Optimal stopping rules for directionally reinforced processes

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Abstract

This paper analyzes optimal single and multiple stopping rules for a class of correlated random walks that provides an elementary model for processes exhibiting momentum or directional reinforcement behavior. Explicit descriptions of optimal stopping rules are given in several interesting special cases with and without transaction costs. Numerical examples are presented comparing optimal strategies to simpler buy and hold strategies.

Keywords: Correlated random walk; directionally reinforced random walk; multiple stopping; buy/sell strategies

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1 Introduction

This paper investigates optimal stopping rules for a class of correlated random walk processes. The walks serve as an elementary model for processes exhibiting momentum or directional reinforcement behavior. Both single and multiple stopping problems are considered, with and without transaction costs. Optimal stopping rules are characterized for several reinforcement schemes which admit structurally tractable optimal strategies.

The processes considered here are defined as follows. Let $X_1, X_2, \ldots$ be $\{-1, 1\}$-valued random variables such that $P(X_1 = 1) = 1/2$, and for $n \geq 0$, $j \geq 1$, and $\varepsilon_1, \ldots, \varepsilon_{n-1} \in \{-1, 1\}$,

$$
\begin{aligned}
&P(X_{n+j+1} = 1 | X_{n+j} = 1, \ldots, X_{n+1} = 1, X_n = -1, X_{n-1} = \varepsilon_{n-1}, \ldots, X_1 = \varepsilon_1) \\
&\quad = P(X_{n+j+1} = 1 | X_{n+j} = 1, \ldots, X_{n+1} = 1, X_n = -1) \\
&\quad = g^+(j),
\end{aligned}
$$

and

$$
\begin{aligned}
&P(X_{n+j+1} = -1 | X_{n+j} = -1, \ldots, X_{n+1} = -1, X_n = 1, X_{n-1} = \varepsilon_{n-1}, \ldots, X_1 = \varepsilon_1) \\
&\quad = P(X_{n+j+1} = -1 | X_{n+j} = -1, \ldots, X_{n+1} = -1, X_n = 1) \\
&\quad = g^-(j).
\end{aligned}
$$

Define the walk $\{S_n\}_{n \in \mathbf{Z}^+}$ by $S_0 = 0$, and for $n \geq 1$,

$$
S_n = \sum_{i=1}^{n} X_i.
$$

Throughout this paper it will be assumed that *the sequences* $\{g^+(n)\}$ *and* $\{g^-(n)\}$ *are nondecreasing, and* $g^+(1), g^-(1) \geq 1/2$. With this assumption, the process $\{S_n\}$ is a random walk which moves one unit per unit of time and for which the current direction of movement is reinforced. The sequences $\{g^+(n)\}$ and $\{g^-(n)\}$ characterize the strength of the reinforcement in the positive and negative directions, respectively. Such processes allow the directional reinforcement to change for successive steps in the same direction instead of remaining constant as in standard definitions of correlated random walks (CRWs). On the other hand, the conditions assumed on the reinforcement sequences are more restrictive than in general CRWs. To help distinguish between the version of CRWs considered here and standard CRWs, $\{S_n\}$ will be called a *directionally reinforced random walk* (DRRW). One objective considered below is, given a finite time horizon N, find a stopping time $\tau \leq N$ which maximizes ES_τ. It is shown that for the special case of $g^-(n) \equiv q \geq 1/2$, the optimal stop rule has a particularly nice form: there exists a time $k^* \leq N$ such that the player does nothing until time $N - k^* - 1$, and then sells at the first downward step after that time. This and other single stop rule results are presented in Section 2.

Multiple stop (buy/sell) rules with or without transaction costs are developed in Section 3. For example, if transaction costs are ignored, then the optimal strategy is simple: always sell as soon as the walk takes a step down and always buy as soon as the walk takes a step up. This holds no matter what form the reinforcement sequences $\{g^+(n)\}$ and $\{g^-(n)\}$ take (as long as they are nondecreasing and $g^+(n), g^-(n) \geq 1/2$). Transaction costs, as to be expected, complicate the analysis. However, some reasonably straightforward rules are obtained under various assumptions on the reinforcement schemes.

Section 4 gives examples comparing the optimal stop rules to simple "buy-and-hold" strategies. Section 5 suggests natural extensions of the analysis.

Basic properties, such as transition probabilities and first passage times, of correlated random walks have been examined in a number of papers, including Gillis (1955), Zhang (1992), Chen and Renshaw (1994), and Böhm (2000). Gambler ruin type problems (in a sense, passive versions of the problem examined here) for CRWs are considered by Mohan (1955), and Mukherjea and Steele (1986,1987). Directionally reinforced random walks were analyzed by Mauldin, Monticino and Von Weizsäcker (1996), and Horváth and Shao (1998). Classical work on multiple stopping problems includes Chow, Robbins and Siegmund (1971), Mosteller and Gilbert (1966), and Haggstrom (1966). While rather general optimality conditions are obtained in these works, explicit strategies are derived mainly in the case of independent observations. Starr (1972) considers an optimal stopping problem in which the underlying process can be modeled by repeated tosses of a coin with probability p of coming up heads and the reward received is based upon the length of the terminal run of heads. Ross (1975), Majumdar and Sakaguchi (1983), Majumdar (1983) and Majumdar (1989) extend Starr (1972) in various ways. For instance, in Majumdar (1983), the underlying process is an urn model containing "plus," "minus" and "cancel" balls and the objective is stop so as to maximize a function of the terminal success or failure run. Neither Starr (1972) (and its extensions) nor the model considered here contain each other as special cases. More recent work on multiple optimal stopping rules includes Kao and Tate (1999), and Bruss and Paindaveine (2000).

2 Optimal single stopping rules

This section investigates the problem of maximizing the expected return given a finite time horizon N. Theorem 2.4 establishes the general form of the optimal stop rule, while Theorem 2.6 and Corollary 2.7 give special cases in which the optimal stop rule has a particularly simple structure. These include the standard correlated random walks.

Let $N < \infty$ be a positive integer and denote the set of stop rules τ such that $1 \leq \tau \leq N$ a.s. by $\mathcal{T}^N$. (A nonnegative random variable τ is a stop rule for the process $\{S_n\}_n$ if for every k, the set $\{\tau = k\}$ is measurable with respect to $\sigma(\{S_1, \ldots, S_k\})$, the σ-algebra generated by the process $\{S_n\}_n$ until time k.) The objective is to find a stop rule τ^* in $\mathcal{T}^N$ such that

$$ES_{\tau^*} = V(N) := \sup_{\tau \in \mathcal{T}^N} ES_\tau.$$

Note that, for convenience, it is assumed that the process is initialized at time 0 - that is, $P(X_1 = 1) = 1/2$. It is straightforward to modify the results given below to account for processes assumed to have begun prior to that time arbitrarily designated time 0.

For $1 \leq j \leq N$, let $\mathcal{T}_j^N \subset \mathcal{T}^N$ be the set of stop rules τ such that $j \leq \tau \leq N$, and define the following quantities. For $1 \leq k \leq N-1$ and $1 \leq n \leq N-k$,

$$\begin{aligned}
W^+(k,n) &= \sup_{\tau \in \mathcal{T}_{N-k+1}^N} E(S_\tau - S_{N-k} | X_{N-k} = 1, \ldots, X_{N-k-n+1} = 1, X_{N-k-n} = -1), \\
V^+(k,n) &= \sup_{\tau \in \mathcal{T}_{N-k}^N} E(S_\tau - S_{N-k} | X_{N-k} = 1, \ldots, X_{N-k-n+1} = 1, X_{N-k-n} = -1) \\
&= W^+(k,n) \vee 0, \\
W^-(k,n) &= \sup_{\tau \in \mathcal{T}_{N-k+1}^N} E(S_\tau - S_{N-k} | X_{N-k} = -1, \ldots, X_{N-k-n+1} = -1, X_{N-k-n} = 1), \\
V^-(k,n) &= \sup_{\tau \in \mathcal{T}_{N-k}^N} E(S_\tau - S_{N-k} | X_{N-k} = -1, \ldots, X_{N-k-n+1} = -1, X_{N-k-n} = 1) \\
&= W^-(k,n) \vee 0.
\end{aligned}$$

Set $V^+(0,n) = V^-(0,n) = W^+(0,n) = W^-(0,n) = 0$ for all n. Intuitively, $W^+(k,n)$ (respectively $W^-(k,n)$) is the maximum expected *additional return* (possibly negative) given that there are k

days left to play, the walk has taken exactly n upward (respectively downward) steps on its current run, and at least one more observation must be taken. $V^+(k,n)$ and $V^-(k,n)$ are the corresponding values if the additional option of stopping immediately is available.

