STOPPING THE MAXIMUM OF A CORRELATED RANDOM WALK, WITH COST FOR OBSERVATION

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Abstract

Let $(S_n)_{n \geq 0}$ be a correlated random walk on the integers, let $M_0 \geq S_0$ be an arbitrary integer, and let $M_n = \max\{M_0, S_1, \ldots, S_n\}$. An optimal stopping rule is derived for the sequence $M_n - nc$, where $c > 0$ is a fixed cost. The optimal rule is shown to be of threshold type: stop the first time that $M_n - S_n \geq \Delta$, where Δ is a certain nonnegative integer. An explicit expression for this optimal threshold is given.

Keywords: Correlated random walk; momentum; stopping rule; optimality principle; linear difference equation

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1. Introduction

Consider a player owning a commodity whose price process exhibits momentum in the following way: If the price goes up at stage n, it will take another step up at stage $n + 1$ with probability p, or a step down with probability $1 - p$. Likewise, if the price takes a step down at stage n, it will take another step down at the next stage with probability q, or a step up with probability $1 - q$. The player may stop at any time, and sell the commodity for the highest price that has occurred since the time of purchase. However, a fixed cost $c > 0$ is incurred for each stage during which the commodity is held. When should the player sell the commodity in order to maximize his expected return (that is, selling price minus cost)?

The process described above is usually referred to in the literature as a *correlated random walk*. More formally, let S_0 be an arbitrary integer and for $n \geq 1$, define $S_n = S_0 + X_1 + \cdots + X_n$, where $\{X_n\}_{n \geq 0}$ is a

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$\{-1, 1\}$-valued Markov chain with one-step transition matrix

$$\begin{array}{cc} & \begin{array}{cc} +1 & \quad -1 \end{array} \\ \begin{array}{c} +1 \\ -1 \end{array} & \left[\begin{array}{cc} p & 1-p \\ 1-q & q \end{array} \right] \end{array}$$

We can think of X_0 as the "initial direction" of the walk, since its value determines the probabilities of an up-step or a down-step at the first stage. To avoid degenerate cases, it will be assumed that $0 < p, q < 1$.

Let M_0 be an arbitrary integer such that $M_0 \geq S_0$, and for $n \geq 1$ define $M_n = \max\{M_0, S_1, \ldots, S_n\}$. The goal of this paper is to find a stopping rule τ that maximizes $E(M_\tau - \tau c)$.

For a random walk without correlation (the case $p+q = 1$), this problem was considered by Ferguson and MacQueen ([4], §4). They showed that there exists a nonnegative integer Δ^* such that the optimal rule is of the form: stop at the first time n for which $M_n - S_n \geq \Delta^*$. They also computed the threshold Δ^* for the special case $p = 1/2$, obtaining $\Delta^* = \lfloor 1/(2c) \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. The present article extends this result to random walks with correlation.

Correlated random walks were introduced by Goldstein [6] to model diffusion in turbulent media. Other early work on the subject is due to Gillis [5] and Mohan [10]. Correlated random walks have subsequently been used to model such diverse phenomena as a gambler's fortune [11], the behavior of a pinball machine [12], and the growth of tree roots [7]. A related class of processes called *directionally reinforced random walks* was introduced by Mauldin, Monticino and von Weizsäcker [9]. In a directionally reinforced random walk, the probabilities of an up- or down step depend not only on the direction of the most recent step, but also on the number of successive steps the walk has just taken in that direction.

In a recent paper, Allaart and Monticino [2] derived optimal single and multiple stopping rules for directionally reinforced random walks in one dimension. This appears to be the first work to consider optimal stopping problems for a class of processes exhibiting momentum. A subsequent article [1] treats the optimal stopping problem for correlated random walks with a discount factor. In that paper, the analysis and the final results are somewhat tedious, due to the "incompatibility" of the linear motion of the walk and the geometric discount factor. In the current setting, however, the linear cost structure allows for a more elegant answer to the optimal stopping problem: as in the case considered by Ferguson and MacQueen, it is optimal to stop at the first time n for which $M_n - S_n \geq \Delta$, for

some nonnegative integer Δ. The calculation of this threshold Δ is fairly straightforward, and follows from the solution of a simple system of linear difference equations.

