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Sensor networks

Mobility Accelerates Consensus Building in Sensor Networks

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Abstract—The primary goal of this letter is to study the impact of mobility in wireless sensor networks modeled as a leader-follower structure. Although it is intuitively known that mobility enhances the convergence rate of consensus-building in a sensor network, analytical reasoning for this intuition is not available in the literature. For filling the gap, this letter provides concrete proofs to demonstrate the benefits of introducing mobility in a sensor network with two leaders in terms of improved convergence rate in consensus building.

Index Terms—Sensor networks, consensus building, leader-follower model, mobility, sensor network.

I. INTRODUCTION

Consensus building in sensor networks has been studied extensively in the literature [1], [2]. Intuitively, it is known that introducing mobility increases the rate of consensus building. For example, it has been shown that convergence rates of gossip algorithms, which are a special case of consensus-building, improve with the addition of mobile nodes [3]. Similarly, it was shown in [4] that consensus models with stochastic disturbances are guaranteed to converge, almost surely, to consensus. However, there is not enough analytical investigation in the literature to demonstrate the benefit of mobile leaders in distributed sensor networks, especially in bipartite graph topology. To fill this gap, this (current) letter concentrates on investigating analytical strategies to demonstrate the faster convergence rates in consensus-building granted by introducing mobile leaders. It demonstrates that mobility, even in its simplest form, improves the convergence rate of consensus algorithms. In the existing literature, consensus-building is only investigated within the unipartite network setting and in an asymptotic sense [3], [4]. However, a bipartite topology representing the leader-follower model assists better with investigating the convergence of consensus-building [5], [6]. In our initial study on consensus-building [5], we showed that introducing mobility increases the rate of convergence in partially connected and disconnected wireless sensor networks. We also showed how the selection of mobile nodes influences the rate of convergence. In our most recent work, we provided analytical proof of the advantages of mobility in networks with an isolated agent or an island of agents [6]. The main contribution of this letter is the study of the long-term behavior of a leader-follower network structure with two mobile leaders in reaching a consensus. Our analysis presented through two theorems demonstrates that the second largest eigenvalue magnitude (SLEM) of the system with mobility is smaller than the SLEM of the system without mobility. Since SLEM is an indicator of network connectivity as well as the rate of convergence, the proofs demonstrate that mobility accelerates the consensus building process. In the next section (Section II), a brief survey of the literature is provided. Section III explains the leader-follower model. The advantages of mobility are analytically explored in Section IV.

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II. BACKGROUND AND LITERATURE SURVEY

Consensus building has extensive applications in contexts such as autonomous computing, wireless sensor networks, and robotics [7]–[9]. Autonomous vehicles apply consensus-building for task management [10], formation control [11], maintaining reliable communication [1], and intersection management [12]. Consensus building depends primarily on the network topology in a sensor network. Leader-follower topology was explored in the literature in a variety of scenarios like limited communication range [13] and faulty agents [14]. The leader-follower network setting in the form of bipartite graphs changes the leader selection problem to partition selection [5]. Consequently, it allows analytical investigation of introducing mobility in networks with isolated partitions [6]. Additionally, consensus building has been explained in terms of subjective probability distribution with graph representation [15]. Linear and nonlinear consensus building in directed and undirected graph topologies, time delay, node/link failure, and the robustness of network topology to variations were investigated thoroughly in the literature [16]–[18]. Finally, transition matrices are used in the literature to demonstrate the connections and the weight of the links among agents. Transition matrices represent uniform and nonuniform [19] weights on the edges. The SLEM of the transition matrix represents the connectivity and rate of convergence in the associated graph.

III. LEADER-FOLLOWER MODEL

The leader-follower model is explored widely in the literature. It is a two-level hierarchical structure where the nodes are divided into two groups; leaders and followers. This structure is best represented by a bipartite graph, which imposes a condition that connections among the peers of a set are prohibited. The leader-follower model finds applications in decentralized detection and estimation problems such as those discussed in [20] and [21], where the objective is to reach global consensus through local interactions. The leader-follower structure or its equivalent bipartite graph representation does not have any effect on the objective of the consensus building problem. Instead, the bipartite representation reduces the number of interactions among the nodes compared to unipartite representation.