The following recursive relations are easily verified by conditioning on the value of X_{N-k}:

$$\begin{aligned} W^+(k+1,n) &= g^+(n)\{V^+(k,n+1)+1\} + (1-g^+(n))\{V^-(k,1)-1\}, \\ W^-(k+1,n) &= (1-g^-(n))\{V^+(k,1)+1\} + g^-(n)\{V^-(k,n+1)-1\}, \end{aligned} \tag{1}$$

and, since it is assumed that $P(X_1 = 1) = 1/2$,

$$\begin{aligned} V(N) &= \frac{1}{2}\{V^+(N-1,1)+1\} + \frac{1}{2}\{V^-(N-1,1)-1\} \\ &= \frac{1}{2}\{V^+(N-1,1)+V^-(N-1,1)\}. \end{aligned} \tag{2}$$

So, in theory, given reinforcement sequences $\{g^+(n)\}$ and $\{g^-(n)\}$, the values of $V(N)$, $V^+(k,n)$, $V^-(k,n)$, $W^+(k,n)$ and $W^-(k,n)$ can be calculated for all N, $1 \le k \le N-1$ and $1 \le n \le N-k$. Examples are given in Section 4.

Several quantities appear repeatedly in the results developed below. Notation for these values are given in the next definition.

Definition 2.1 *Define the array of numbers* $\{\gamma(k,n)\}_{k\ge 0, n\ge 1}$ *by*

$$\gamma(0,n) := 1, \qquad n = 1,2,\ldots,$$

and

$$\gamma(k,n) := g^+(n)\{1+\gamma(k-1,n+1)\}, \qquad k,n \ge 1.$$

For $k = 0,1,2,\ldots,$ *let* $\beta(k) := \gamma(k,1)$, *and* $\alpha(k) := 1-(\beta(k)+1)^{-1}$.

Remark 2.2 An intuitive view of $\beta(k)$ can be given as follows. Let T be a random variable such that $P(T=1) = 1-g^+(1)$ and for $k \ge 2$, $P(T=k) = g^+(1)\cdots g^+(k-1)(1-g^+(k))$. T can be interpreted as the length of an upward run in a walk with infinite time horizon. For each integer $k \ge 1$, define the random variable T_k by

$$T_k = \begin{cases} T-1, & \text{if} \quad T \le k, \\ k+1, & \text{if} \quad T > k. \end{cases}$$

Then $\beta(k) = ET_k$. So, given the walk has taken exactly one step to the right, $\beta(k)$ is the expected additional return (including the initial step to the right) under the strategy of stopping the first time the walk takes a step to the left or at time k, whichever comes first. An explicit expression for $\beta(k)$ is

$$\beta(k) = \sum_{j=1}^{k-1}\prod_{i=1}^{j} g^+(i) + 2\prod_{i=1}^{k} g^+(i).$$

The following lemma collects some useful properties of the sequences defined above. The proofs are straightforward and therefore omitted.

Lemma 2.3 *(i)* $\gamma(k,n) \ge 1$ *for all* $k \ge 0$ *and* $n \ge 1$;

(ii) $\{\beta(k)\}_k$ *is non-decreasing, and* $\beta(k+1)=\beta(k)$ *iff* $g^+(k+1)=1/2$*, in which case* $\beta(k+1)=\beta(1)=1$.

(iii) $\{\alpha(k)\}_k$ *is non-decreasing.*

Define the random variables

$$R^-(n)=n-\max\{j\le n: X_j=1\},\qquad n=1,2,\ldots,$$

with the convention that $\max(\emptyset)=0$. $R^-(n)$ is the length of the current downward run at time n, or zero if the walk is currently on an upward run.

Theorem 2.4 *The optimal stop rule* τ^* *is given by*

$$\tau^*=\inf\{j\ge 1: R^-(j)>0 \text{ and } g^-(R^-(j))\ge\delta(N-j)\}\wedge N,$$

where for $k=1,2,\ldots,$

$$\delta(k)=1-(V^+(k-1,1)+2)^{-1}.$$

The constants $\delta(k)$ *satisfy* $\delta(k)\ge\alpha(k-1)$*, with equality for* $k\le 3$.

Proof. If $R^-(j)=0$ at some time $j<N$, then the walk is on an upward run of some length $n\ge 1$. Hence the maximum expected additional return after time j is

$$\begin{aligned}W^+(N-j,n)&=g^+(n)\{V^+(N-j-1,n+1)+1\}+(1-g^+(n))\{V^-(N-j-1,1)-1\}\\&\ge 2g^+(n)-1+g^+(n)\{V^+(N-j-1,n+1)\}\\&\ge 0.\end{aligned}$$

So it is optimal to continue playing. On the other hand, if $R^-(j)=n>0$, then the optimal expected return after time j is $W^-(N-j,n)$. Thus, making the substitution $N-j=k$, the optimality of τ^* will follow by establishing the equivalence

$$W^-(k,n)>0\qquad\text{iff}\qquad g^-(n)<\delta(k). \tag{3}$$

Suppose first that $g^-(n)<\delta(k)$. Then

$$\begin{aligned}W^-(k,n)&=(1-g^-(n))\{V^+(k-1,1)+1\}+g^-(n)\{V^-(k-1,n+1)-1\}\\&\ge(1-g^-(n))\{V^+(k-1,1)+1\}+g^-(n)(-1)\\&=V^+(k-1,1)+1-g^-(n)\{V^+(k-1,1)+2\}\\&=\{V^+(k-1,1)+2\}\{\delta(k)-g^-(n)\}\\&>0.\end{aligned}$$

The converse will be proved by induction on k. Since $W^-(1,n)=1-2g^-(n)\le 0$ for all n, the implication is trivial for $k=1$. Assume that for some positive integer k and for each n,

$$g^-(n)\ge\delta(k)\text{ implies }W^-(k,n)\le 0.$$

To complete the inductive step, assume that $g^-(n)\ge\delta(k+1)$. Since the sequence $\{\delta(k)\}$ is easily seen to be non-decreasing, this implies $g^-(n+1)\ge\delta(k)$. Hence by the induction hypothesis, $W^-(k,n+1)\le 0$. Substituting this in (1) yields $W^-(k+1,n)\le 0$. Thus (3) holds, and the optimality of τ^* follows.

That $\delta(k) \geq \alpha(k-1)$ follows from the more general fact that

$$W^+(k,n) \geq \gamma(k,n) - 1. \tag{4}$$

To prove (4), note that $W^+(0,n) = 0 = \gamma(0,n) - 1$, and if (4) holds for some non-negative k and all n, then

$$\begin{aligned} W^+(k+1,n) &= g^+(n)\{V^+(k,n+1)+1\} + (1-g^+(n))\{V^-(k,1)-1\} \\ &\geq g^+(n)\gamma(k,n+1) + (1-g^+(n))(-1) \\ &= g^+(n)\{1+\gamma(k,n+1)\} - 1 \\ &= \gamma(k+1,n) - 1. \end{aligned} \tag{5}$$

Thus, (4) follows by induction. Taking $n = 1$ and using Definition 2.1 gives the lower bound $\delta(k) \geq \alpha(k-1)$. Finally, a simple calculation shows that $V^+(1,1) = \beta(1)-1$ and $V^+(2,1) = \beta(2)-1$. So $\delta(k) = \alpha(k-1)$ for $k \leq 3$. $\square$

Remark 2.5 For $k \geq 4$, $\delta(k)$ can in general be strictly greater than $\alpha(k-1)$. To see this, fix $k \geq 4$, and assume $g^+(k-3) > 1/2$. Now let $\{g^-(n)\}$ be a sequence such that $1/2 \leq g^-(1) < \alpha(k-3)$ (this is possible since $g^+(k-3) > 1/2$ implies $\alpha(k-3) > 1/2$), and $g^-(n) = \alpha(k-1)$ for $n = 2, 3, \ldots$. Clearly, the sequence $\{g^-(n)\}$ is nondecreasing, and for any $n \geq 2$, $g^-(n) \geq \alpha(k-1)$. However, (3) (with $n = 1$) implies $V^-(k-2,1) > 0$, and it is easily checked, using the equations (1) and (4), that this implies $V^-(k,n) > 0$. Hence $\alpha(k-1) \leq g^-(n) < \delta(k)$.

The calculation of $\delta(k)$, though easy for $k \leq 3$, becomes increasingly complex for larger k. This can make calculation of the optimal stop rule tedious for large N. The next two results concern special cases where the optimal rule has a simple structure.

Theorem 2.6 *(i) If $g^-(1) \geq \alpha(K-1)$ for some integer $K \geq 2$, then*

$$V^+(k,n) = \gamma(k,n) - 1, \tag{6}$$

and

$$V^-(k,n) = 0, \tag{7}$$

for all $1 \leq k \leq K$ and $n \geq 1$.

(ii) If $g^-(1) \geq \alpha(N-2)$, then $V(N) = (\beta(N-1)-1)/2$, and the optimal stop rule τ^* is given by

$$\tau^* = \inf\{n \geq 1 : X_n = -1\} \wedge N.$$

That is, stop at the first downward step, or at time N, whichever comes first.