2. Form of the optimal stopping rule

For $n \geq 0$, let $Y_n = M_n - nc$, and let $\mathcal{F}_n$ be the sigma algebra generated by the random triplets $(M_0, S_0, X_0), \ldots, (M_n, S_n, X_n)$. For $s \in \mathbf{Z}$, $m \geq s$ and $x \in \{-1, 1\}$, define

$$V(m, s, x) = \sup_\tau E[M_\tau - \tau c | M_0 = m, S_0 = s, X_0 = x], \qquad (2.1)$$

where τ ranges over the set of all stopping rules adapted to the filtration $\{\mathcal{F}_n\}_{n \geq 0}$ such that $0 \leq \tau < \infty$ almost surely. Call a stopping rule *optimal* for the triple (m, s, x) if it attains the supremum in (2.1).

The existence of an optimal rule depends on the magnitude of c relative to the drift δ of the walk, where

$$\delta = \frac{p - q}{2 - (p + q)}.$$

It can be shown that, independently of the initial values, $E\, S_n / n \to \delta$ as $n \to \infty$. (See, for instance, §4 of [2]. The argument given there is easily generalized to arbitrary first-step probabilities.)

Lemma 2.1. *Assume $c > \delta$. Then*

(i) $E(\sup_n Y_n) < \infty$, *and*

(ii) $\lim_{n \to \infty} Y_n = -\infty$ *almost surely.*

Consequently, there exists an optimal stopping rule, and it is given by the Principle of Optimality: Stop at the first time n at which

$$Y_n = \operatorname*{ess\,sup}_{\tau \geq n} E[Y_\tau | \mathcal{F}_n].$$

Proof. Clearly, the initial conditions do not affect the asymptotic properties of the walk, so without loss of generality assume that $S_0 = M_0 = 0$ and $X_0 = -1$. Let $T_1, T_2, \ldots$ be the times at which the walk reverses its direction. That is, let $T_0 = 0$, and for $k \geq 1$, let $T_k = \inf\{n > T_{k-1} : X_n = -X_{T_{k-1}}\}$. Then the random variables $(T_k - T_{k-1})_{k \geq 1}$ are independent, and the distribution of $T_k - T_{k-1}$ is geometric with parameter $1 - p$ if k is even, and $1 - q$ if k is odd.

Now define

$$Z_k = S_{T_{2k}} - S_{T_{2k-2}} - c(T_{2k} - T_{2k-2}), \qquad k = 1, 2, \dots$$

and let $S'_n = \sum_{k=1}^n Z_k$, $n \in \mathbb{N}$. From the representation

$$Z_k = (T_{2k} - T_{2k-1}) - (T_{2k-1} - T_{2k-2}) - c(T_{2k} - T_{2k-2})$$
$$= (1-c)(T_{2k} - T_{2k-1}) - (1+c)(T_{2k-1} - T_{2k-2}),$$

it follows that the Z_k's are i.i.d. Moreover,

$$EZ_1 = (1-c)\, E(T_2 - T_1) - (1+c)\, E(T_1 - T_0)$$
$$= \frac{1-c}{1-p} - \frac{1+c}{1-q} = \frac{2-p-q}{(1-p)(1-q)}(\delta - c)$$
$$< 0.$$

Since $E\, Z_k^2 < \infty$, Theorem 5 of Kiefer and Wolfowitz [8] implies that $E(\sup S'_n) < \infty$. Moreover, $S'_n \to -\infty$ almost surely by the Strong Law of Large Numbers. Since for $n = T_{2k}$ we have $S_n - nc = S'_k$, and since the process $S_n - nc$ takes on its local maximum values at the points $n = T_{2k} - 1$, it follows that $E(\sup(S_n - nc)) < \infty$ and $S_n - nc \to -\infty$ almost surely.

Now define, for $k \geq 1$, $N_k := \inf\{n : S_n = k\}$ ($= \infty$ if no such n exists). Assume that for a particular sample path, $S_n - nc \to -\infty$. If $N_k < \infty$ for all $k \geq 1$, then $M_{N_k} - N_k c = S_{N_k} - N_k c \to -\infty$. And if $N_k = \infty$ for some k, then M_n is bounded, so $Y_n \to -\infty$ since $c > 0$. This establishes part (ii) of the lemma. Part (i) follows since $Y_n \leq \max_{1 \leq j \leq n}(S_j - jc)$, and so $E(\sup Y_n) = E(\sup(S_n - nc)) < \infty$. The remaining statements of the lemma follow from Theorem 4.5 of Chow *et al.* [3].