Graphs are suitable representations of networks. For a leader-follower structure, the set of all of the leader nodes is represented by $\mathcal{L} = \{1, \ldots, l\}$ and the set of all of the followers by $\mathcal{F} = \{1, \ldots, f\}$. We let $n = l + f$ denote the total number of nodes. We allow an edge between two distinct nodes only if one is from $\mathcal{L}$ and the other is from $\mathcal{F}$, though not every node from $\mathcal{L}$ is necessarily connected to every node from $\mathcal{F}$. In addition, we assume the graph contains self-loops from and to the leader nodes. That is, $(i, i) \in \mathcal{E}$ if and only if $i \in \mathcal{L}$. Here, $\mathcal{E}$ denotes the set of all edges of the graph. Hence, the graph is bipartite except for the self-loops. We assume moreover that the graph is connected, so there is a path from any node to any other node.

A. Transition Matrix

A transition matrix represents the states of the network with a discrete-time Markov chain. This representation allows for studying the long-run behavior of the network. In this representation, the state of the network at time t is denoted by the column vector $\mathcal{X}_t$. Each state depends on its immediate previous state, via the relationship

$$\mathcal{X}_{t+1} = \mathcal{P}\mathcal{X}_t \tag{1}$$

The transition matrix $\mathcal{P}$ is a nonnegative stochastic matrix. As a result, the sum of every row is equal to one. We now explain how the transition matrix $\mathcal{P}$ is constructed. Let $H = [h_{ij}]$ be the $l \times n$ incidence matrix of the graph; that is, $h_{ij} = 1$ if $(i, j) \in \mathcal{E}$, and $= 0$ otherwise. Here, the l rows and the first l columns of H correspond to nodes from $\mathcal{L}$. So, in block form, $H = [I_l \; \tilde{H}]$, where I_l denotes the $l \times l$ identity matrix, and $\tilde{H}$ is an $l \times f$ matrix.

At each discrete time step, two things happen: First, each follower node transmits its current estimate to each of the leaders connected to it, and each leader node updates its estimate with the average of all the values it receives (including its own). Next, each leader transmits its new estimate to each of its followers, and each follower node updates its estimate with the average of all the values it receives. Thus,

$$\mathcal{P} = K_1 H^T K_2 H, \tag{2}$$

where $K_1 = (\mathrm{diag}(\mathbf{1}_{1\times l}H))^{-1}$ and $K_2 = (\mathrm{diag}(H\mathbf{1}_{n\times 1}))^{-1}$. (Here, $\mathrm{diag}(\vec{u})$ denotes the diagonal matrix whose diagonal is the vector $\vec{u}$). See [22] for more details.

Let $c = \mathbf{1}_{1\times l}H\mathbf{1}_{n\times 1}$ be the sum of all the entries in H. Since the matrix $\mathcal{P}$ is stochastic, its largest eigenvalue is 1, and it has corresponding right eigenvector $\vec{v} = \mathbf{1}_{n\times 1}$. An easy calculation shows that $\vec{w}^T = c^{-1}\mathbf{1}_{1\times l}H$ is a corresponding left eigenvector. The vectors $\vec{v}$ and $\vec{w}^T$ are normalized so that $\vec{w}^T\vec{v} = 1$. Because the graph is connected and contains self-loops, the matrix $\mathcal{P}$ is irreducible and aperiodic. As a consequence of the Perron–Frobenius Theorem

$$\lim_{t\to\infty} \mathcal{P}^t = \mathcal{P}_\infty := \vec{v}\vec{w}^T. \tag{3}$$