Proof. (6) and (7) will be proved simultaneously, by induction on k. Both statements are readily checked for $k = 1$. Suppose then that both (6) and (7) hold for some k, $1 \leq k \leq K-1$, and for all n. By the induction assumption, equality obtains in (5). So

$$W^+(k+1,n) = \gamma(k+1,n) - 1.$$

For each $n \geq 1$ and $k \leq K-1$, $g^-(n) \geq g^-(1) \geq \alpha(K-1) \geq \alpha(k)$. By this, Definition 2.1 and the induction assumption,

$$\begin{aligned} W^-(k+1,n) &= (1-g^-(n))(V^+(k,1)+1) + g^-(n)(V^-(k,n+1)-1) \\ &= (1-g^-(n))\gamma(k,1) + g^-(n)(-1) \\ &= \beta(k) - g^-(n)(\beta(k)+1) \\ &\leq \beta(k) - \alpha(k)(\beta(k)+1) \\ &= 0. \end{aligned}$$

This completes the proof of (i). Part (ii) follows easily from (i) and (2). □

Corollary 2.7 *(i) Suppose that there is a $1/2 \leq p \leq 1$ such that $g^-(n) = p$ for all n. Let*

$$k^* = \inf\{k : \alpha(k) > p\} \wedge (N-1).$$

Then the optimal stop rule τ^* is given by

$$\tau^* = \inf\{n \geq N - k^* : X_n = -1\} \wedge N.$$

In other words, if $k^* < N-1$ it is optimal to play at least until day $N - k^* - 1$, and after that day to stop as soon as the walk takes a step down. If $k^* \geq N-1$, it is optimal to stop the first time the walk takes a downward step (as in Theorem 2.6).

(ii) (Symmetric CWR) *If $g^+(n) = g^-(n) = p$ for all $n \geq 1$ and some $p \geq 1/2$, then it is optimal to stop the first time the walk takes a downward step, and*

$$V(N) = (p - 1/2)\frac{1-p^{N-1}}{1-p} \to \frac{2p-1}{2(1-p)} \quad (as \quad N \to \infty).$$

Proof. By the definition of k^*, $g^-(1) = p \geq \alpha(k^*-1)$. Hence, by Theorem 2.6 (i), $V^-(k,n) = 0$ for all $k \leq k^*$ and $n \geq 1$. On the other hand, if $k > k^*$, $g^-(n) = p < \alpha(k-1)$. So by (3), $V^-(k,n) > 0$ for all $n \geq 1$. This proves (i). For the proof of (ii), a straightforward calculation shows that $\beta(k) - 1 = (2p-1)(1-p^k)/(1-p)$ and $\alpha(k) = 1 - (1-p)/(1-(2p-1)p^k)$. Since $p \geq 1/2$ implies $\alpha(k) \leq p$, the result now follows from Theorem 2.6. □

3 Multiple stops: buy/sell strategies

This section investigates the return of a player who is allowed to make multiple stops, alternately buying and selling a commodity. Each time the player buys or sells, a fixed transaction cost $c \geq 0$ is charged. If the player stops at times

$$0 \leq \tau_1 < \tau_2 < \cdots < \tau_{2m} \leq N,$$

then the commodity is bought on day τ_{2j-1}, and sold on day τ_{2j}, for $j = 1, \ldots, m$. Thus, the total return $P(N)$ is given by

$$P(N) = \sum_{j=1}^{m} (S_{\tau_{2j}} - S_{\tau_{2j-1}}) - 2mc.$$

The relations (8)-(10) below can be used to recursively derive the optimal expected return for any time horizon N. And thus optimal strategies can, at least in theory, be (recursively) specified. However, for arbitrary reinforcement sequences such specification is in general unenlightening. Here the emphasis is on developing optimal strategies in some interesting special cases which illustrate the relationship between the structure of the optimal strategy and the form of the reinforcement sequences and transaction costs.

First the zero-cost case is treated, where the optimal rule is very straightforward: buy at the start of an upward run, and sell at the start of a downward run. The optimal expected return is calculated explicitly for the case of a standard correlated random walk. (See Theorem 3.3.)

Next, the case of arbitrary transaction costs is considered under the assumption that $g^+(n) = g^-(n) = p$ for some constant p. In this case the process $\{S_n\}$ is a standard symmetric correlated random walk. Here the optimal strategy is to act precisely as in the zero-cost case, except that after a certain critical time $N - k^*$, depending on the cost c and the reinforcement p, the commodity should not be bought again. Intuitively, the expected additional return over the remaining k^* days would not compensate for the transaction costs. For this case, too, a closed-form expression for the optimal expected return is given.

The last two results in this section describe the optimal strategy for arbitrary cost and reinforcement, under the condition that the reinforcement probabilities exceed certain threshold values. When c is small, these thresholds are close to $1/2$. For instance, Theorem 3.8 assumes that $g^+(1) \geq c+1/2$. In this case the positive reinforcement is strong enough to compensate for the burden of transaction costs, making it always optimal to buy when the walk takes a step up. Moreover, there is a fixed, critical negative run length M such that the optimal strategy calls for selling as soon as a downward run of length M has occurred. Theorem 3.9 assumes $g^-(1) \geq (2c+1)/2(c+1)$. In this case, it is optimal to sell at the start of a downward run. (The rule specifying when to buy is more complicated).

3.1 Notation and preliminaries

Before the main results can be stated, a fair amount of notation needs to be developed. First, define the following states in the "buy/sell" process.

$(H, k, n, +)$: The commodity is currently being held ("holding" state), there are k days left to play, and the walk has taken exactly n upward steps on its current run.

$(F, k, n, +)$: The commodity is not currently held ("free" state), there are k days left to play, and the walk has taken exactly n upward steps on its current run.

States $(H, k, n, -)$ and $(F, k, n, -)$ are defined analogously, except that the walk has taken exactly n downward steps on its current run.

If the process is in state $(H, k, n, +)$ and the commodity is sold, then the process moves to states $(F, k-1, n+1, +)$ or $(F, k-1, 1, -)$, with respective probabilities $g^+(n)$ and $1 - g^+(n)$. On the other hand, if the process is in state $(H, k, n, +)$ and no action is taken, then the process moves to states $(H, k-1, n+1, +)$ or $(H, k-1, 1, -)$, with respective probabilities $g^+(n)$ and $1-g^+(n)$. The obvious corresponding state transitions hold for the other state and action (buy, sell, do nothing) combinations.

Let $P(N)$ be the expected gain under the optimal strategy given time horizon N, and let $P_B(N)$ (respectively $P_F(N)$) be the optimal expected gain given time horizon N and that the commodity is bought (respectively not bought) on day zero. Note that $P_B(N)$ may be negative, while $P_F(N) \geq 0$.

For $1 \leq k \leq N-1$ and $1 \leq n \leq N-k$, let $W^+_{HH}(k, n)$ (respectively $W^+_{HS}(k, n)$) be the maximum expected additional gain - possibly negative - given that there are k days left to play, the walk has taken exactly n upward steps on its current run, the commodity is currently being hold and is held at least one more day (respectively sold immediately). Let $W^+_{FB}(k, n)$ (respectively

$W_{FF}^{+}(k,n)$) be defined analogously, except given that the commodity is not being held and is then bought (respectively not bought). Quantities $W_{HH}^{-}(k,n), W_{HS}^{-}(k,n), W_{FB}^{-}(k,n)$, and $W_{FF}^{-}(k,n)$ are defined analogously, replacing "upward" with "downward".

Next, define quantities $V_H^{+}(k,n)$, $V_H^{-}(k,n)$, $V_F^{+}(k,n)$, and $V_F^{-}(k,n)$ by

$$\begin{aligned} V_H^{\pm}(k,n) &= \max\{W_{HH}^{\pm}(k,n), W_{HS}^{\pm}(k,n)\}, \\ V_F^{\pm}(k,n) &= \max\{W_{FB}^{\pm}(k,n), W_{FF}^{\pm}(k,n)\}. \end{aligned} \tag{8}$$

From the above definitions the following identities hold.

$$W_{HH}^{\pm}(k,n) = W_{FB}^{\pm}(k,n) + c, \qquad \text{and} \qquad W_{HS}^{\pm}(k,n) = W_{FF}^{\pm}(k,n) - c. \tag{9}$$

Moreover, we have the recursive relations

$$\begin{aligned} W_{HH}^{+}(k+1,n) &= g^{+}(n)\{V_H^{+}(k,n+1)+1\} + (1-g^{+}(n))\{V_H^{-}(k,1)-1\}, \\ W_{HS}^{+}(k+1,n) &= g^{+}(n)V_F^{+}(k,n+1) + (1-g^{+}(n))V_F^{-}(k,1) - c, \\ W_{HH}^{-}(k+1,n) &= (1-g^{-}(n))\{V_H^{+}(k,1)+1\} + g^{-}(n)\{V_H^{-}(k,n+1)-1\}, \\ W_{HS}^{-}(k+1,n) &= (1-g^{-}(n))V_F^{+}(k,1) + g^{-}(n)V_F^{-}(k,n+1) \; - c. \end{aligned} \tag{10}$$

The following intuitively plausible lemma is used to obtain most of the results in this section. It verifies that in the holding state after a step up it is always optimal to hold; and in the free state after a step down it is always optimal not to buy. Moreover, in all cases, it is optimal not to buy at time zero, but to wait at least until the first upward step. Note that the proof of the lemma does not require monotonicity of the sequences $g^{+}(n)$ and $g^{-}(n)$.