Corollary 2.1. *If $c > \delta$, the rule*

$$\tau = \inf\{n \geq 0 : M_n = V(M_n, S_n, X_n)\}$$

is optimal.

Proof. The process (M_n, S_n, X_n) is a time-homogeneous Markov chain. Thus, on the set $\{M_n = m, S_n = s, X_n = x\}$,

$$\operatorname*{ess\,sup}_{\tau \geq n} E[Y_\tau | \mathcal{F}_n] = \sup_{\tau \geq n} E[M_\tau - \tau c | M_n = m, S_n = s, X_n = x]$$
$$= \sup_{\tau \geq n} E[M_{\tau - n} - \tau c | M_0 = m, S_0 = s, X_0 = x]$$
$$= \sup_{\tau' \geq 0} E[M_{\tau'} - \tau' c | M_0 = m, S_0 = s, X_0 = x] - nc$$
$$= V(m, s, x) - nc.$$

Applying Lemma 2.1 completes the proof.

Next, define the short-hand notation

$$\mathrm{E}_{m,s,x}[.] := \mathrm{E}[.|M_0 = m, S_0 = s, X_0 = x].$$

Observe that

$$V(m, s, x) - m = \sup_\tau \mathrm{E}_{0,0,x}[m \vee (s + M_\tau) - \tau c] - m$$
$$= \sup_\tau \mathrm{E}_{0,0,x}[\{M_\tau - (m - s)\}^+ - \tau c],$$

and hence the optimal rule (if one exists) depends on m and s only through their difference. We can therefore arbitrarily set $m = 0$, and consider the quantities

$$W^+(k) = V(0, -k, 1), \qquad W^-(k) = V(0, -k, -1) \qquad (k \geq 0).$$

It is clear that the functions $W^+(k)$ and $W^-(k)$ are nonnegative and nonincreasing. Hence if we define

$$\Delta^+ = \inf\{k \geq 0 : W^+(k) = 0\},$$

and

$$\Delta^- = \inf\{k \geq 0 : W^-(k) = 0\}$$

(with the convention that $\inf(\emptyset) = \infty$), then the optimal rule is:

> *Stop the first time $n \geq 0$ at which either $M_n - S_n \geq \Delta^+$ and $X_n = 1$, or $M_n - S_n \geq \Delta^-$ and $X_n = -1$.*

Moreover, the optimal values $V(m, s, \pm 1)$ can be obtained from the functions $W^+(k)$ and $W^-(k)$ through the identities

$$V(m, s, 1) = W^+(m - s) + m, \qquad V(m, s, -1) = W^-(m - s) + m.$$

Lemma 2.2.

$$\Delta^- \leq \Delta^+ + 1. \tag{2.2}$$

Proof. It is to be shown that $W^-(k) = 0$ whenever $k \geq \Delta^+ + 1$. By conditioning on X_1 it can be seen that, for $k \geq 1$,

$$W^-(k) = \max\{0, qW^-(k + 1) + (1 - q)W^+(k - 1) - c\}.$$

Now let $k \geq \Delta^+ + 1$ and suppose that $W^-(k) > 0$. Then, using the definition of Δ^+ and the fact that both W^- and W^+ are nonincreasing,

$$W^-(k) = qW^-(k + 1) + (1 - q)W^+(k - 1) - c$$
$$= qW^-(k + 1) - c \leq qW^-(k) - c < W^-(k),$$

a contradiction. Hence $W^-(k) = 0$.