The limit $\mathcal{P}_\infty$ is a rank-one matrix whose rows are proportional to the vector of column sums of H. The system eventually achieves consensus in the sense that

$$\lim_{t\to\infty} \mathcal{X}_t = \mathcal{P}_\infty \mathcal{X}_0.$$

In this last vector, all entries are equal. The rate of convergence in (3) is given by the SLEM $\lambda_2(\mathcal{P})$ of $\mathcal{P}$. Precisely

$$\|\mathcal{P}^t - \mathcal{P}_\infty\| \sim C|\lambda_2(\mathcal{P})|^t \qquad \text{as } t \to \infty \tag{4}$$

where $\|\cdot\|$ denotes any matrix norm, and C is a constant depending only on $\mathcal{P}$ and the choice of norm. Thus, the smaller $\lambda_2(\mathcal{P})$ is, the faster the system reaches consensus.

IV. MOBILITY IN LEADER-FOLLOWER MODEL

In our previous works [5], [23], we empirically represented that mobility increases the connectivity in a network and enhances the rate of convergence. The analytical reasoning for the advantages of mobility in sensor networks with disconnected partitions is represented in [6]. Furthermore, the discussion remained on the advantages of mobility in a sensor network where the consensus is achievable. However, introducing mobility to this system would increase the rate of consensus.

The main focus of this letter is to study the long-term behavior of the network with two mobile leaders in reaching an agreement. Thus from here on we assume that $l = 2$. This case is easier to analyze since the transition matrix $\mathcal{P}$ has only one eigenvalue other than 0 or 1, which must therefore be the SLEM, and we can in fact give an explicit expression for $\lambda_2(\mathcal{P})$ (see Theorem 1 below).

Throughout the rest of this letter, I_k denotes the $k \times k$ identity matrix, for any $k \in \mathbb{N}$. Let Π be the $n \times n$ permutation matrix, which corresponds to switching the two leaders. In block form

$$\Pi = \begin{bmatrix} \Pi_2 & 0 \\ 0 & I_{n-2} \end{bmatrix}, \qquad \Pi_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{5}$$

Define a second transition matrix $\mathcal{P}'$ by

$$\mathcal{P}' := \Pi^{-1}\mathcal{P}\Pi.$$

We assume the leaders switch positions at the end of every time step. Thus, (1) is replaced by

$$\mathcal{X}_{t+1} = \begin{cases} \mathcal{P}\mathcal{X}_t, & \text{if } t \text{ is odd} \\ \mathcal{P}'\mathcal{X}_t, & \text{if } t \text{ is even.} \end{cases}$$

In particular, $\mathcal{X}_{2t} = (\mathcal{P}'\mathcal{P})^t\mathcal{X}_0$, so in order to investigate the effect of mobility on the rate of consensus building, we must compute $\lambda_2(\mathcal{P}'\mathcal{P})$ and compare it to $\lambda_2(\mathcal{P}^2)$. Observe that

$$\mathcal{P}'\mathcal{P} = (\Pi^{-1}\mathcal{P}\Pi)\mathcal{P} = (\Pi\mathcal{P})^2$$

since $\Pi^{-1} = \Pi$. Therefore, we must compare $\lambda_2(\mathcal{P})$ with $\lambda_2(\Pi\mathcal{P})$. The latter is always smaller in absolute value, as we will show in Theorem 2 below. First, we give an explicit expression for $\lambda_2(\mathcal{P})$.

Theorem 1: Let H be a $2 \times n$ matrix of 0's and 1's that has no columns of two zeros; let $[r_1 \; r_2]^T = H\mathbf{1}_{n\times 1}$ be the vector of row sums of H; and let n_{ij} be the number of columns equal to $[i \;\; j]^T$ in H, for $i, j = 0, 1$. Let $K_1 = (diag(\mathbf{1}_{1\times 2}H))^{-1}$ and $K_2 = \begin{bmatrix} \frac{1}{r_1} & 0 \\ 0 & \frac{1}{r_2} \end{bmatrix}$. Define the $n \times n$ matrix $\mathcal{P}$ by (2). Then

$$\lambda_2(\mathcal{P}) = \frac{n(r_1 + r_2) - (r_1^2 + r_2^2)}{2r_1r_2} = \frac{1}{2}\left(\frac{n_{10}}{r_1} + \frac{n_{01}}{r_2}\right).$$

