ_+ and ϱ_- solve the system of equations

$$\varrho_+ = \beta p + \beta(1-p)\varrho_-\varrho_+, \tag{4.4}$$

$$\varrho_- = \beta(1-q) + \beta q\varrho_-\varrho_+. \tag{4.5}$$

From this it follows that $\varrho_+ \geq \varrho_-$ if and only if $p + q \geq 1$. Finally, in the special case $\beta = 1$, the above equations yield the simple expressions

$$\varrho_+ = \min\left\{\frac{p}{q}, 1\right\}, \qquad \varrho_- = \min\left\{\frac{1-q}{1-p}, 1\right\}. \tag{4.6}$$

If $\beta < 1$, ϱ_+ and ϱ_- can easily be computed from (4.4) and (4.5). In particular, ϱ_+ is the smallest root of the equation

$$\beta q\varrho_+^2 - [1 + \beta^2(p+q-1)]\varrho_+ + \beta p = 0. \tag{4.7}$$

We will frequently use the stopping rule

$$\sigma := \inf\{n \geq 1 : X_n = -1\}.$$

That is, σ is the time of the first down-step. Under $P_{s,+}$, σ has a geometric distribution with parameter $1-p$, and hence, for $a^{s-1} \geq K$,

$$W_{s,+}(\sigma) = \sum_{n=1}^{\infty} \beta^n (a^{s+n-2} - K) p^{n-1}(1-p) = \begin{cases} \beta(1-p)\left(\frac{a^{s-1}}{1-\beta pa} - \frac{K}{1-\beta p}\right), & \text{if } \beta pa < 1 \\ \infty, & \text{if } \beta pa \geq 1. \end{cases} \tag{4.8}$$

4.1 The case $\varrho_+ a > 1$

We are now ready to solve problem (4.1). Note first that if $\varrho_+ a > 1$, then

$$W_{k,\pm}(\tau_m) = \varrho_\pm \varrho_+^{m-k-1}(a^m - K) \to \infty \qquad \text{as } m \to \infty,$$

so that $V \equiv \infty$. One might ask whether there exists an 'optimal' rule; that is, a $\tau \in \mathcal{T}$ such that $W_{k,\pm}(\tau) = \infty$. The answer is "yes" if $\beta pa \geq 1$ (let $\tau = \inf\{n \geq 1 : X_{n-1} = 1,\ X_n = -1\}$), and "yes, provided randomized stopping rules are permitted", if $\beta pa < 1$. (Let N be a random variable independent of the process $\{(S_n, X_n)\}$, having a geometric distribution with parameter $\varepsilon > 0$ such that $\varrho_+ a(1-\varepsilon) > 1$, and consider the randomized rule $\tau = \inf\{n : S_n \geq N\}$.)

4.2 The case $\varrho_+ a = 1$

In this subsection we show that if $\varrho_+ a = 1$, then the value function V is finite but no optimal stopping rule exists. The following result is helpful for the analysis of both the present case and the case $\varrho_+ a < 1$, to be discussed in the next subsection.

Lemma 4.1. *If $\varrho_+ a \leq 1$, then $\lim_{n\to\infty} \beta^n a^{S_n} = 0$ almost surely.*

Proof. If $\beta = 1$, then $p < q$ by (4.6). Thus the walk has a negative drift, and $S_n \to -\infty$ almost surely, or equivalently, $a^{S_n} \to 0$ almost surely.

Assume therefore that $\beta < 1$. Setting $\beta^\infty \equiv 0$, we have

$$\limsup_{n\to\infty} \beta^n a^{S_n} = \limsup_{j\to\infty} \beta^{\tau_j} a^j. \tag{4.9}$$

Without loss of generality, assume the walk starts in state $(0, +)$. Then

$$\tau_j = \tau_1 + (\tau_2 - \tau_1) + \cdots + (\tau_j - \tau_{j-1}) \stackrel{d}{=} \tau_1^{(1)} + \tau_1^{(2)} + \cdots + \tau_1^{(j)},$$

where $\tau_1^{(1)}, \ldots, \tau_1^{(j)}$ are independent copies of τ_1, and we arbitrarily put $\infty - \infty := 0$. Putting $Y_k := (\log \beta)\tau_1^{(k)} + \log a$ for $k \in \mathbb{N}$, we can therefore write

$$\log\left(\beta^{\tau_j} a^j\right) = Y_1 + \cdots + Y_j,$$

a partial sum of i.i.d. random variables, which can possibly take the value $-\infty$. If we can show that $\mathrm{E}_{0,+} Y_1 < 0$, then the strong law of large numbers implies that $\log(\beta^{\tau_j} a^j) \to -\infty$ almost surely, which by (4.9) implies the conclusion of the lemma. Since the function $x \mapsto \beta^x$ is strictly convex and τ_1 is not constant, Jensen's inequality gives

$$\beta^{\mathrm{E}_{0,+}(\tau_1)} < \mathrm{E}_{0,+}\left(\beta^{\tau_1}\right) = \varrho_+ \leq a^{-1}.$$

Taking logarithms on both sides shows that $\mathrm{E}_{0,+} Y_1 < 0$, as required. □

Theorem 4.2. *If $\varrho_+ a = 1$, then $V < \infty$ but no optimal stopping rule exists; more specifically:*

(i) If $p + q \geq 1$, then for all $k \in \mathbf{Z}$,

$$V(k,+) = (\varrho_+/\varrho_-)a^k \qquad \text{and} \qquad V(k,-) = a^k,$$

and the rule $\tau = \inf\{n > \tau_m : X_n = -1\}$ is ε-optimal as $m \to \infty$.

(ii) If $p + q \leq 1$, then for all $k \in \mathbf{Z}$,

$$V(k,+) = a^k \qquad \text{and} \qquad V(k,-) = (\varrho_-/\varrho_+)a^k,$$

and the rule $\tau = \tau_m$ is ε-optimal as $m \to \infty$.