In view of Lemma 2.2, the first condition in the optimal rule above cannot occur before the second does, unless it occurs at time 0. As a result, the optimal rule is either $\tau \equiv 0$ (if $X_0 = 1$ and $M_0 - S_0 \geq \Delta^+$), or else $\tau = \sigma(\Delta^-)$, where for $K \geq 0$,

$$\sigma(K) = \inf\{n \geq 0 : M_n - S_n \geq K \text{ and } X_n = -1\}.$$

3. Calculation of the optimal threshold

The rest of the paper is devoted to calculating the expected returns from the rules $\sigma(K)$, and maximizing over K. To this end, define for $K \geq 0$ and $i \geq 0$,

$$f_i^+(K) = \mathrm{E}_{0,-i,1}\, Y_{\sigma(K)},$$

and

$$f_i^-(K) = \mathrm{E}_{0,-i,-1}\, Y_{\sigma(K)}.$$

Proposition 3.1. *(i) For $i \geq K$, $f_i^-(K) = 0$. For $0 \leq i < K$,*

$$f_i^-(K) = \begin{cases} \dfrac{c}{\delta}(K-i) + \dfrac{q}{q-p}\cdot\dfrac{1-q}{1-p}\left(\dfrac{c}{\delta}-1\right)\left[\left(\dfrac{p}{q}\right)^K - \left(\dfrac{p}{q}\right)^i\right], & p \neq q, \\[4mm] \left[1 - \left(3 - \dfrac{1}{p}\right)c\right](K-i) - \left(\dfrac{1-p}{p}\right)c(K^2 - i^2), & p = q. \end{cases}$$

$$(3.1)$$

(ii) For $K \in \{0,1\}$ and $i \geq 0$,

$$f_i^+(K) = (p^{i+1} - c)/(1-p).$$

For $K \geq 2$,

$$f_i^+(K) = \begin{cases} (f_{i+1}^-(K) - q f_{i+2}^-(K) + c)/(1-q), & 0 \leq i \leq K-2 \\[4mm] p^{i-(K-2)}\left[f_{K-2}^+(K) + \dfrac{c}{1-p}\right] - \dfrac{c}{1-p}, & i \geq K-1. \end{cases}$$

(Full expressions for $f_i^+(K)$ are unenlightening, and in any case are not needed for the derivation of the optimal rule. They are therefore omitted.)

Remark 3.1. One would expect the expressions for $f_i^-(K)$ in equation (3.1) to be continuous at $p = q$. That this is indeed the case can be

seen by applying standard limit computation techniques. The technical details are somewhat tedious, and are omitted here since continuity is not of paramount importance in what follows.

Proof of Proposition 3.1. Fix $K \geq 0$, and let $a_i := f_i^+(K)$, and $b_i := f_i^-(K)$. Then a_i and b_i satisfy the difference equations

$$a_i = pa_{i-1} + (1-p)b_{i+1} - c, \qquad i \geq 1 \qquad (3.2)$$
$$b_i = (1-q)a_{i-1} + qb_{i+1} - c, \qquad 1 \leq i < K \qquad (3.3)$$

as well as the boundary conditions

$$a_0 = p(a_0 + 1) + (1-p)b_1 - c, \qquad (3.4)$$
$$b_0 = (1-q)(a_0 + 1) + qb_1 - c, \qquad (3.5)$$

and

$$b_i = 0, \qquad i \geq K. \qquad (3.6)$$

Now consider three cases.

Case 1. $K = 0$. In this case, statement (i) is trivial. To see (ii), note that $b_i = 0$ for all $i \geq 1$, and so (3.2) and (3.4) imply that $a_0 = (p-c)/(1-p)$ and, inductively, that $a_i = (p^{i+1} - c)/(1-p)$ for $i \geq 0$.

Case 2. $K = 1$. In this case, statement (ii) follows by the same argument as in Case 1. In particular, $a_0 = (p-c)/(1-p)$, and substituting this into (3.5) yields the expression for $b_0 = f_0^-(1)$ given in part (i) of the proposition.

Case 3. $K \geq 2$. We derive a second order difference equation for the b_i's as follows. By (3.3), we have

$$(1-q)a_i = b_{i+1} - qb_{i+2} + c, \qquad 0 \leq i \leq K - 2.$$

On the other hand, (3.2)-(3.5) imply that

$$(1-q)a_i = pb_i + (1-p-q)(b_{i+1} - c), \qquad 0 \leq i \leq K - 1.$$

It follows that

$$qb_{i+2} - (p+q)b_{i+1} + pb_i = (2 - p - q)c, \qquad 0 \leq i \leq K - 2. \qquad (3.7)$$