Lemma 3.1 *(a) For all k and n,*

(i) $W_{HH}^{+}(k,n) \geq W_{HS}^{+}(k,n)$, *and*

(ii) $W_{FF}^{-}(k,n) \geq W_{FB}^{-}(k,n)$.

In other words, in state $(H,\cdot,\cdot,+)$ it is always optimal to continue holding, and in state $(F,\cdot,\cdot,-)$ it is always optimal not to buy (i.e. to stay "free").

(b) $P_F(N) \geq P_B(N)$.

Proof. Statements (i) and (ii) will be proved simultaneously, by induction on k. First, let $k=1$. Then $W_{HH}^{+}(1,n) = 2g^{+}(n) - 1 - c \geq -c = W_{HS}^{+}(1,n)$, and $W_{FF}^{-}(1,n) = 0 \geq 1 - 2g^{-}(n) - 2c = W_{FB}^{-}(1,n)$. Next, assume that (i) and (ii) hold for some k and all n. Then

$$\begin{aligned} W_{HH}^{+}(k,n+1) &\geq \max\{W_{HS}(k,n+1), W_{FB}^{+}(k,n+1)+c\} \\ &\geq \max\{W_{FF}^{+}(k,n+1)-c, W_{FB}^{+}(k,n+1)-c\} \\ &= V_F^{+}(k,n+1) - c, \end{aligned} \tag{11}$$

where the first inequality follows from the induction assumption, the second by (10), and the equality by (9).

Thus,

$$\begin{aligned} W_{HH}^{+}(k+1,n) &= g^{+}(n)V_H^{+}(k,n+1) + (1-g^{+}(n))V_H^{-}(k,1) + 2g^{+}(n) - 1 \\ &\geq g^{+}(n)W_{HH}^{+}(k,n+1) + (1-g^{+}(n))W_{HS}^{-}(k,1) + 2g^{+}(n) - 1 \\ &= g^{+}(n)W_{HH}^{+}(k,n+1) + (1-g^{+}(n))\{W_{FF}^{-}(k,1) - c\} + 2g^{+}(n) - 1 \\ &\geq g^{+}(n)\{V_F^{+}(k,n+1) - c\} + (1-g^{+}(n))\{V_F^{-}(k,1) - c\} \\ &= W_{HS}^{+}(k+1,n), \end{aligned}$$

where the last inequality follows from (11), the induction assumption and $g^+(n) \geq 1/2$. The proof of (ii) is analogous.

Part (b) of the lemma follows easily from part (a). First observe that for all k and n,

$$V_H^-(k,n) \leq W_{FF}^-(k,n) + c.$$

This follows since $V_H^-(k,n) = \max\{W_{HH}^-(k,n), W_{HS}^-(k,n)\} = \max\{W_{FB}^-(k,n)+c, W_{FF}^-(k,n)-c\} \leq W_{FF}^-(k,n) + c$, where the inequality follows by (ii). Thus,

$$\begin{aligned} P_B(N) &= \frac{1}{2}\{V_H^+(N-1,1) + 1 + V_H^-(N-1,1) - 1\} - c \\ &= \frac{1}{2}\{W_{FB}^+(N-1,1) + V_H^-(N-1,1) - c\} \\ &\leq \frac{1}{2}\{W_{FB}^+(N-1,1) + W_{FF}^-(N-1,1)\} \\ &\leq \frac{1}{2}\{V_F^+(N-1,1) + V_F^-(N-1,1)\} = P_F(N). \qquad \square \end{aligned}$$

The inequalities established in the next lemma are used throughout the rest of the section.

Lemma 3.2 *(i) If $W_{FB}^+(k,n+1) \geq W_{FF}^+(k,n+1)$, then*

$$W_{FB}^+(k+1,n) - W_{FF}^+(k+1,n) \geq 2g^+(n)(1+c) - (2c+1), \tag{12}$$

with equality if and only if $W_{HS}^-(k,1) \geq W_{HH}^-(k,1)$.

(ii) If $W_{HS}^-(k,n+1) \geq W_{HH}^-(k,n+1)$, then

$$W_{HS}^-(k+1,n) - W_{HH}^-(k+1,n) \geq 2g^-(n)(1+c) - (2c+1), \tag{13}$$

with equality if and only if $W_{FB}^+(k,1) \geq W_{FF}^+(k,1)$.

Proof. Suppose $W_{FB}^+(k,n+1) \geq W_{FF}^+(k,n+1)$. Using (9), (10) and Lemma 3.1 it follows that

$$\begin{aligned} &W_{FB}^+(k+1,n) - W_{FF}^+(k+1,n) \\ &\quad = g^+(n)\{V_H^+(k,n+1) - V_F^+(k,n+1) + 1\} \\ &\qquad\qquad + (1-g^+(n))\{V_H^-(k,1) - V_F^-(k,1) - 1\} - c \\ &\quad \geq g^+(n)\{W_{HH}^+(k,n+1) - W_{FB}^+(k,n+1) + 1\} \\ &\qquad\qquad + (1-g^+(n))\{W_{HS}^-(k,1) - W_{FF}^-(k,1) - 1\} - c \\ &\quad = 2g^+(n)(1+c) - (2c+1), \end{aligned}$$

with equality if and only if $W_{HS}^-(k,1) \geq W_{HH}^-(k,1)$. The proof of (ii) follows similarly. $\square$

3.2 The case $c = 0$

If no transaction costs are charged, then the optimal strategy is simple.

Proposition 3.3 *Assume that $c = 0$.*

(a) An optimal strategy is to buy at time 0, and then: continue holding the commodity in state $(H,\cdot,\cdot,+)$, sell in state $(H,\cdot,\cdot,-)$, buy in state $(F,\cdot,\cdot,+)$, and do not buy in state $(F,\cdot,\cdot,-)$.

In other words, it is always optimal to buy at time 0, and subsequently to sell as soon as the walk takes a step down, and to buy again as soon as a step up is taken, and so on, repeating this strategy until time N. At time N the commodity must be sold if it is still being held.

(b) (Symmetric CRW) *If for some $1/2 \le p \le 1$, $g^+(n) = g^-(n) = p$ for all $n \ge 1$, then*

$$P(N) = (N-1)(p-1/2).$$

Proof. Since $c = 0$, Lemma 3.1 and (9) give

$$W^+_{FB}(k,n) = W^+_{HH}(k,n) \ge W^+_{HS}(k,n) = W^+_{FF}(k,n), \tag{14}$$

and

$$W^-_{HS}(k,n) = W^-_{FF}(k,n) \ge W^-_{FB}(k,n) = W^-_{HH}(k,n). \tag{15}$$

It follows that $V^\pm_F(k,n) = V^\pm_H(k,n)$ for all k and n. In particular,

$$\begin{aligned} P_B(N) &= \frac{1}{2}\{V^+_H(N-1,1)+1\} + \frac{1}{2}\{V^-_H(N-1,1)-1\} \\ &= \frac{1}{2}\{V^+_F(N-1,1) + V^-_F(N-1,1)\} \\ &= P_F(N). \end{aligned}$$

This, together with Lemma 3.1, (14) and (15) gives part (a) - for instance, in the event of a downward step in the holding state, (15) implies that it is optimal to sell, etc.

To prove part (b), first note that since the reinforcement sequences are constant, the values of $V^+_H(k,n)$, $V^+_F(k,n)$, $V^-_H(k,n)$ and $V^-_F(k,n)$ are independent of n. Let v_k denote the common value of $V^+_H(k,n) = V^+_F(k,n)$, and let w_k denote the common value of $V^-_H(k,n) = V^-_F(k,n)$. Then

$$v_{j+1} = p(v_j+1) + (1-p)(w_j-1), \tag{16}$$

and

$$w_{j+1} = (1-p)v_j + pw_j, \tag{17}$$

which implies $v_{j+1} + w_{j+1} = v_j + w_j + (2p-1)$ for each j. Since $v_1 + w_1 = 2p-1$, it follows inductively that $v_k + w_k = k(2p-1)$ for each k. In particular, $P(N) = P_B(N) = (v_{N-1}+w_{N-1})/2 = (N-1)(p-1/2)$. □

Remark 3.4 Notice that the proof of Proposition 3.3 shows that the optimal strategy is not unique. In fact, it does not matter whether the commodity is bought at time zero, or at the first upward step. Also, the only structural assumption on the reinforcement probabilities used is that $g^+(n), g^-(n) \ge 1/2$ for all n. Otherwise, the sequences $g^+(n)$ and $g^-(n)$ may be completely arbitrary.