Moreover, $0 \le \lambda_2(\mathcal{P}) \le 1$, and $\lambda_2(\mathcal{P}) < 1$ if H has at least one column of $[1 \;\; 1]^T$, and $\lambda_2(\mathcal{P}) > 0$ if H has at least one column of $[1 \;\; 0]^T$ or $[0 \;\; 1]^T$.

Theorem 2: Let H, K_1, K_2, and $\mathcal{P}$ be as in Theorem 1, but assume in addition that the first two columns of H are $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$. Let Π be the permutation matrix from (5). Then

1) $\lambda_2(\Pi\mathcal{P}) = \lambda_2(\mathcal{P}) - (\frac{1}{r_1} + \frac{1}{r_2})$; and
2) $|\lambda_2(\Pi\mathcal{P})| \le |\lambda_2(\mathcal{P})|$, with equality if and only if $r_1 = r_2 = n - 1$.

According to Theorem 2, the SLEM of the system with mobility is smaller than the SLEM of the system without mobility. Consequently, introducing mobility improves the connectivity of the system and enhances the rate of convergence.

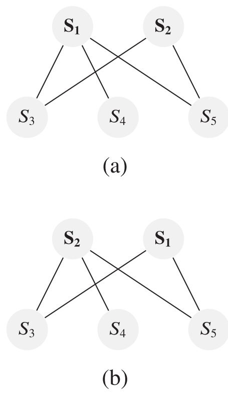

Fig. 1. Leaders switch locations. (a) First instant. (b) Second instant.

Remark 1: In the two-leader case considered here, the analysis was straightforward because the SLEMs of both $\mathcal{P}$ and $\Pi\mathcal{P}$ could be explicitly calculated and were easily compared. In networks with three or more leaders, a simple and natural way to model mobility is to let the leaders change positions in a cyclical pattern. One then again has to compare $\lambda_2(\mathcal{P})$ and $\lambda_2(\Pi\mathcal{P})$, where Π is the permutation matrix corresponding to the cyclical permutation of the leaders. However, even in the case of three leaders, $\lambda_2(\mathcal{P})$ is typically irrational, and $\lambda_2(\Pi\mathcal{P})$ is typically complex, and the corresponding eigenvectors are no longer the same. We have not yet been able to give a general proof that $|\lambda_2(\Pi\mathcal{P})| < |\lambda_2(\mathcal{P})|$. Doing so is a goal for future work.

V. EXAMPLE

Fig. 1 represents a sensor network with two leaders: S_1 and S_2. In the first instant, node S_1 connects to S_3, S_4, S_5, and node S_2 connects to S_3, S_5. In the second instant, sensors S_1 and S_2 switch their locations. The matrices H and $\mathcal{P}$ corresponding to the first instant are

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

and

$$\mathcal{P} = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{8} & \frac{1}{6} & \frac{7}{24} & \frac{1}{8} & \frac{7}{24} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{6} & \frac{7}{24} & \frac{1}{8} & \frac{7}{24} \end{bmatrix}.$$

The normalized left eigenvector associated with the matrix $\mathcal{P}$ corresponding to the eigenvalue 1 is

$$\vec{w}^T = \left[\frac{1}{7}, \frac{1}{7}, \frac{2}{7}, \frac{1}{7}, \frac{2}{7}\right]$$

and the entries of $\mathcal{X}_t$ all converge to the consensus value $\vec{w}^T\mathcal{X}_0$. Observe that the weights in $\vec{w}^T$ corresponding to follower nodes are proportional to their degrees in the graph, as should not be surprising. The SLEMs of the matrices $\mathcal{P}$ and $\Pi\mathcal{P}$ are $5/12$ and $-1/6$, respectively, illustrating that indeed, $|\lambda_2(\Pi\mathcal{P})| < \lambda_2(\mathcal{P})$. Let T_ε denote the first time t at which $\|\mathcal{X}_t - \mathcal{X}_\infty\| < \varepsilon$, where $\mathcal{X}_\infty = \lim_{t\to\infty}\mathcal{X}_t$ is the consensus value and ε is a very small error tolerance. Equation (4) implies that, in the system without mobility