Proof. We prove statement (i); the proof of (ii) is analogous but somewhat simpler. First, multiply both sides of (4.4) by a to get

$$\varrho_- = \frac{1 - \beta p a}{\beta(1-p)}. \tag{4.10}$$

Let $\sigma_m := \inf\{n > \tau_m : X_n = -1\}$. Note that under $\mathrm{P}_{m,+}$, σ_m is simply σ. Hence, for sufficiently large m,

$$W_{k,+}(\sigma_m) = \mathrm{E}_{k,+}\, \beta^{\tau_m} \cdot W_{m,+}(\sigma) = \varrho_+^{m-k} \beta(1-p)\left(\frac{a^{m-1}}{1 - \beta p a} - \frac{K}{1 - \beta p}\right),$$

by (4.8). Since $\varrho_+ a = 1$, it follows using (4.10) that

$$\lim_{m\to\infty} W_{k,+}(\sigma_m) = \frac{\beta(1-p)}{1 - \beta p a}\varrho_+^{1-k} = \frac{\varrho_+}{\varrho_-}a^k.$$

Similarly, $W_{k,-}(\sigma_m) = (\varrho_-/\varrho_+)W_{k,+}(\sigma_m) \to a^k$ as $m \to \infty$.

Now define the function $f : \mathbf{Z} \times \{+,-\} \to \mathbb{R}_+$ by

$$f(k,+) = (\varrho_+/\varrho_-)a^k, \qquad f(k,-) = a^k.$$

The above developments imply that $V \geq f$. In order to establish the reverse inequality, observe first that

$$f(k,+) = \beta[pf(k+1,+) + (1-p)f(k-1,-)],$$

and

$$f(k,-) = \beta[(1-q)f(k+1,+) + qf(k-1,-)].$$

These equalities may be verified using (4.4) and (4.5) and the substitution $a = 1/\varrho_+$. It follows that the process $\beta^n f(S_n, X_n)$, $n \in \mathbf{Z}_+$ is a martingale.

Now let $\tau \in \mathcal{T}$, and put $\beta^\infty f(S_\infty, X_\infty) \equiv 0$. Then:

$$\begin{aligned}
\mathrm{E}\left[\beta^\tau (a^{S_\tau} - K)^+\right] &\leq \mathrm{E}\left[\beta^\tau f(S_\tau, X_\tau)\right] = \int_\Omega \beta^\tau f(S_\tau, X_\tau)\, d\,\mathrm{P} \\
&= \int_\Omega \lim_{n\to\infty} \beta^{\tau\wedge n} f(S_{\tau\wedge n}, X_{\tau\wedge n})\, d\,\mathrm{P} \\
&\leq \liminf_{n\to\infty} \int_\Omega \beta^{\tau\wedge n} f(S_{\tau\wedge n}, X_{\tau\wedge n})\, d\,\mathrm{P} = \mathrm{E}\left[f(S_0, X_0)\right].
\end{aligned}$$

Here the second equality follows from Lemma 4.1, the second inequality by Fatou's lemma, and the last equality by the optional sampling theorem. Taking the supremum over all $\tau \in \mathcal{T}$ shows that $V \leq f$.

Finally, since $f(k,+) \geq f(k,-) > (a^k - K)^+$, it is clear that no optimal stopping rule exists. □

4.3 The case $\varrho_+ a < 1$

Assume for the remainder of this section that $\varrho_+ a < 1$. Our first aim is to show that in this case an optimal stopping rule exists, and to determine its general form.

Lemma 4.3. *If* $\varrho_+ a < 1$, *then* $\mathrm{E}(\sup_n \beta^n a^{S_n}) < \infty$.

Proof. Assume without loss of generality that $S_0 = 0$ and $X_0 = 1$. Then

$$\begin{aligned}
\mathrm{E}\left(\sup_n \beta^n a^{S_n}\right) &= \mathrm{E}\left(\sup_{j:\,\tau_j<\infty} \beta^{\tau_j} a^j\right) \leq \mathrm{E}\left(\sum_{j:\,\tau_j<\infty} \beta^{\tau_j} a^j\right) \\
&= \sum_{j=1}^\infty \mathrm{E}(\beta^{\tau_j} a^j) = \sum_{j=1}^\infty (\varrho_+ a)^j < \infty. \quad \square
\end{aligned}$$

By Lemmas 4.3 and 4.1, $Y_n := \beta^n(a^{S_n} - K)^+$ satisfies conditions A1 and A2′ of Section 3. Hence $V < \infty$, and an optimal rule exists and is given by the principle of optimality. More precisely, the stopping rule

$$\tau^* := \inf\{n \geq 0 : a^{S_n} - K \geq V(S_n, X_n)\}$$

is optimal, and it is the smallest (fastest) among all optimal rules.

Lemma 4.4. *There exist unique numbers r and l in $\mathbf{Z} \cup \{+\infty\}$ such that*

(i) $V(s,+) \leq a^s - K$ if and only if $s \geq r$, and

(ii) $V(s,-) \leq a^s - K$ if and only if $s \geq l$.

Proof. Since any stopping rule τ that stops with positive probability on a level s with $a^s < K$ is clearly suboptimal, we may write

$$\sup_{\tau \geq 1} \mathrm{E}_{s,+}\left[\beta^\tau (a^{S_\tau} - K)^+\right] = \sup_{\tau \geq 1} \mathrm{E}_{0,+}\left[\beta^\tau (a^{s+S_\tau} - K)\right].$$

Thus, $a^s - K \geq V(s,+)$ if and only if

$$\left(1 - \mathrm{E}_{0,+}\, \beta^\tau a^{S_\tau}\right) a^s \geq (1 - \mathrm{E}_{0,+}\, \beta^\tau) K$$

for every $\tau \in \mathcal{T}$ with $\mathrm{P}(\tau \geq 1) = 1$. This yields the first statement. The second follows in the same way. □

It is now clear that τ^* is of one of the following three types. Let

$$\tau_{r,l} := \inf\{n : S_n \geq r \text{ and } X_n = 1, \text{ or } S_n \geq l \text{ and } X_n = -1\} \quad (r, l \in \mathbf{Z}),$$
$$\tau_{r,\infty} := \inf\{n : S_n \geq r \text{ and } X_n = 1\} \qquad (r \in \mathbf{Z}),$$
$$\tau_{\infty,l} := \inf\{n : S_n \geq l \text{ and } X_n = -1\} \qquad (l \in \mathbf{Z}).$$

Then we can write $\tau^* = \tau_{r,l}$, where one (but not both) of the thresholds r and l may be $+\infty$. The real work lies in the exact determination of r and l. The following intuitively obvious fact is in this regard helpful; the proof is analogous to that of Lemma 5.1 in Allaart [1].