Remark 3.5 The optimal expected return $P(N)$ can be calculated in the slightly more general case that $g^+(n) = p$ and $g^-(n) = q$, $n \ge 1$, for $1/2 \le p, q \le 1$. In this case, (17) is replaced by

$$w_{j+1} = (1-q)v_j + qw_j. \tag{18}$$

Solving (16) and (18) using generating functions and the method of partial fractions yields

$$\begin{aligned} P(N) &= \frac{p-1/2}{(2-p-q)^2}\left[2(1-q)(2-p-q)(N-1) + (q-p)\{1-(p+q-1)^{N-1}\}\right] \\ &\sim \frac{(2p-1)(1-q)}{2-p-q}N, \qquad \text{as} \quad N \to \infty. \end{aligned}$$

3.3 Symmetric CRW with arbitrary transaction costs

In this subsection it will be assumed that $\{S_n\}$ is a standard, symmetric correlated random walk. That is, $g^+(n) = g^-(n) = p$ for all $n \geq 1$ and some fixed $p \geq 1/2$. In this case, it is optimal to utilize the strategy from Proposition 3.3 up till a certain critical time, and after that time to sell immediately after a downward step, and never buy. Intuitively, after the critical time there is not sufficient time left to gain enough to compensate for the transaction costs.

Theorem 3.6 *Suppose that $g^+(n) = g^-(n) = p$ for all $n \geq 1$ and some fixed $p \geq 1/2$. Define*

$$k^* = \inf \left\{ k \geq 0 : c \leq (p - 1/2) \left(\frac{1 - p^k}{1 - p} \right) \right\}.$$

(a) If $k^* \geq N$ the optimal strategy is to do nothing. If $k^* < N$, the optimal strategy is to act as follows:

(i) Do not buy at time 0.

(ii) In state $(F, k, \cdot, +)$: buy if and only if $k \geq k^*$.

(iii) In state $(F, \cdot, \cdot, -)$: do not buy.

(iv) In state $(H, \cdot, \cdot, +)$: hold.

(v) In state $(H, \cdot, \cdot, -)$: sell.

In other words, it is optimal to buy as soon as the walk takes a step up, and to sell again as soon as it takes a step down, repeating this strategy until day $N - k^*$. If on that day the commodity is held, it should be sold after the next downward step or on day N, whichever comes first. After day $N - k^*$ it is always optimal not to buy.

(b) If $k^* \geq N$ then $P(N) = 0$. Otherwise,

$$P(N) = \frac{1}{2} \left[(N - k^* - 1)\{2p - 1 - 2(1 - p)c\} + \frac{(2p - 1)(1 - p^{k^*})}{1 - p} - 2c \right].$$

(c) $\lim_{N \to \infty} P(N)/N = \max\{0, p - 1/2 - (1 - p)c\}$.

Before proving the theorem, note that since the reinforcement probabilities $g^+(n)$ and $g^-(n)$ do not depend on n, neither do the quantities $W^+_{HH}(k,n)$, $W^+_{HS}(k,n)$, $V^+_H(k,n)$, etc. Therefore the notation can be simplified to $W^+_{HH}(k)$, $W^+_{HS}(k)$, $V^+_H(k)$, etc.

Proof of Theorem 3.6. The optimality of parts (i),(iii) and (iv) follows from Lemma 3.1. To prove the optimality of (ii) and (v), the following expressions are established.

$$W^+_{FB}(k) - W^+_{FF}(k) = \begin{cases} (2p - 1) \left(\frac{1 - p^k}{1 - p} \right) - 2c, & 1 \leq k \leq k^* \\ 2p - 1 - 2(1 - p)c, & k^* < k \leq N - 1, \end{cases} \tag{19}$$

and

$$W^-_{HS}(k) - W^-_{HH}(k) = \begin{cases} (2p - 1)p^{k-1}, & 1 \leq k \leq k^* \\ 2p - 1 - 2(1 - p)c, & k^* < k \leq N - 1. \end{cases} \tag{20}$$

Since $k^* < \infty$ if and only if $c < (2p - 1)/2(1 - p)$, the definition of k^* and (19) imply that $W^+_{FB}(k) \geq W^+_{FF}(k)$ if and only if $k \geq k^*$. And (20) implies that $W^-_{HS}(k) \geq W^-_{HH}(k)$ for all k. The optimality of (ii) and (v) follows.

Formulas (19) and (20) are proved by induction. Note first that $W^+_{FB}(1)-W^+_{FF}(1) = (2p-1-2c)$, and $W^-_{HS}(1) - W^-_{HH}(1) = -c-(1-2p-c) = 2p-1$. Assume next that (19) and (20) hold for some $k < k^*$. In particular, $W^-_{HS}(k) \geq W^-_{HH}(k)$. And since $k < k^*$, $W^+_{FB}(k) < W^+_{FF}(k)$ by the definition of k^* and the induction hypothesis. Thus

$$\begin{aligned}
&W^+_{FB}(k+1) - W^+_{FF}(k+1) \\
&\qquad = p\{W^+_{HH}(k) - W^+_{FF}(k)\} + (1-p)\{W^-_{HS}(k) - W^-_{FF}(k)\} + 2p - 1 - c \\
&\qquad = p\{W^+_{FB}(k) - W^+_{FF}(k) + c\} + (1-p)(-c) + 2p - 1 - c \\
&\qquad = p\left[(2p-1)\left(\frac{1-p^k}{1-p}\right) - c\right] + pc + 2p - 1 - 2c \\
&\qquad = (2p-1)\left(\frac{1-p^{k+1}}{1-p}\right) - 2c,
\end{aligned}$$

and

$$\begin{aligned}
&W^-_{HS}(k+1) - W^-_{HH}(k+1) \\
&\qquad = (1-p)\{W^+_{FF}(k) - W^+_{HH}(k)\} + p\{W^-_{FF}(k) - W^-_{HS}(k)\} + 2p - 1 - c \\
&\qquad = (1-p)\{W^+_{FF}(k) - W^+_{FB}(k) - c\} + pc + 2p - 1 - c \\
&\qquad = (1-p)\left[c - (2p-1)\left(\frac{1-p^k}{1-p}\right)\right] + 2p - 1 - (1-p)c \\
&\qquad = (2p-1)p^k.
\end{aligned}$$

This establishes the first case of (19) and (20). Note that both expressions have now been proved for $k = k^*$.

If $k^* \geq N-1$, the proof of (19) and (20) is complete, since the second case ($k^* < k \leq N-1$) can not occur. Suppose then that $k^* \leq N-2$. In particular, this means that $2p-1-2(1-p)c \geq 0$. Substituting $k = k^*$ in (19) and using the definition of k^* gives

$$W^+_{FB}(k^*) - W^+_{FF}(k^*) = (2p-1)(1-p^{k^*})/(1-p) - 2c \geq 0. \tag{21}$$

Substituting $k = k^*$ in (20) gives

$$W^-_{HS}(k^*) - W^-_{HH}(k^*) = (2p-1)p^{k^*-1} \geq 0.$$

Thus, the hypotheses of both parts of Lemma 3.2 are satisfied for $k = k^*$, and (12) and (13) hold with equality. Note that $2g^+(n)(1+c) - (2c+1) = 2g^-(n)(1+c) - (2c+1) = 2p(1+c) - (2c+1) = 2p-1-2(1-p)c \geq 0$. The second case in (19) and (20) therefore follows by induction and Lemma 3.2.

To prove (b) for the case $k^* < N$, note that, for $k > k^*$, $V^+_F(k) = W^+_{FB}(k)$ and $V^-_H(k) = W^-_{HS}(k)$. Hence

$$P(N) = P_F(N) = \frac{1}{2}[V^+_F(N-1) + V^-_F(N-1)] = \frac{1}{2}[W^+_{FB}(N-1) + W^-_{FF}(N-1)].$$

If $k^* \leq N-2$, the relations (9) and (10) can be used to obtain

$$\begin{aligned} P(N) &= \frac{1}{2}[p\{V_H^+(N-2)+1\} + (1-p)\{V_H^-(N-2)-1\} \\ &\qquad + (1-p)V_F^+(N-2) + pV_F^-(N-2) - c] \\ &= \frac{1}{2}[2p-1-c+pW_{HH}^+(N-2)+(1-p)W_{HS}^-(N-2) \\ &\qquad + (1-p)W_{FB}^+(N-2)+pW_{FF}^-(N-2)] \\ &= \frac{1}{2}[2p-1-2(1-p)c+W_{FB}^+(N-2)+W_{FF}^-(N-2)]. \end{aligned}$$

Iterating yields

$$P(N) = \frac{1}{2}[(N-k^*-1)\{2p-1-2(1-p)c\} + W_{FB}^+(k^*) + W_{FF}^-(k^*)]. \tag{22}$$

Since with fewer than k^* days left it is optimal not to buy it follows that $W_{FF}^-(k^*) = W_{FF}^+(k^*) = 0$. This, together with (21) and (22) finishes the proof of (b). Finally, observe that (c) is a direct consequence of (b) since $k^* < \infty$ if and only if $2p-1 > 2(1-p)c$. □

3.4 Results for more general cases

Finally, this subsection gives optimal strategies when the reinforcement sequences $g^+(n)$ and $g^-(n)$ exceed some threshold. Note that when c is small these thresholds are close to $1/2$ and so in this case the extra conditions are fairly mild.