Suppose first that $p \neq q$. Using the transformation

$$d_i = b_{i+1} - b_i + \frac{c}{\delta}, \qquad (3.8)$$

(3.7) can be written as

$$d_{i+1} = (p/q)d_i, \qquad 0 \le i \le K - 2,$$

so that $d_i = (p/q)^i d_0$ for $i = 0, 1, \ldots, K - 1$. The value of d_0 is obtained by eliminating a_0 from (3.4) and (3.5). This yields

$$d_0 = b_1 - b_0 + \frac{c}{\delta} = \frac{c}{\delta} - \left(\frac{1-q}{1-p}\right)(p - c) - (1 - q - c)$$

$$= \left(\frac{1-q}{1-p}\right)\left(\frac{c}{\delta} - 1\right),$$

and hence,

$$d_i = \left(\frac{1-q}{1-p}\right)\left(\frac{c}{\delta} - 1\right)\left(\frac{p}{q}\right)^i, \qquad 0 \le i \le K - 1.$$

Using this expression and (3.8), it follows inductively that

$$b_i - b_0 = \sum_{\nu=0}^{i-1}\left[\frac{1-q}{1-p}\left(\frac{c}{\delta} - 1\right)\left(\frac{p}{q}\right)^{\nu} - \frac{c}{\delta}\right]$$

$$= \frac{1-q}{1-p}\left(\frac{c}{\delta} - 1\right) \cdot \frac{1 - (p/q)^i}{1 - p/q} - \left(\frac{c}{\delta}\right)i.$$

Finally, since $b_K = 0$, we obtain

$$b_i = \frac{c}{\delta}(K - i) + \frac{q}{q - p}\left(\frac{1-q}{1-p}\right)\left(\frac{c}{\delta} - 1\right)\left[\left(\frac{p}{q}\right)^K - \left(\frac{p}{q}\right)^i\right], \qquad 0 \le i \le K.$$

Suppose next that $p = q$. Let $d_i' = b_{i+1} - b_i$. Then (3.7) reduces to

$$d_{i+1}' - d_i' = 2\left(\frac{1-p}{p}\right)c, \qquad 0 \le i \le K - 2.$$

The initial condition becomes $d_0' = 2c - 1$, and thus,

$$d_i' = 2\left(\frac{1-p}{p}\right)ic + 2c - 1, \qquad 0 \le i \le K - 1,$$

and

$$b_i - b_0 = \sum_{\nu=0}^{i-1}\left[2c\left(\frac{1-p}{p}\right)\nu + 2c - 1\right]$$

$$= i(i - 1)\left(\frac{1-p}{p}\right)c + (2c - 1)i.$$

Finally, since $b_K = 0$,

$$b_i = (1 - 2c)(K - i) - \left(\frac{1-p}{p}\right) c\{K(K-1) - i(i-1)\}$$

$$= \left[1 - \left(3 - \frac{1}{p}\right) c\right] (K - i) - \left(\frac{1-p}{p}\right)(K^2 - i^2)c, \qquad 0 \le i \le K.$$

This completes the proof of statement (i) in Case 3. The expressions for $f_i^+(K)$ in statement (ii) follow from (3.3) if $0 \le i \le K - 2$, and from iteration of (3.2) if $i \ge K - 1$ (recalling that $b_{i+1} = 0$ in the latter case).

The following immediate consequence of Proposition 2.1 states that if $c \le \delta$, the stop rules $\sigma(K)$ yield arbitrarily large returns if K is chosen large enough. This is intuitively plausible, since the average growth of the walk per time unit exceeds the cost for one observation.

Corollary 3.1. *If $c \le \delta$, then $V(m, s, x) = \infty$ for all m, s, and x. Moreover,*

$$\lim_{K \to \infty} \mathrm{E}_{m,s,x}[M_{\sigma(K)} - \sigma(K)c] = \infty.$$

Proof. Since $c > 0$, the hypothesis implies that $p > q$. Hence by (3.1), $f_i^-(K) \to \infty$ as $K \to \infty$. The same is then true for $f_i^+(K)$.