Lemma 4.5. *(i) If $p + q \geq 1$, then $V(k,+) \geq V(k,-)$ for all $k \in \mathbf{Z}$, and it holds that $r \geq l$.*

(ii) If $p + q \leq 1$, then $V(k,+) \leq V(k,-)$ for all $k \in \mathbf{Z}$, and it holds that $r \leq l$.

In view of this Lemma we consider the cases of positive and negative correlation separately. First, define the constants

$$\mu_+ := \mathrm{E}_{0,+}\, a^{S_1} = pa + (1-p)a^{-1},$$
$$\mu_- := \mathrm{E}_{0,-}\, a^{S_1} = (1-q)a + qa^{-1}.$$

4.3.1 The positively correlated case

Lemma 4.6. *Suppose $p+q \geq 1$ and $\varrho_+ a < 1$. Then:*
(i) $l < \infty$; and
(ii) $r < \infty$ if and only if $\beta\mu_+ < 1$ or $\beta = \mu_+ = 1$.

Proof. Since r and l cannot both be infinite, $l < \infty$ by Lemma 4.5. Next, $r = \infty$ if and only if in every state $(s, +)$ with $s > l$, the rule $\sigma = \inf\{n \geq 1 : X_n = -1\}$ is strictly better than stopping. By (4.8) and some algebra,

$$W_{s,+}(\sigma) - (a^s - K) = \frac{\beta\mu_+ - 1}{1 - \beta p a} a^s + \frac{1-\beta}{1-\beta p} K.$$

It follows that $W_{s,+}(\sigma) - (a^s - K)$ is negative for all large enough s if $\beta\mu_+ < 1$; is zero if $\beta = \mu_+ = 1$; and is strictly positive for all s in the remaining cases. This yields statement (ii). □

Remark 4.7. If $\beta = \mu_+ = 1$, the above proof shows that in any state $(s, +)$ with $s \geq r$ we are indifferent between stopping and continuing. Thus, for any $m \geq r$ (including $m = \infty$), the rule $\tau_{m,l}$ is optimal. However, $\tau_{r,l}$ is the *fastest* among these optimal rules, and is therefore the rule τ^*.

Next, define the quantities

$$F(s; m) := W_{s+1,+}(\tau_{m,s}), \qquad m \in \mathbf{Z} \cup \{\infty\}, \quad s < m.$$

Then $F(s; \infty) = W_{s+1,+}(\sigma)$, so (4.8) gives

$$F(s; \infty) = \beta(1-p)\left(\frac{a^s}{1-\beta p a} - \frac{K}{1-\beta p}\right), \tag{4.11}$$

provided $a^s > K$. And for $m \in \mathbf{Z}$, $s < m$ with $a^s > K$, we have

$$F(s; m) = \sum_{n=1}^{m-s-1} \beta^n (a^{s+n-1} - K) p^{n-1}(1-p) + (\beta p)^{m-s-1}(a^m - K), \tag{4.12}$$

where the empty sum is taken to be zero.

In the theorem below, c_1 and c_2 denote the constants

$$c_1 := \frac{a(1-\varrho_-)}{1-\varrho_- a}, \qquad c_2 := \frac{a^2(1-\varrho_-\varrho_+)}{1-\varrho_-\varrho_+ a^2}.$$

The inequalities $\varrho_+ \geq \varrho_-$ and $\varrho_+ a < 1$ imply that $c_1 < c_2$.

Theorem 4.8. *Assume $p+q \geq 1$ and $\varrho_+ a < 1$. Then $\tau^* = \tau_{r,l}$, where r and l in $\mathbf{Z} \cup \{\infty\}$ are determined as follows:*

(i) Suppose either $\beta\mu_+ > 1$, or $\beta\mu_+ = 1$ and $\beta < 1$. Then $r = \infty$, and l is the smallest integer s such that

$$\frac{a^s}{K} \geq 1 + \frac{(a-1)(\varrho_+ - \beta p)}{(1-\varrho_+ a)(1-\beta p)}. \tag{4.13}$$

(ii) Suppose $\beta\mu_+ < 1 < \mu_+$. Then r is the smallest integer s such that

$$\frac{a^s}{K} \geq 1 + \max\left\{\frac{\beta(\mu_+ - 1)}{1-\beta\mu_+}, \frac{\varrho_+(a-1)}{1-\varrho_+ a}\right\}. \tag{4.14}$$

Furthermore:

$$l = \begin{cases} r & \text{if } a^r < c_1 K \\ r-1 & \text{if } c_1 K \leq a^r < c_2 K \\ s^* & \text{if } a^r \geq c_2 K, \end{cases}$$

where s^* is the largest integer $s \leq r-2$ such that

$$a^{s-1} - K < \varrho_- \varrho_+ F(s;r). \tag{4.15}$$

(iii) Suppose $\mu_+ \leq 1$. Then r is the smallest integer s such that

$$\frac{a^s}{K} \geq 1 + \frac{\varrho_+(a-1)}{1-\varrho_+ a}, \tag{4.16}$$

and if $a^r < c_1 K$, then $l = r$; else $l = r-1$.

Remark 4.9. If $l \in \{r-1, r\}$ and $S_0 < l$, then $\tau_{r,l} = \tau_r$ because the second condition in the definition of $\tau_{r,l}$ can not occur before the first one does. Thus, the distinction between the possibilities $r = l$ and $r = l-1$ is relevant only if the process (S_n, X_n) begins in state $(r-1, -)$.