Lemma 3.7 *(i) If $g^+(1) \geq c + 1/2$, then $W_{FB}^+(k,n) \geq W_{FF}^+(k,n)$ for all k and n. In other words, in state $(F,\cdot,\cdot,+)$ it is optimal to buy.*

(ii) If $g^-(1) \geq (2c+1)/2(c+1)$, then $W_{HS}^-(k,n) \geq W_{HH}^-(k,n)$ for all k and n. In other words, in state $(H,\cdot,\cdot,-)$ it is best to sell.

Proof. Only the proof of (ii) is outlined here. The proof of (i) is analogous.

Observe that, for each n, $W_{HS}^-(1,n) - W_{HH}^-(1,n) = -c - (1-2g^-(n)-c) = 2g^-(n) - 1 \geq 0$. Part (ii) now follows by induction, using Lemma 3.2 and the fact that $g^-(n) \geq g^-(1)$. □

Theorem 3.8 *Suppose $g^+(1) \geq c + 1/2$, and let*

$$M := \inf\{n \geq 1 : g^-(n) \geq (2c+1)/2(c+1)\}.$$

(M is allowed to take the value ∞.)

Then

(i) $W_{FB}^+(k,n) \geq W_{FF}^+(k,n)$ for all k and n.

(ii) $W_{HS}^-(1,n) \geq W_{HH}^-(1,n)$ for all n.

(iii) For $k \geq 2, W_{HS}^-(k,n) \geq W_{HH}^-(k,n)$ iff $n \geq M$.

Thus, the optimal strategy is to act as follows:

- *Do not buy at time 0.*
- *In state $(F,\cdot,\cdot,+)$: buy.*

- *In state $(F,\cdot,\cdot,-)$: do not buy.*
- *In state $(H,\cdot,\cdot,+)$: hold.*
- *In state $(H,k,n,-)$: sell if $k=1$ or ($k\geq 2$ and $n\geq M$); hold otherwise.*

Proof. Parts (i) and (ii) follow from Lemma 3.7 and its proof. To prove (iii), note first that by (i), (ii), and Lemma 3.2 it follows that for all n

$$W^-_{HS}(2,n) - W^-_{HH}(2,n) = 2g^-(n)(1+c) - (2c+1). \tag{23}$$

And (23) is non-negative if and only if $n \geq M$. Assume the equivalence in (iii) holds for some $k\geq 2$. For $n\geq M-1$, $W^-_{HS}(k,n+1)\geq W^-_{HH}(k,n+1)$, so again by Lemma 3.2,

$$W^-_{HS}(k+1,n) - W^-_{HH}(k+1,n) = 2g^-(n)(1+c) - (2c+1),$$

which is again non-negative if and only if $n\geq M$. If, on the other hand, $n < M-1$, then $W^-_{HS}(k,n+1) < W^-_{HH}(k,n+1)$, and hence

$$\begin{aligned}
W^-_{HS}(k+1,n) &- W^-_{HH}(k+1,n) \\
&= (1-g^-(n))\{V^+_F(k,1) - V^+_H(k,1) - 1\} \\
&\qquad + g^-(n)\{V^-_F(k,n+1) - V^-_H(k,n+1) + 1\} - c \\
&= (1-g^-(n))\{W^+_{FB}(k,1) - W^+_{HH}(k,1) - 1\} \\
&\qquad + g^-(n)\{W^-_{FF}(k,n+1) - W^-_{HH}(k,n+1) + 1\} - c \\
&< (1-g^-(n))\{W^+_{FB}(k,1) - W^+_{HH}(k,1) - 1\} \\
&\qquad + g^-(n)\{W^-_{FF}(k,n+1) - W^-_{HS}(k,n+1) + 1\} - c \\
&= 2g^-(n)(1+c) - (2c+1) \\
&< 0.
\end{aligned}$$

The proof now follows by induction. □

Recall the definition of $\gamma(k,n)$ from Definition 2.1.

Theorem 3.9 *Suppose $g^-(1)\geq (2c+1)/2(c+1)$, and define the (possibly infinite) thresholds*

$$M_1 := \inf\{n\geq 1 : g^+(n) \geq (2c+1)/2(c+1)\},$$

and

$$M_2 := \inf\{n\geq 1 : g^+(n)\geq c+1/2\}.$$

(Note that $M_2\geq M_1$.) Then

(i) $W^-_{HS}(k,n)\geq W^-_{HH}(k,n)$ for all k and n, and

(ii) $W^+_{FB}(k,n)\geq W^+_{FF}(k,n)$ if and only if $n\geq M_1$ and $\gamma(k,n)\geq 2c+1$.

Thus it is optimal to act as follows:

- *Do not buy at time 0.*
- *In state $(F,k,n,+)$: buy if and only if $n\geq M_1$ and $\gamma(k,n)\geq 2c+1$.*

- *In state $(F,\cdot,\cdot,-)$: do not buy.*
- *In state $(H,\cdot,\cdot,+)$: hold.*
- *In state $(H,\cdot,\cdot,-)$: sell.*

The condition $\gamma(k,n) \geq 2c+1$ is automatically satisfied when $n \geq M_1$ and $k+n \geq M_2+1$.

The proof uses the following lemmas. The first is easily checked using the definition of $\gamma(k,n)$.

Lemma 3.10 *(i) If $n < M_1$ and $\gamma(k,n+1) < 2c+1$, then $\gamma(k+1,n) < 2c+1$.*

(ii) If $n \geq M_1$ and $\gamma(k,n+1) \geq 2c+1$, then $\gamma(k+1,n) \geq 2c+1$.

Lemma 3.11 *If $M_1 \leq n \leq M_2 - k$ and $\gamma(k,n) < 2c+1$, then*

$$W_{FB}^{+}(k,n) - W_{FF}^{+}(k,n) = \gamma(k,n) - (2c+1) < 0. \tag{24}$$

Proof. Since $W_{FB}^{+}(1,n) - W_{FF}^{+}(1,n) = 2g^{+}(n) - 1 - 2c = \gamma(1,n) - (2c+1) < 0$ for $n \leq M_2 - 1$, (24) follows for $k = 1$. Assume (24) holds for some $k \geq 1$. Let $M_1 \leq n \leq M_2 - (k+1)$, and assume $\gamma(k+1,n) < 2c+1$. Then $\gamma(k,n+1) < 2c+1$ by Lemma 3.10 (ii). And since $n+1 \leq M_2 - k$, the induction hypothesis implies $W_{FB}^{+}(k,n+1) - W_{FF}^{+}(k,n+1) = \gamma(k,n+1) - (2c+1) < 0$. This gives $V_F^{+}(k,n+1) = W_{FB}^{+}(k,n+1)$. Now applying (10) and the recursive definition of $\gamma(k+1,n)$, it follows that $W_{FB}^{+}(k+1,n) - W_{FF}^{+}(k+1,n) = \gamma(k+1,n) - (2c+1) < 0$. □

Lemma 3.12 *If $n \geq M_1$ and $k+n \geq M_2+1$, then $\gamma(k,n) \geq 2c+1$.*

Proof. The condition $k+n \geq M_2+1$ implies $\gamma(1,k+n-1) = 2g^{+}(k+n-1) \geq 2c+1$. If $k=1$ the proof is complete. Assume $k \geq 2$. Proceeding by induction, suppose that $\gamma(j,k+n-j) \geq 2c+1$ for some $j \leq k-1$. Since

$$k+n-(j+1) \geq n \geq M_1,$$

Lemma 3.10 (ii) yields $\gamma(j+1,k+n-(j+1)) \geq 2c+1$. Substituting $k-1$ for j gives the desired result. □

Proof of Theorem 3.9. Part (i) follows from Lemma 3.7. For the proof of (ii), observe that $W_{FB}^{+}(1,n) - W_{FF}^{+}(1,n) = \gamma(1,n) - (2c+1)$. So (ii) holds for $k=1$. Assume that the equivalence (ii) holds for some $k \geq 1$ and for all $n \geq 1$. There are three possibilities.