$$T_\varepsilon \sim \frac{\log\varepsilon}{\log|\lambda_2(\mathcal{P})|} = \frac{\log\varepsilon}{\log|5/12|} \approx 1.14\log(1/\varepsilon)$$

whereas in the system with mobility

$$T_\varepsilon \sim \frac{\log\varepsilon}{\log|\lambda_2(\Pi\mathcal{P})|} = \frac{\log\varepsilon}{\log|-1/6|} \approx 0.56\log(1/\varepsilon)$$

as $\varepsilon \to 0$. So here, the system with mobility converges roughly twice as fast.

This example illustrates that a small movement in the network leads to significant savings in the transition time T_ε, suggesting that the transient behavior and, hence, the rate of convergence can be significantly improved when a systematic mobility pattern is introduced in the leader-follower topology. Swapping of leaders at different speeds can be observed in formation control applications involving biological (e.g., bird flocking) as well as mechanical systems (e.g., robots and manned and unmanned aircraft systems).

VI. CONCLUSION

This letter provides concrete proofs to demonstrate the benefits of introducing mobility in a sensor network with two leaders in terms of improved convergence rate in consensus building. A future goal is to extend this analysis to networks with three or more mobile leaders. Such analysis may provide better insights into localized and decentralized detection problems such as those discussed in [20] and [21].

APPENDIX
PROOFS OF THEOREMS 1 AND 2

Lemma 1: Let C be an $n \times m$ matrix and D an $m \times n$ matrix, then CD and DC have the same nonzero eigenvalues.

The elementary proof is omitted.

Proof of Theorem 1: Note that K_1H^T is $n \times 2$ and K_2H is $2 \times n$. By Lemma 1, $\mathcal{P}$ has the same nonzero eigenvalues as the 2×2 matrix

$$Q := K_2HK_1H^T.$$

Observe that Q does not depend on the arrangement of the columns of H. To see this, let P be any $n \times n$ permutation matrix, and replace H with $\hat{H} := HP$. Then, K_1 must be replaced by $\hat{K}_1 := P^{-1}K_1P$ while K_2 is unaffected; hence we obtain the 2×2 matrix

$$\hat{Q} := K_2\hat{H}\hat{K}_1\hat{H}^T = K_2HPP^{-1}K_1PP^TH^T = K_2HK_1H^T = Q.$$

This holds since $P^T = P^{-1}$ for any permutation matrix P. As a result, we may assume the columns of H are ordered as follows:

$$H = \begin{bmatrix} \underbrace{\begin{matrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{matrix}}_{n_{11}} & \underbrace{\begin{matrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{matrix}}_{n_{10}} & \underbrace{\begin{matrix} 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \end{matrix}}_{n_{01}} \end{bmatrix}$$

or in block form, $H = [H_{11}\ H_{10}\ H_{01}]$, where H_{ij} is $2 \times n_{ij}$. We can also write K_1 in block form as

$$K_1 = \begin{bmatrix} \frac{1}{2}I_{n_{11}} & 0 \\ 0 & I_{n-n_{11}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}I_{n_{11}} & 0 & 0 \\ 0 & I_{n_{10}} & 0 \\ 0 & 0 & I_{n_{01}} \end{bmatrix}$$