Remark 4.10. In the undiscounted case ($\beta = 1$), the condition $\varrho_+ a < 1$ is equivalent to $pa < q$, by (4.6). This implies in particular that $p < q$, so that $\varrho_+ = p/q$, and $\varrho_- = (1-q)/(1-p)$.

Proof of Theorem 4.8. (i) Suppose first that either $\beta\mu_+ > 1$, or $\beta\mu_+ = 1$ and $\beta < 1$. Then $r = \infty$ by Lemma 4.6, so $\tau^* = \tau_{\infty,l}$. Now the inequality

$$a^{s-1} - K < W_{s-1,-}(\tau_{\infty,s}) \tag{4.17}$$

must hold for $s = l$, but fail for all $s > l$. If the walk starts at level $s-1$, the rule $\tau_{\infty,s}$ tells us to first wait until the walk reaches $s+1$, and then stop after the first down-step. Thus, by (4.3) and (4.8),

$$\begin{aligned} W_{s-1,-}(\tau_{\infty,s}) &= \mathrm{E}_{s-1,-}(\beta^{\tau_{s+1}}) W_{s+1,+}(\sigma) \\ &= \varrho_- \varrho_+ \beta(1-p) \left(\frac{a^s}{1-\beta p a} - \frac{K}{1-\beta p} \right). \end{aligned}$$

Using (4.4) and regrouping terms shows that (4.17) is equivalent to

$$\frac{a^{s-1}}{K} < \frac{(1-\varrho_+)(1-\beta p a)}{(1-\varrho_+ a)(1-\beta p)} = 1 + \frac{(a-1)(\varrho_+ - \beta p)}{(1-\varrho_+ a)(1-\beta p)}.$$

Hence, l is the smallest integer s satisfying (4.13).

(ii) Assume next that $\beta\mu_+ < 1 < \mu_+$. By Lemma 4.6, $r < \infty$. Furthermore, the inequality $a^r - K \geq W_{r,+}(\tau)$ must hold for all $\tau \in \mathcal{T}$. Taking $\tau \equiv 1$ in this inequality gives

$$a^r - K \geq \beta\, \mathrm{E}_{r,+}(a^{S_1} - K) = \beta(a^r \mu_+ - K), \tag{4.18}$$

and taking $\tau = \tau_{r+1}$ gives

$$a^r - K \geq \varrho_+(a^{r+1} - K). \tag{4.19}$$

Thus, (4.14) holds for $s = r$. Now either $l \geq r-1$ or $l < r-1$. In the first case, the rule $\tau = \tau_r$ is optimal when starting from state $(r-1,+)$; in the second case, the rule $\tau \equiv 1$ is optimal when starting from that state. Thus, either (4.18) or (4.19) fails when r is replaced with $r-1$. But then (4.14) fails for $s = r-1$, so r is as claimed.

It remains to determine l. By Lemma 4.5, $l \leq r$. Equality holds if and only if in state $(r-1,-)$ it is strictly better to continue than to stop. Thus, $l = r$ if and only if

$$a^{r-1} - K < W_{r-1,-}(\tau_r) = \varrho_-(a^r - K),$$

or equivalently, $a^r < c_1 K$. Similarly, we can conclude by Remark 4.9 that $l \geq r-1$ if and only if

$$a^{r-2} - K < W_{r-2,-}(\tau_r) = \varrho_- \varrho_+(a^r - K),$$

or equivalently, $a^r < c_2 K$. Finally, suppose $a^r \geq c_2 K$, so that $l \leq r-2$. Now from state $(l-1,-)$ the optimal rule must first reach the level $l+1$, and after that, stop as soon as it either reaches level r or takes a down-step. It follows that

$$a^{l-1} - K < W_{l-1,-}(\tau_{r,l}) = \varrho_- \varrho_+ F(l; r).$$

On the other hand, in any state $(s, -)$ with $s \geq l$ it is optimal to stop, so the above inequality must fail if l is replaced throughout with any larger integer. Thus, l is the greatest integer $s \leq r - 2$ satisfying (4.15).

(iii) Finally, assume $\mu_+ \leq 1$, and suppose by way of contradiction that $l \leq r - 2$. Then in state $(r - 1, +)$ the rule $\tau \equiv 1$ must be strictly optimal, so $a^{r-1} - K < \beta\, \mathrm{E}_{r-1,+}(a^{S_1} - K)^+$. But, since $a^{r-2} \geq a^l > K$, we have

$$\beta\, \mathrm{E}_{r-1,+}(a^{S_1} - K)^+ = \beta(a^{r-1}\mu_+ - K) \leq a^{r-1} - K,$$

a contradiction. Therefore, $l \geq r - 1$. The values of r and l now follow as in case (ii), but since $\mu_+ \leq 1$, (4.14) reduces to (4.16). □

4.3.2 The negatively correlated case

The optimal rule is somewhat simpler in the negatively correlated case. The following lemma is analogous to Lemma 4.6, but note that the finiteness conditions in the two lemmas are not entirely symmetric.

Lemma 4.11. *Suppose $p + q \leq 1$ and $\varrho_+ a < 1$. Then:*
(i) $r < \infty$; and
(ii) $l < \infty$ if and only if $\beta\mu_- < 1$.

Proof. Since r and l cannot both be infinite, $r < \infty$ by Lemma 4.5. To prove statement (ii), let $\delta_s := V(s, -) - (a^s - K)$, for $s \in \mathbf{Z}$. We first derive the recursion

$$\delta_s = \max\{0, (\beta\mu_- - 1)a^s + (1 - \beta)K + \beta q \delta_{s-1}\}, \qquad s \geq r. \tag{4.20}$$

To see this, note that $s \geq r$ implies that $V(s + 1, +) = a^{s+1} - K$, and so

$$V(s, -) = \max\{a^s - K, \beta(1 - q)(a^{s+1} - K) + \beta q V(s - 1, -)\}.$$

Subtracting $a^s - K$ and rearranging terms gives (4.20).