Case 1. $n+1 < M_1$. By the induction hypothesis, $W_{FB}^{+}(k,n+1) < W_{FF}^{+}(k,n+1)$. Using (10) it follows that

$$\begin{aligned}
&W_{FB}^{+}(k+1,n) - W_{FF}^{+}(k+1,n) \\
&\quad = g^{+}(n)\{W_{HH}^{+}(k,n+1) - W_{FF}^{+}(k,n+1) + 1\} \\
&\qquad\qquad + (1-g^{+}(n))\{W_{HS}^{-}(k,1) - W_{FF}^{-}(k,1) - 1\} - c \\
&\quad = g^{+}(n)\{W_{FB}^{+}(k,n+1) - W_{FF}^{+}(k,n+1) + 2c + 2\} - (2c+1) \\
&\quad < 2g^{+}(n)(1+c) - (2c+1) \\
&\quad < 0.
\end{aligned}$$

Case 2. $M_1 \leq n+1 < M_2 + 1 - k$. There are two subcases.

a) $\gamma(k,n+1) < 2c+1$. By Lemma 3.11,

$$W_{FB}^{+}(k,n+1) - W_{FF}^{+}(k,n+1) = \gamma(k,n+1) - (2c+1) < 0.$$

Hence

$$
\begin{aligned}
W^+_{FB}(k+1,n) - W^+_{FF}(k+1,n) \\
&= g^+(n)\{W^+_{FB}(k,n+1) - W^+_{FF}(k,n+1) + 2 + 2c\} - (2c+1) \\
&= g^+(n)\{1 + \gamma(k,n+1)\} - (2c+1) \\
&= \gamma(k+1,n) - (2c+1).
\end{aligned}
$$

The equivalence (ii) now follows immediately for $M_1 \le n < M_2 - k$. If $n = M_1 - 1$, Lemma 3.10 (i) shows that $W^+_{FB}(k+1,n) < W^+_{FF}(k+1,n)$.

b) $\gamma(k,n+1) \ge 2c+1$. By Lemma 3.10 (ii), $\gamma(k+1,n) \ge 2c+1$. The induction hypothesis yields $W^+_{FB}(k,n+1) \ge W^+_{FF}(k,n+1)$. So $W^+_{FB}(k+1,n) - W^+_{FF}(k+1,n) = 2g^+(n)(1+c) - (2c+1)$ by Lemma 3.2. It thus follows that $W^+_{FB}(k+1,n) \ge W^+_{FF}(k+1,n)$ if and only if $n \ge M_1$.

Case 3. $n+1 \ge (M_2+1-k) \vee (M_1+1)$. Since $k+(n+1) \ge M_2+1$, the induction hypothesis and Lemma 3.12 imply $W^+_{FB}(k,n+1) \ge W^+_{FF}(k,n+1)$. Applying Lemma 3.2 gives

$$W^+_{FB}(k+1,n) - W^+_{FF}(k+1,n) = 2g^+(n)(1+c) - (2c+1) \ge 0,$$

since $n \ge M_1$. □

4 Optimal versus *buy-and-hold*

In this section, some numeric examples are given comparing the optimal strategies derived in the previous sections to *buy-and-hold* strategies. Buy-and-hold strategies buy the commodity on day 0 and hold it until day N when it is sold. The return of such a strategy (ignoring transaction costs) is just the expected value on day N of a DRRW which starts at 0. Let $E(N)$ denote the expected value on day N of a DRRW starting at 0 with reinforcement sequences $\{g^+(n)\}$ and $\{g^-(n)\}$. Then

$$E(0) = E(1) = 0,$$

and, for $N > 1$,

$$E(N) = \frac{1}{2}\{E^+(N-1,1)+1\} + \frac{1}{2}\{E^-(N-1,1)-1\},$$

where $E^+(k,n)$ and $E^-(k,n)$ denote the expected value of the DRRW after k days, given that the walk is currently at 0 and has taken exactly n steps in the positive or negative direction, respectively. For $k \ge 1$,

$$
\begin{aligned}
E^+(k+1,n) &= g^+(n)\{E^+(k,n+1)+1\} + (1-g^+(n)\{E^-(k,1)-1\}, \\
E^-(k+1,n) &= (1-g^-(n))\{E^+(k,1)+1\} + g^-(n)\{E^-(k,n+1)-1\}.
\end{aligned}
\tag{25}
$$

So, for any given reinforcement sequences, $E(N)$ can be obtained recursively. For constant reinforcement sequences, a closed form solution is available. Suppose that for fixed $1/2 \le p, q \le 1$, $g^+(n) = p$ and $g^-(n) = q$ for all $n \ge 1$. Then $E^+(k,n)$ and $E^-(k,n)$ do not depend on n and the relations (25) become

$$
\begin{aligned}
E^+(k+1) &= p(E^+(k)+1) + (1-p)(E^-(k)-1), \\
E^-(k+1) &= (1-q)(E^+(k)+1) + q(E^-(k)-1).
\end{aligned}
$$

Thus, in this case, for $N \geq 2$,

$$E(N) = (p-q) + E(N-1) + \frac{1}{2}(p-q)\{E^+(N-2) - E^-(N-2)\}. \tag{26}$$

Letting $F(k) = E^+(k) - E^-(k)$ it is straightforward to derive and solve the relation $F(k) = 2r + rF(k-1)$, yielding

$$F(k) = 2r\left(\frac{1-r^k}{1-r}\right),$$

where $r = p + q - 1$. Now applying standard techniques to (26) gives

$$E(N) = \frac{p-q}{1-r}\left(N - \frac{1-r^N}{1-r}\right). \tag{27}$$

Notice that (27) could also have been established by using explicit expressions for the distribution of a correlated random walk after N steps. (See Böhm (2000)). The method used here, however, is more straightforward.

Example 4.1 Suppose $g^+(n) \equiv .75$ and $g^-(n) \equiv .70$. The return for the single stop strategy is given in Figure 1 along with the buy-and-hold expected return, $E(N)$, for $N = 1, \ldots, 25$. Notice that for $N > 10$ both $E(N)$ and $V(N)$ appear nearly linear with the same slope. For constant reinforcements sequences $g^+(n) \equiv p > q \equiv g^-(n)$, this is indeed the case, as the following proposition shows.

Proposition 4.2 *If $g^+(n) \equiv p > q \equiv g^-(n)$, then*

$$\lim_{N\to\infty} \frac{E(N)}{N} = \frac{p-q}{1-r} = \lim_{N\to\infty} \frac{V(N)}{N}. \tag{28}$$

Proof. The first equality in (28) is obvious from (27). To see the second equality, first note that from the proof of Corollary 2.7 it follows that $\lim_{k\to\infty} \alpha(k) = p$. Hence there exists a k^* such that $\alpha(k) > q$ for all $k \geq k^*$. Thus, by Corollary 2.7, for large N it is optimal to continue holding at least until time $N - k^* - 1$. Since $V^+(k,n)$ and $V^-(k,n)$ do not depend on n, they can be abbreviated to $V^+(k)$ and $V^-(k)$. By Corollary 2.7, $V^+(k) > 0$ and $V^-(k) > 0$ for all $k > k^*$. Hence for $N > k^*$, (2) gives

$$V(N) = (p-q) + V(N-1) + \frac{1}{2}(p-q)\{V^+(N-2) - V^-(N-2)\}.$$

Thus, for large N, $V(N)$ satisfies the same difference equation as $E(N)$. Solving the equation yields

$$V(N) = C_1 + \frac{p-q}{1-r}N + C_2 r^N,$$

where C_1 and C_2 are constants depending only on $V^+(k^*)$ and $V^-(k^*)$. This completes the proof. □

Thus, any advantage of a single stop optimal strategy over a buy-and-hold strategy is obtained near the end of the time period. Close to the end of the time period, the optimal strategy will sell at the start of a downward run, while the buy-and-hold strategy will continue holding to the very end. In general, this advantage is greatest when p and q are both close to 1. For instance, Figure 2 shows $E(N)$ and $V(N)$ for $g^+(n) \equiv .95$ and $g^-(n) \equiv .93$.

Example 4.3 below compares the buy-and-hold strategy to multiple stop optimal strategies for several transaction costs. When $c > 0$, the expected return of the buy-and-hold strategy (buy at time 0 and sell at time N) is given by $E(N) - 2c$. As might be expected, generally the larger the transaction costs, the closer the value of the buy-and-hold strategy is to that of the optimal strategy. (However, it can be more complicated than this. For large relative transaction costs, it may be optimal to buy only when the walk is likely to continue on a long upward run. And so a strategy which waits for such an opportunity will do better than a strategy which buys at time 0.) On the other hand, when transaction costs are included, the expected return for multiple (and single) stop strategies is always greater than or equal to 0, while the expected return of a buy-and-hold strategy may be negative.

Unlike single stop strategies, the value of the optimal multiple stop strategy may asymptotically diverge from the buy-and-hold strategy (see Figure 3). This can happen when long upward and downward runs are likely and transaction costs are low enough that successive buying (on an uptrend) and selling (on a downtrend) is advantageous. The advantage is especially evident for small transaction costs when the overall trend of the DRRW is downward - for example, when $g^+(n) \equiv p < q \equiv g^-(n)$, as shown in Figure 6.