so

$$K_1H^T = \begin{bmatrix} \frac{1}{2}H_{11} & H_{10} & H_{01} \end{bmatrix}^T$$

and

$$Q = K_2HK_1H^T$$

$$= K_2 \begin{bmatrix} H_{11} & H_{10} & H_{01} \end{bmatrix} \begin{bmatrix} \frac{1}{2}H_{11} & H_{10} & H_{01} \end{bmatrix}^T$$
$$= K_2 \left(\frac{1}{2} H_{11}H_{11}^T + H_{10}H_{10}^T + H_{01}H_{01}^T \right)$$
$$= K_2 \left\{ \frac{1}{2} \begin{bmatrix} n_{11} & n_{11} \\ n_{11} & n_{11} \end{bmatrix} + \begin{bmatrix} n_{10} & 0 \\ 0 & n_{01} \end{bmatrix} \right\}$$
$$= \begin{bmatrix} \frac{n_{11}+2n_{10}}{2r_1} & \frac{n_{11}}{2r_1} \\ \frac{n_{11}}{2r_2} & \frac{n_{11}+2n_{01}}{2r_2} \end{bmatrix} = \begin{bmatrix} \frac{r_1-r_2+n}{2r_1} & \frac{r_1+r_2-n}{2r_1} \\ \frac{r_1+r_2-n}{2r_2} & \frac{r_2-r_1+n}{2r_2} \end{bmatrix}.$$

Since Q is stochastic, one of its eigenvalues is 1, and the other is

$$\lambda_2(Q) = \det(Q)$$
$$= \frac{r_1 - r_2 + n}{2r_1} \cdot \frac{r_2 - r_1 + n}{2r_2} - \frac{(r_1 + r_2 - n)^2}{4r_1r_2}$$
$$= \frac{r_1(n - r_1) + r_2(n - r_2)}{2r_1r_2} = \frac{1}{2}\left(\frac{n_{10}}{r_1} + \frac{n_{01}}{r_2} \right).$$

The last equation shows clearly that $0 \le \lambda_2(Q) \le 1$, as $n_{10} \le r_1$ and $n_{01} \le r_2$. If H has at least one column $[1 \ \ 1]^T$, then $n_{10} < r_1$ and $n_{01} < r_2$, so $\lambda_2(Q) < 1$. If either $n_{10} > 0$ or $n_{01} > 0$, then $\lambda_2(Q) > 0$. The same is then true for $\lambda_2(\mathcal{P})$. ∎

Proof of Theorem 2: Let $\widetilde{\mathcal{P}} := \Pi\mathcal{P} = \Pi K_1 H^T K_2 H$. According to Lemma 1, $\widetilde{\mathcal{P}}$ has the same nonzero eigenvalues as $\widetilde{Q} := K_2 H \Pi K_1 H^T$. Note that $\widetilde{Q} = K_2 H(\Pi - I_n)K_1 H^T + Q$, where Q is the matrix from the proof of Theorem 1. Now

$$\Pi - I_n = \begin{bmatrix} -1 & 1 & \mathbf{0}_{2\times(n-2)} \\ 1 & -1 & \\ \mathbf{0}_{(n-2)\times 2} & & \mathbf{0}_{(n-2)\times(n-2)} \end{bmatrix}, \qquad K_1 = \begin{bmatrix} I_2 & 0 \\ 0 & \tilde{K}_1 \end{bmatrix}$$

so

$$K_2 H(\Pi - I_n)K_1 H^T = K_2 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{r_1} & \frac{1}{r_1} \\ \frac{1}{r_2} & \frac{-1}{r_2} \end{bmatrix}.$$

We claim that $\vec{y} = [r_2 \ \ -r_1]^T$ is an eigenvector of Q for $\lambda_2(Q)$. Recall from the proof of Theorem 1 that

$$Q = \begin{bmatrix} \frac{n_{11}+2n_{10}}{2r_1} & \frac{n_{11}}{2r_1} \\ \frac{n_{11}}{2r_2} & \frac{n_{11}+2n_{01}}{2r_2} \end{bmatrix}, \qquad \lambda_2(Q) = \frac{1}{2}\left(\frac{n_{10}}{r_1} + \frac{n_{01}}{r_2} \right).$$