Now suppose $\beta\mu_- \geq 1$. By Lemma 4.5, $\delta_{r-1} \geq V(r-1, +) - (a^{r-1} - K) > 0$, while (4.20) implies that $\delta_s > 0$ whenever $\delta_{s-1} > 0$. Hence $V(s, -) > a^s - K$ for all s, and so $l = \infty$.

On the other hand, if $\beta\mu_- < 1$, then, for all large enough s, (4.20) gives $\delta_s \leq \max\{0, \delta_{s-1}\}$, so δ_s is bounded. But then, again for large enough s, (4.20) shows that $\delta_s = 0$. Thus, $l < \infty$. □

Next, define the function

$$G(s;r) := W_{s-1,-}(\tau_{r,s}), \qquad s \geq r.$$

From state $(s-1,-)$, the rule $\tau_{r,s}$ calls for stopping after the first up-step, unless the walk begins with $s-r$ or more consecutive down-steps, in which case it goes below r and we must wait until the walk is back in r. Thus,

$$G(s;r) = \sum_{n=1}^{s-r} \beta^n (a^{s-n+1} - K) q^{n-1}(1-q) + (\beta q)^{s-r} \varrho_-(a^r - K). \tag{4.21}$$

Theorem 4.12. *Assume $p+q \leq 1$, and $\varrho_+ a < 1$. Then $\tau^* = \tau_{r,l}$, where r and l in $\mathbf{Z} \cup \{\infty\}$ are determined as follows:*

(i) The number r is the smallest integer s such that

$$\frac{a^s}{K} \geq 1 + \frac{\varrho_+(a-1)}{1-\varrho_+ a}.$$

(ii) If $\beta\mu_- \geq 1$, then $l = \infty$. Otherwise, l is the largest integer $s \geq r$ such that

$$a^{s-1} - K < G(s;r). \tag{4.22}$$

Proof. Since $r \leq l$ in the negatively correlated case, the value of r follows in the same way as in case (iii) of Theorem 4.8. The question of finiteness of l is answered by Lemma 4.11. Assume $l < \infty$. Then

$$a^{l-1} - K < W_{l-1,-}(\tau_{r,l}) = G(l;r),$$

while the inequality must fail if l is replaced throughout by any greater integer. Thus, l is the largest integer s for which (4.22) holds. □

Remark 4.13. Note that if $r \leq l$, then $\tau_{r,l} = \tau_r$ on the set $\{S_0 < r\}$. Thus, in the negatively correlated case, if the walk begins below the threshold r, the optimal rule becomes a simple threshold rule.

4.3.3 Explicit expressions for the value functions

The calculations of expected returns in the proofs above are easily extended to complete expressions for the value function V. If $r > l+1$, then we have

$$V(k,+) = \begin{cases} \varrho_+^{l-k+1} F(l;r), & k \leq l \\ F(k-1;r), & l+1 \leq k < r \\ a^k - K, & k \geq r, \end{cases}$$

where $F(s;r)$ is given by (4.11) or (4.12), as appropriate. Of course, the last case ($k \geq r$) does not occur when $r = \infty$.

On the other hand, if $r \leq l+1$, then

$$V(k,+) = \begin{cases} \varrho_+^{r-k}(a^r - K), & k < r \\ a^k - K, & k \geq r. \end{cases}$$

Next, $V(k,-)$ can be given in terms of $V(k,+)$ by

$$V(k,-) = \begin{cases} (\varrho_-/\varrho_+)V(k,+), & k < \min\{r,l\} \\ G(k+1;r), & r \leq k < l \\ a^k - K, & k \geq l, \end{cases}$$

where $G(k+1;r)$ is given by (4.21). Here the second case ($r \leq k < l$) is void if $r \geq l$, and the third case ($k \geq l$) is void if $l = \infty$.

5 Optimal rules for the Russian option

In this section we solve Problem 2. Let $M_n := \max\{M_0, S_1, \ldots, S_n\}$ for $n \in \mathbb{N}$, where M_0 is an $\mathcal{F}_0$-measurable, integer-valued random variable with $\mathrm{P}(M_0 \geq S_0) = 1$. Our goal is to compute

$$V(m,s,x) := \sup_{\tau \in \mathcal{T}} \mathrm{E}_{m,s,x}\left[\beta^\tau a^{M_\tau}\right], \tag{5.1}$$

where $\mathrm{E}_{m,s,x}$ denotes expectation given that $M_0 = m, S_0 = s$ and $X_0 = x$. We can think of the function V as giving the value of a *Russian option* when the price process follows a multiplicative CRW on the set $\{a^k : k \in \mathbf{Z}\}$. The Russian option was introduced by Shepp and Shiryaev [17], who solved the corresponding optimal stopping problem in the continuous-time model of geometric Brownian motion. Their work has been extended by several authors, most notably to the finite-horizon case (e.g. Ekström [10]).

Since $V(m,s,x) = a^m V(0, s-m, x)$ when $m \geq s$, it is sufficient to determine the renormalized value functions

$$\tilde{V}(k,x) := V(0,-k,x), \qquad k \in \mathbf{Z}_+,\ x \in \{-1,1\}.$$

Assume from now on that $M_0 \equiv 0$ and $S_0 \leq 0$, and define $Z_n := M_n - S_n$, for $n \in \mathbf{Z}_+$. The process $\{(Z_n, X_n)\}_{n \in \mathbf{Z}_+}$ is a bivariate Markov chain that contains all the information needed to determine an optimal stopping rule. For $\tau \in \mathcal{T}$, let

$$W_{k,x}(\tau) := \mathrm{E}\left[\beta^\tau a^{M_\tau} | M_0 = 0, Z_0 = k, X_0 = x\right],$$

so that $\tilde{V}(k,x) = \sup_\tau W_{k,x}(\tau)$.

Recall the definition of ϱ_+ from (4.2). As in the previous section, it is immediate that $V \equiv \infty$ when $\varrho_+ a > 1$; again, the stopping times τ_m witness this fact. We will see that for this problem, $V \equiv \infty$ even when $\varrho_+ a = 1$. However, a result analogous to that of Theorem 4.2, with a finite value but no optimal stopping rule, does present itself when $\varrho_+ a < 1$ and $\beta = 1$.