Example 4.3 Let $g^+(n) \equiv .85$ and $g^-(n) \equiv .75$. The return for the multiple stop optimal strategies when $c = .05$, $.5$ and 1 are shown in Figures 3, 4 and 5 respectively. For these reinforcement sequences the return from the single stop optimal strategy is nearly the same as the buy-and-hold strategy.

5 Future work

Some natural questions which would be interesting to pursue include the following.

- Characterize reinforcement sequences for which $V(N)/E(N) \to 1$.
- Characterize reinforcement sequences and transaction costs for which $P(N)/E(N) \to 1$.
- Determine optimal strategies when the unit step of the DRRW is replaced by a random step size. In this case, it is natural to let the reinforcement probabilities depend not only on the length of the current run, but also on the step sizes during the current run.
- Determine optimal strategies under a constraint on the number of trades (buying or selling) that can be made within a given time period.
- Determine optimal strategies when future returns are discounted.
- Include an explicit drift component in the DRRW, and determine optimal strategies.

The last four items in the above list are extensions that make the model more realistic, at the usual price of complicating the analysis. Of course, in each case, recursive relations analogous to (1) and (10) can be derived, and so the optimal strategies can always be calculated by a computer program. But are there cases where the optimal strategies can be expressed by a simple rule or formula?

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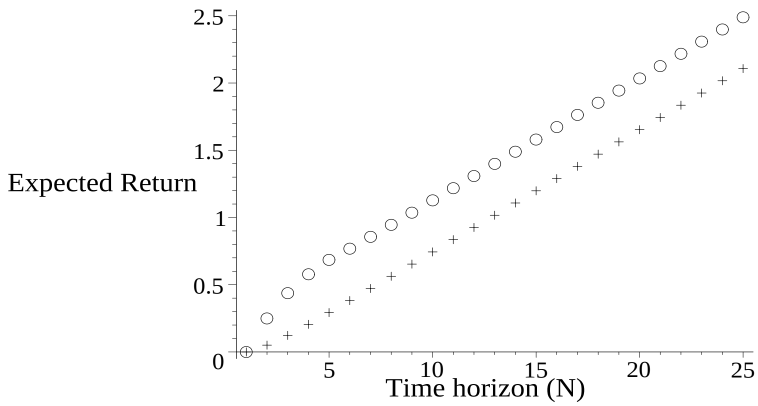

Figure 1: Expected return for optimal strategy and buy-and-hold strategy for $g^{+}(n) \equiv .75$ and $g^{-}(n) \equiv .70$. $E(N)$ values are denoted by crosses and $V(N)$ values by circles.

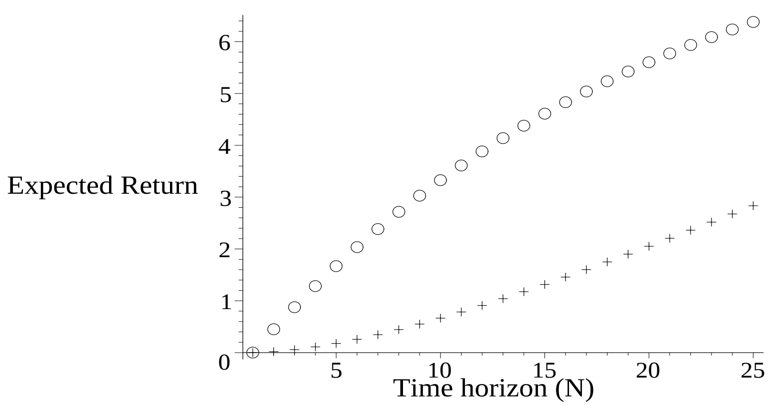

Figure 2: Expected return for optimal strategy and buy-and-hold strategy for $g^{+}(n) \equiv .95$ and $g^{-}(n) \equiv .93$. $E(N)$ values are denoted by crosses and $V(N)$ values by circles.

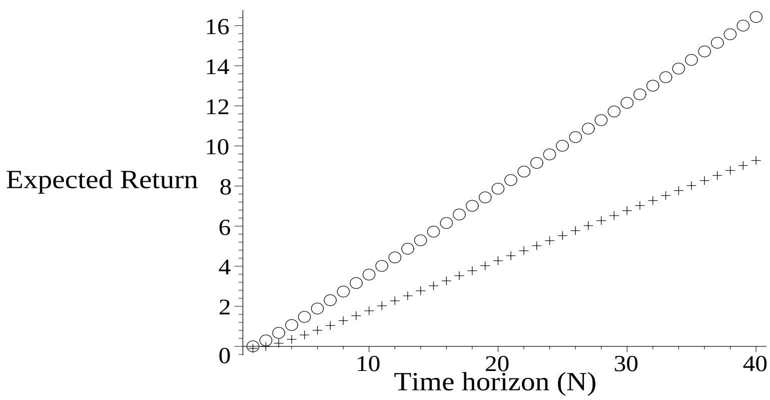

Figure 3: Expected return for optimal multiple stop strategy and buy-and-hold strategy for $g^+(n) \equiv .85$, $g^-(n) \equiv .75$ and $c = .05$. $E(N)$ values are denoted by crosses and $P(N)$ values by circles.

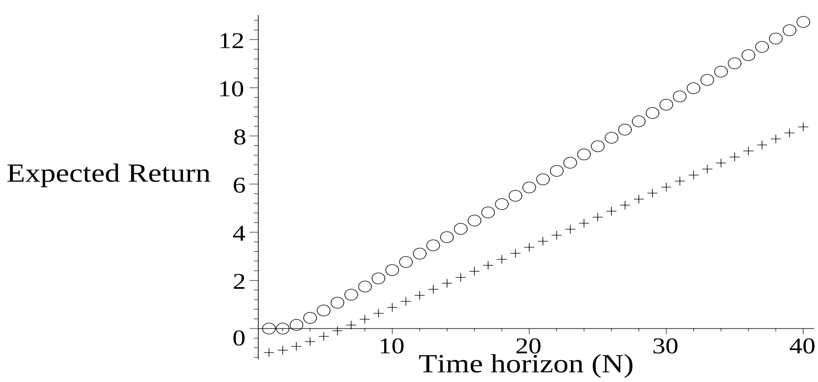

Figure 4: Expected return for optimal multiple stop strategy and buy-and-hold strategy for $g^+(n) \equiv .85$, $g^-(n) \equiv .75$ and $c = .5$. $E(N)$ values are denoted by crosses and $P(N)$ values by circles.

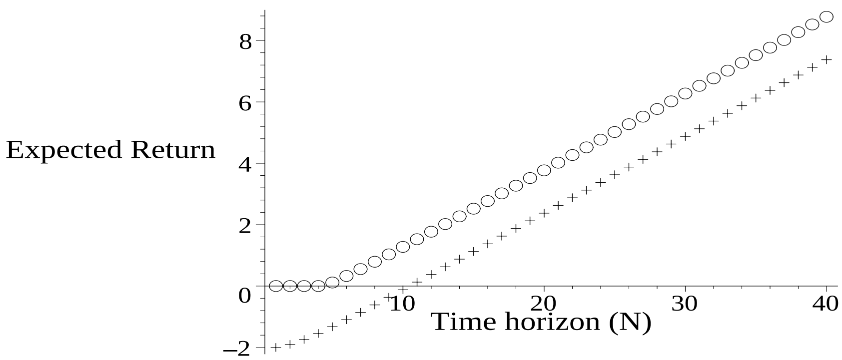

Figure 5: Expected return for optimal multiple stop strategy and buy-and-hold strategy for $g^+(n) \equiv .85$, $g^-(n) \equiv .75$ and $c = 1$. $E(N)$ values are denoted by crosses and $P(N)$ values by circles.

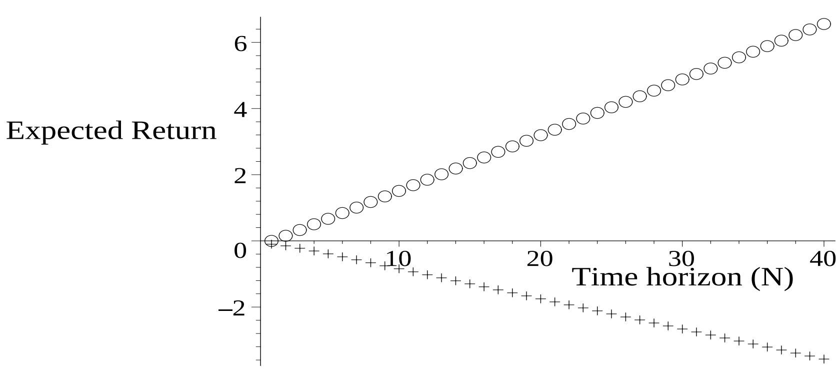

Figure 6: Expected return for optimal multiple stop strategy and buy-and-hold strategy for $g^+(n) \equiv .7$, $g^-(n) \equiv .75$ and $c = .05$. $E(N)$ values are denoted by crosses and $P(N)$ values by circles.