Since $n_{11} + n_{10} = r_1$ and $n_{11} + n_{01} = r_2$, this implies

$$Q - \lambda_2(Q)I_2 = \begin{bmatrix} \frac{r_2-n_{01}}{2r_2} & \frac{n_{11}}{2r_1} \\ \frac{n_{11}}{2r_2} & \frac{r_1-n_{10}}{2r_1} \end{bmatrix}.$$

Therefore, $(Q - \lambda_2(Q)I_2) \begin{bmatrix} r_2 \\ -r_1 \end{bmatrix} = 0$, establishing the claim. As a result

$$\widetilde{Q}\vec{y} = K_2 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} r_2 \\ -r_1 \end{bmatrix} + Q\vec{y} = K_2 \begin{bmatrix} -(r_1 + r_2) \\ r_1 + r_2 \end{bmatrix} + \lambda_2(Q)\vec{y}$$
$$= \lambda_2(Q)\vec{y} - \frac{r_1 + r_2}{r_1 r_2} \begin{bmatrix} r_2 \\ -r_1 \end{bmatrix} = \left(\lambda_2(Q) - \left(\frac{1}{r_1} + \frac{1}{r_2} \right) \right) \vec{y}.$$

Thus, $\lambda_2(\mathcal{P}) - (\frac{1}{r_1} + \frac{1}{r_2})$ is an eigenvalue of $\widetilde{Q}$, and hence of $\widetilde{\mathcal{P}}$. (Recall from the proof of Theorem 1 that $\lambda_2(\mathcal{P}) = \lambda_2(Q)$.)

Clearly then, $\lambda_2(\widetilde{\mathcal{P}}) = \lambda_2(\mathcal{P}) - (\frac{1}{r_1} + \frac{1}{r_2}) < \lambda_2(\mathcal{P})$. Furthermore, since

$$\lambda_2(\mathcal{P}) = \frac{1}{2}\left(\frac{n_{10}}{r_1} + \frac{n_{01}}{r_2} \right) \ge \frac{1}{r_1} + \frac{1}{r_2}$$

we have

$$\lambda_2(\widetilde{\mathcal{P}}) + \lambda_2(\mathcal{P}) \ge 2\lambda_2(\mathcal{P}) - \frac{1}{r_1} + \frac{1}{r_2} \ge 0.$$

Hence, $|\lambda_2(\widetilde{\mathcal{P}})| \le |\lambda_2(\mathcal{P})|$, with equality if and only if $n_{10} = n_{01} = 1$, or equivalently, $r_1 = r_2 = n - 1$. ∎

REFERENCES

[1] S. S. Stankoviċ, M. Beko, and M. S. Stankoviċ, "A robust consensus seeking algorithm," in *Proc. IEEE EUROCON 18th Int. Conf. Smart Technol.*, Jul. 2019, pp. 1–6.

[2] L. Conradt and C. List, "Group decision in humans and animals: A survey," *Philos. Trans. Roy. Soc.*, vol. 364, no. 1518, pp. 719–742, 2009.

[3] A. D. Sarwate and A. G. Dimakis, "The impact of mobility on gossip algorithms," *IEEE Trans. Inf. Theory*, vol. 58, no. 3, pp. 1731–1742, Mar. 2012.

[4] T. C. Aysal and K. E. Barner, "Convergence of consensus models with stochastic disturbances," *IEEE Trans. Inf. Theory*, vol. 56, no. 8, pp. 4101–4113, Aug. 2010.

[5] R. Norouzi-Kandalan, R. Sing, K. Namuduri, M. Varanasi, and B. Buckles, "Impact of mobility on convergence rate in a wireless sensor network," in *Proc. Conf. Inf. Sci. Syst.*, 2017, pp. 1–6, doi: 10.1109/VTCFall.2019.8891121.

[6] R. Norouzi-Kandalan, M. Varanasi, B. Buckles, and K. Namuduri, "Impact of leadership and mobility on consensus-building in sensor networks," *Sensors*, vol. 20, 2020, Art. no. 1081.