5.1 The undiscounted case

Suppose $\beta = 1$. Since a^{M_n} is nondecreasing, we have trivially that $\tilde{V}(k,\pm) = \lim_{n\to\infty} \mathrm{E}_{0,-k,\pm}\, a^{M_n} = \mathrm{E}_{0,-k,\pm}\, a^M$, where $M := \sup_n M_n$. If $pa < q$ (that is, $\varrho_+ a < 1$), these expectations are finite and easily calculated: Under $\mathrm{P}_{0,-k,+}$, M takes the value zero with probability $\mathrm{P}_{0,-k,+}(\tau_1 = \infty) = 1 - \gamma^{k+1}$, and has a geometric distribution with parameter $1-\gamma$ conditionally on $M > 0$, where $\gamma := p/q$. Thus,

$$\begin{aligned}\tilde{V}(k,+) &= \sum_{m=0}^{\infty} a^m\, \mathrm{P}_{0,-k,+}(M = m) \\ &= 1 - \gamma^{k+1} + \sum_{m=1}^{\infty} a^m \gamma^{m+k}(1-\gamma) = 1 + \frac{\gamma(a-1)}{1-\gamma a}\gamma^k.\end{aligned}$$

Similarly,

$$\tilde{V}(k,-) = 1 + \frac{(1-q)(a-1)}{(1-p)(1-\gamma a)}\gamma^k.$$

5.2 The discounted case

Suppose for now that $\varrho_+ a < 1$ and $\beta < 1$. If the stronger inequality $\beta a \le 1$ holds, then $\beta^\tau a^{M_\tau} \le \beta^\tau a^{M_0+\tau} \le a^{M_0}$, so the rule $\tau \equiv 0$ is trivially optimal. Thus, we may assume additionally that $\beta a > 1$. Since $\beta^n a^{S_n}$ can attain its local maxima only at time 0 and at times when $S_n = M_n$, Lemmas 4.1 and 4.3 guarantee the existence of an optimal rule and the finiteness of V. In particular, the rule

$$\tau^* := \inf\{n \ge 0 : \tilde{V}(Z_n, X_n) \le 1\}$$

is optimal, and it is the smallest (fastest) among all optimal rules. Since $\tilde{V}(k,x)$ is nonincreasing in k, there exist unique nonnegative integers d_+ and d_- (one of which could conceivably be infinite) such that

$$\tau^* = \inf\{n \ge 0 : Z_n \ge d_+ \text{ and } X_n = 1, \text{ or } Z_n \ge d_- \text{ and } X_n = -1\}.$$

In the analysis below, we include $(0,-)$ as a possible initial state of the process (Z_n, X_n) for convenience. It follows as in the proof of Lemma 5.1 of Allaart [1], that $\tilde{V}(k,+) \geq \tilde{V}(k,-)$ for all $k \geq 0$ if $p+q \geq 1$, while the reverse inequality holds if $p+q \leq 1$. Thus, $d_+ \geq d_-$ if the correlation is positive, while $d_+ \leq d_-$ if the correlation is negative. On the other hand, regardless of the sign of $p+q-1$, we always have

$$d_+ \geq d_- - 1. \tag{5.2}$$

This is obvious if $d_- \leq 1$. For $d_- \geq 2$ it can be seen as follows. From state $(d_- - 1, -)$ of the process (Z_n, X_n) one can move to state $(d_-, -)$, where it is optimal to stop; or to state $(d_- - 2, +)$. Since it is optimal to continue in state $(d_- - 1, -)$, it must be optimal to continue in state $(d_- - 2, +)$ as well. Thus, $d_+ \geq d_- - 1$.

In view of (5.2), d_- can be determined by considering the simpler stopping rules

$$\tilde{\tau}_d := \inf\{n \geq 0 : Z_n \geq d\}, \qquad d \in \mathbf{Z}_+,$$

and maximizing the expected return $W_{0,-}(\tilde{\tau}_d)$ over d. Once d_- is known, it will be straightforward to determine d_+.

Let λ_1 and λ_2, $\lambda_1 \leq \lambda_2$, be the roots of the equation

$$\beta q \lambda^2 - [1 + \beta^2(p+q-1)]\lambda + \beta p = 0. \tag{5.3}$$

It is not difficult to see that $\lambda_1 = \varrho_+ \leq 1 \leq \lambda_2$, with both inequalities being strict when $\beta < 1$. (See the Appendix.) Define the constants

$$\begin{gathered} A := 1 - \frac{\beta^2(1-p)(1-q)}{1-\beta p a}, \\ \varepsilon_1 := \lambda_1^{-1}(\beta q \lambda_2 - A), \qquad \varepsilon_2 := \lambda_2^{-1}(A - \beta q \lambda_1). \end{gathered} \tag{5.4}$$

The constant A is well defined, since $\beta p a < \varrho_+ a < 1$.

Lemma 5.1. *Assume $\varrho_+ a < 1 < \beta a$. Then:*
(i) $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$; and
(ii) we have

$$W_{k,-}(\tilde{\tau}_d) = \frac{\varepsilon_1 \lambda_1^k + \varepsilon_2 \lambda_2^k}{\varepsilon_1 \lambda_1^d + \varepsilon_2 \lambda_2^d}, \qquad k = 0, 1, \ldots, d. \tag{5.5}$$

Proof. The proof of (i) is given in the Appendix. Statement (ii) is obvious when $d = 0$, so assume $d \geq 1$. Let

$$f_k := W_{k,-}(\tilde{\tau}_d), \qquad g_k := W_{k,+}(\tilde{\tau}_d) \qquad (k = 0, 1, \ldots, d).$$

Then f_k and g_k satisfy the system of difference equations

$$g_k = \beta p g_{k-1} + \beta(1-p) f_{k+1}, \qquad k = 1, \ldots, d-1, \tag{5.6}$$

$$f_k = \beta(1-q) g_{k-1} + \beta q f_{k+1}, \qquad k = 1, \ldots, d-1, \tag{5.7}$$

with boundary conditions

$$f_0 = \beta(1-q) a g_0 + \beta q f_1, \tag{5.8}$$

$$g_0 = \beta p a g_0 + \beta(1-p) f_1, \tag{5.9}$$

$$f_d = g_d = 1. \tag{5.10}$$

From (5.6) and (5.7) we can derive the second order difference equation

$$\beta q f_k - [1 + \beta^2(p+q-1)] f_{k-1} + \beta p f_{k-2} = 0, \tag{5.11}$$

valid for $k \geq 3$. The characteristic equation of (5.11) is (5.3), so its general solution is of the form

$$f_k = C_1 \lambda_1^k + C_2 \lambda_2^k, \qquad k \geq 1. \tag{5.12}$$

Now, by (5.10), $C_1 \lambda_1^d + C_2 \lambda_2^d = 1$. To obtain a second equation, rewrite (5.9) as $(1 - \beta p a) g_0 = \beta(1-p) f_1$, and substitute this into (5.7) with $k = 1$. This gives $A f_1 = \beta q f_2$, which by (5.12) and (5.4) can be expressed as $C_2 = (\varepsilon_2/\varepsilon_1) C_1$. Routine algebra now yields

$$C_1 = \frac{\varepsilon_1}{\varepsilon_1 \lambda_1^d + \varepsilon_2 \lambda_2^d}, \qquad C_2 = \frac{\varepsilon_2}{\varepsilon_1 \lambda_1^d + \varepsilon_2 \lambda_2^d}. \tag{5.13}$$

Thus, for $k \geq 1$, (5.5) follows from (5.12) and (5.13). That f_0 obeys the same expression can be seen by combining (5.8) and (5.9). This yields

$$f_0 = \left(\frac{\beta^2 (1-p)(1-q) a}{1 - \beta p a} + \beta q \right) f_1 = \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1 \lambda_1 + \varepsilon_2 \lambda_2} f_1, \tag{5.14}$$

where the last equality follows using (5.4) and the relationships $\lambda_1 \lambda_2 = p/q$, and $\lambda_1 + \lambda_2 = [1 + \beta^2(p+q-1)]/\beta q$. Hence, (5.5) holds for $k = 0$ as well. □

Theorem 5.2. *Assume $\beta < 1$, and $\varrho_+ a < 1 < \beta a$. Then d_- is the smallest nonnegative integer d such that*

$$\left(\frac{\lambda_1}{\lambda_2} \right)^d \leq \frac{\varepsilon_2 (\lambda_2 - 1)}{\varepsilon_1 (1 - \lambda_1)},$$

and

$$\tilde{V}(k, -) = \begin{cases} \dfrac{\varepsilon_1 \lambda_1^k + \varepsilon_2 \lambda_2^k}{\varepsilon_1 \lambda_1^{d_-} + \varepsilon_2 \lambda_2^{d_-}}, & k = 0, 1, \ldots, d_- \\ 1, & k \geq d_-. \end{cases}$$

Proof. The first statement follows by solving the inequality $W_{0,-}(\tilde{\tau}_{d+1}) \leq W_{0,-}(\tilde{\tau}_d)$ using (5.5). The second statement is immediate. $\square$

Now that d_- is known, it is relatively easy to find d_+ and an expression for $\tilde{V}(k,+)$. Note first that if $X_0 = 1$, then the state $(0,-)$ can never be visited, and so the distinction between the possibilities $d_- = 0$ and $d_- = 1$ is irrelevant. Therefore, set $d := \max\{d_-, 1\}$, and define f_k and g_k as in the proof of Lemma 5.1. If $d = 1$, then $f_1 = 1$, and (5.9) gives

$$g_{d-1} = g_0 = \frac{\beta(1-p)}{1-\beta pa}. \tag{5.15}$$

If $d \geq 2$, then (5.6) and (5.7) give

$$\begin{aligned} g_{d-1} &= \beta p g_{d-2} + \beta(1-p) f_d = \frac{p}{1-q}(f_{d-1} - \beta q) + \beta(1-p) \\ &= \frac{p}{1-q} f_{d-1} - \beta\left(\frac{p+q-1}{1-q}\right), \end{aligned} \tag{5.16}$$

where f_{d-1} is given by (5.5). It can be seen, using (5.15) and a calculation analogous to the one yielding (5.14), that (5.16) gives the correct expression for g_{d-1} even in the case $d = 1$.

Suppose now that $(Z_0, X_0) = (k,+)$ with $k \geq d-1$, and consider the rule $\tau = \inf\{n \geq 1 : X_n = -1 \text{ and } Z_n \geq d\}$. This rule has expected return

$$\begin{aligned} W_{k,+}(\tau) &= \sum_{n=1}^{k-d+1} \beta^n p^{n-1}(1-p) \cdot 1 + \beta^{k-d+1} p^{k-d+1} \cdot g_{d-1} \\ &= \eta + (g_{d-1} - \eta)(\beta p)^{k-d+1}, \end{aligned}$$

where $\eta := \beta(1-p)/(1-\beta p) < 1$. Now if $g_{d-1} \leq 1$, then $d_+ = d - 1$. Otherwise, $W_{k,+}(\tau)$ decreases to η, and d_+ is the smallest k such that $W_{k,+}(\tau) \leq 1$. We summarize the results in

Theorem 5.3. *Assume $\beta < 1$, and $\varrho_+ a < 1 < \beta a$. Let $d := \max\{d_-, 1\}$, and define*

$$v := \frac{p}{1-q}\left(\frac{\varepsilon_1 \lambda_1^{d-1} + \varepsilon_2 \lambda_2^{d-1}}{\varepsilon_1 \lambda_1^d + \varepsilon_2 \lambda_2^d}\right) - \beta\left(\frac{p+q-1}{1-q}\right), \qquad \eta := \frac{\beta(1-p)}{1-\beta p}.$$