[7] S. Ali, S. Ahmed, and S. N. Khan Marwat, "A practical approach to consensus based control of multi-agent systems," in *Proc. Int. Symp. Recent Adv. Elect. Eng.*, Oct. 2018, pp. 1–5.

[8] L. Yang and H. Li, "Vehicle-to-vehicle communication based on a peer-to-peer network with graph theory and consensus algorithm," *IET Intell. Transport Syst.*, vol. 13, no. 2, pp. 280–285, 2019.

[9] J. Lozano Domínguez and T. J. Mateo Sanguino, "Review on v2x, i2x, and p2x communications and their applications: A comprehensive analysis over time," *Sensors (Basel, Switzerland)*, vol. 19, no. 12, 2019, Art. 2756. [Online]. Available: https://www.ncbi.nlm.nih.gov/pubmed/31248189

[10] L. Jin, "Research on distributed task allocation algorithm for unmanned aerial vehicles based on consensus theory," in *Proc. Chin. Control Decis. Conf.*, May 2016, pp. 4892–4897.

[11] Z. Yan, D. Xu, T. Chen, W. Zhang, and Y. Liu, "Leader-follower formation control of UUVs with model uncertainties, current disturbances, and unstable communication," *Sensors*, vol. 18, 2018, Art. no. 662.

[12] C. Wuthishuwong and A. Traechtler, "Consensus-based local information coordination for the networked control of the autonomous intersection management," *Complex Intell. Syst.*, vol. 3, no. 1, pp. 17–32, Mar. 2017. [Online]. Available: https://doi.org/10.1007/s40747-016-0032-6

[13] Y. Ruan and S. K. Jayaweera, "Leader-following consensus in vehicle platoons with an inter-vehicle communication network," in *Proc. 8th Int. Conf. Telecommun. Syst. Serv. Appl.*, Oct. 2014, pp. 1–6.

[14] M. I. Khan and R. Hossain, "Efficient and fast convergent consensus algorithms for faulty nodes tracking in distributed wireless sensor networks," in *Proc. 3rd Int. Conf. Elect. Eng. Inf. Commun. Technol*, Sep. 2016, pp. 1–6.

[15] M. H. DeGroot, "Reaching a consensus," *J. Amer. Stat. Assoc.*, vol. 69, no. 345, pp. 118–121, 1974.

[16] R. Olfati-Saber and R. M. Murray, "Consensus protocols for networks of dynamic agents," in *Proc. IEEE Amer. Control Conf.*, 2003, pp. 951–956.

[17] R. Olfati-Saber and R. M. Murray, "Consensus problem in network of agents with switching topology and time-delay," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[18] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent system," in *Proc. IEEE*, vol. 95, no. 1, pp. 215–233, Jan. 2007.

[19] S. Boyd, P. Diaconis, and L. Xiao, "Fastest mixing Markov chain on a graph," *SIAM Rev.*, vol. 46, no. 4, pp. 667–689, 2004.

[20] D. Ciuonzo, P. S. Rossi, and P. Willett, "Generalized rao test for decentralized detection of an uncooperative target," *IEEE Signal Process. Lett.*, vol. 24, no. 5, pp. 678–682, May 2017.

[21] X. Cheng, D. Ciuonzo, and P. S. Rossi, "Multibit decentralized detection through fusing smart and dumb sensors based on Rao test," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 56, no. 2, pp. 1391–1405, Apr. 2020, 2019.

[22] Y. Wan, K. Namuduri, S. Akula, and M. Varanasi, "The impact of multi-group multi-layer network structure on the performance of distributed consensus building strategies," *Int. J. Robost Nonlinear Control*, vol. 23, pp. 653–662, Apr. 2012.

[23] R. Norouzi-Kandalan, S. Alla, and N. H. Rezaeian, "Impact of mobility on consensus building in the leader-follower model," in *Proc. IEEE 90th Veh. Technol. Conf.*, Sep. 2019, pp. 1–6.