(i) If $v \leq 1$, then $d_+ = d-1$. Otherwise, d_+ is the smallest integer $k \geq d$ such that

$$(\beta p)^{k-d+1} \leq \frac{1-\eta}{v-\eta}.$$

(ii) We have

$$\tilde{V}(k,+) = \begin{cases} \frac{\varepsilon_1(1-\beta q\lambda_1)\lambda_1^{k+1}+\varepsilon_2(1-\beta q\lambda_2)\lambda_2^{k+1}}{\beta(1-q)(\varepsilon_1\lambda_1^d+\varepsilon_2\lambda_2^d)}, & 0 \le k < d-1 \\ \eta + (v-\eta)(\beta p)^{k-d+1}, & d-1 \le k < d_+ \\ 1, & k \ge d_+. \end{cases}$$

5.2.1 The borderline case: $\varrho_+ a = 1$

To complete the analysis, suppose finally that $\varrho_+ a = 1$. It then follows (see the Appendix) that $\varepsilon_2 = 0$, so $W_{k,-}(\tilde{\tau}_d) = \lambda_1^{k-d} \to \infty$ as $d \to \infty$. Similarly, $W_{k,+}(\tilde{\tau}_d) \to \infty$. Thus, $V \equiv \infty$.

Appendix

Proof of Lemma 5.1,(i). Let

$$f(\lambda) := \beta q\lambda^2 - [1 + \beta^2(p+q-1)]\lambda + \beta p. \tag{5.17}$$

The graph of f is a parabola opening upward. Moreover, $f(0) = \beta p > 0$, and $f(1) = (1-\beta)[\beta(p+q-1)-1] \le 0$, with strict inequality if $\beta < 1$. It follows that $\lambda_1 \le 1 \le \lambda_2$, both inequalities being strict when $\beta < 1$. Finally, comparison of (5.17) and (4.7) shows that $\lambda_1 = \varrho_+$.

To show that $\varepsilon_1 > 0$ when $\beta a > 1$, calculate

$$\beta q\lambda_2 - A = \beta q\lambda_2 - 1 + \frac{\beta^2(1-p)(1-q)}{1-\beta pa} > \beta q\lambda_2 - 1 + \beta^2(1-q).$$

Thus it suffices to show that

$$\lambda_2 \ge \frac{1-\beta^2(1-q)}{\beta q}. \tag{5.18}$$

But (5.18) follows from the easily verified fact that

$$f\left(\frac{1-\beta^2(1-q)}{\beta q}\right) = -\beta(1-\beta^2)\left(\frac{p}{q}\right)(1-q) \le 0.$$

Next, suppose $\varrho_+ a < 1$. Then $\lambda_1 = \varrho_+ < 1/a < 1 \le \lambda_2$, and hence $f(1/a) < 0$. Equivalently, $\beta pa^2 - [1+\beta^2(p+q-1)]a + \beta q < 0$, which can be reworked to

$$\frac{\beta q}{a} < 1 - \frac{\beta^2(1-p)(1-q)}{1-\beta pa}.$$

Therefore, $A > \beta q/a > \beta q\varrho_+ = \beta q\lambda_1$, and hence $\varepsilon_2 > 0$. □

Note that the above proof also shows that $\varepsilon_2 = 0$ exactly when $\varrho_+ a = 1$.

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References

[1] Allaart, P. C. (2004a). Optimal stopping rules for correlated random walks with a discount. *J. Appl. Probab.* **41**, no. 2, 483-496.

[2] Allaart, P. C. (2004b). Stopping the maximum of a correlated random walk, with cost for observation. *J. Appl. Probab.* **41**, no. 4, 998-1007.

[3] Allaart, P. C. and Monticino, M. G. (2001). Optimal stopping rules for directionally reinforced processes. *Adv. Appl. Probab.* **33**, no. 2, 483-504.

[4] Allaart, P. C. and Monticino, M. G. (2008). Optimal buy/sell rules for correlated random walks *J. Appl. Probab.* **45**, 33-44.

[5] Chen, A. and Renshaw, E. (1994). The general correlated random walk. *J. Appl. Probab.* **31**, 869-884.

[6] Chow, Y. S., Robbins, H. and Siegmund, D. (1971). *Great expectations: the theory of optimal stopping.* Houghton Mifflin, Boston, MA.

[7] Darling, D. A., Liggett, T. and Taylor, H. M. (1972). Optimal stopping for partial sums. *Ann. Math. Statist.* **43**, 1363-1368.

[8] Dayanik, S. and Karatzas, I. (2003). On the optimal stopping problem for one-dimensional diffusions. *Stoch. Process. Appl.* **107**, no. 2, 173-212.

[9] Dubins, L. E. and Teicher, H. (1967). Optimal stopping when the future is discounted. *Ann. Math. Statist.* **38**, 601-605.

[10] Ekström, E. (2004). Russian options with a finite time horizon. *J. Appl. Probab.* **41**, no. 2, 313-326.

[11] Ferguson, T. S. and MacQueen, J. B. (1992). Some time-invariant stopping rule problems. *Optimization* **23**, 155-169.

[12] Goldstein, S. (1951). On diffusion by discontinuous movements, and on the telegraph equation. *Quart. J. Mech.* **4**, 129-156.

[13] Henderson, R. and Renshaw, E. (1980). Spatial stochastic models and computer simulation applied to the study of tree root systems. *Compstat* **80**, 389-395.

[14] Kramkov, D. O. and Shiryaev, A. N. (1994). On the rational pricing of the "Russian option" for the symmetrical binomial model of a (B, S)-market. *Theory of Probab. Appl.* **39**, no. 1, 153-162.

[15] Mordecki, E. (2002). Optimal stopping and perpetual options for Lévy processes. *Finance Stoch.* **6**, no. 4, 473-493.

[16] Renshaw, E. and Henderson, R. (1981). The correlated random walk. *J. Appl. Probab.* **18**, 403-414.

[17] Shepp, L. and Shiryaev, A. N. (1993). The Russian option: reduced regret. *Ann. Appl. Probab.* **3**, no. 3, 631–640.

[18] Shiryaev, A. N. (1978). *Optimal stopping rules.* Springer-Verlag, Berlin.