Theorem 3.1. *Suppose $c > \delta$. Then*

(i) $V(m, s, x) < \infty$ *for all m, s, and x.*

(ii) *The optimal rule is*

$$\tau = \begin{cases} 0 & \text{if } X_0 = 1 \text{ and } M_0 - S_0 \ge \Delta^+, \\ \sigma(\Delta^-) & \text{otherwise.} \end{cases}$$

(iii) $\Delta^- = \max\{0, \lceil \kappa \rceil\}$, *where*

$$\kappa = \begin{cases} \log_{p/q} \left(\dfrac{1-p}{1-q} \cdot \dfrac{c}{c-\delta}\right), & \text{if } p \ne q \\[4mm] \dfrac{p}{1-p} \left(\dfrac{1}{2c} - 1\right), & \text{if } p = q, \end{cases}$$

and $\lceil \kappa \rceil$ denotes the smallest integer greater than or equal to κ.

(iv)

$$\Delta^+ = \begin{cases} \inf\{k \geq 0 : p^{k+1} \leq c\}, & \text{if } \Delta^- \leq 1 \\[3mm] \inf\{k \geq 0 : p^{k-(\Delta^--2)} \leq [1 + (1-p)W/c]^{-1}\}, & \text{if } \Delta^- \geq 2, \end{cases}$$

where $W := W^+(\Delta^- - 2) = (f_{\Delta^--1}^-(\Delta^-) + c)/(1-q)$.

(v) The optimal expected returns are $W^+(i) = f_i^+(\Delta^-)$ if $i < \Delta^+$, and $= 0$ otherwise; and $W^-(i) = f_i^-(\Delta^-)$, for $i \geq 0$.

Proof. Notice that Δ^- is the (common) value of K that maximizes each of the functions $f_i^-(K)$ $(i \geq 0)$. This value can be found by looking at the first-order differences of $f := f_0^-$ (say). For the case $p \neq q$, this gives

$$f(K+1) - f(K) = \frac{c}{\delta} - \frac{1-q}{1-p}\left(\frac{c}{\delta} - 1\right)\left(\frac{p}{q}\right)^K.$$

Investigate the cases $p < q$ and $p > q$ separately to conclude that $f(K+1) - f(K)$ is decreasing and eventually negative. (If $p < q$ this follows since $\delta < 0$.) Hence f is maximized at the smallest K for which $f(K+1) \leq f(K)$. This inequality reduces to

$$\left(\frac{p}{q}\right)^K \geq \frac{1-p}{1-q} \cdot \frac{c}{c-\delta}$$

if $p > q$, and to

$$\left(\frac{p}{q}\right)^K \leq \frac{1-p}{1-q} \cdot \frac{c}{c-\delta}$$

if $p < q$. In both cases, the expression for Δ^- given in the statement of the theorem follows.

On the other hand, if $p = q$ we have

$$f(K+1) - f(K) = 1 - \left(3 - \frac{1}{p}\right)c - \left(\frac{1-p}{p}\right)(2K+1)c,$$

and the inequality $f(K+1) \leq f(K)$ simplifies to

$$K \geq \frac{p}{1-p}\left(\frac{1}{2c} - 1\right). \tag{3.9}$$

This establishes the expression for Δ^- in the case $p = q$.

The expressions for Δ^+ follow immediately from Proposition 3.1 and the observation that Δ^+ is the smallest i for which $f_i^+(\Delta^-) \leq 0$.

Remark 3.2. As earlier in Proposition 3.1, the expressions for κ in Theorem 3.1 can be shown to be continuous at $p = q$. The details are omitted.

Corollary 3.2. *If $c \geq p$, the rule $\tau \equiv 0$ is optimal for any triplet (m, s, x) of initial conditions different from $(0, 0, -1)$.*

Proof. Observe first that the hypothesis $c \geq p$ implies that $c > \delta$, since

$$p - \delta = \frac{(1-p)(p+q)}{2 - p - q} > 0.$$

Thus the optimal rule is given by Theorem 3.1. It suffices to show that $\Delta^- \leq 1$, for then $\Delta^+ = \inf\{k : p^{k+1} \leq c\} = 0$.

Suppose first that $p > q$. Since $(1-q)\delta \leq p - q$, it follows that

$$\frac{1-p}{1-q} \cdot \frac{c}{c-\delta} \leq \frac{1-p}{1-q} \cdot \frac{p}{p-\delta} \leq \frac{(1-p)p}{p(1-q) - (p-q)} = \frac{p}{q}.$$

Hence part (iii) of Theorem 3.1 implies that $\Delta^- \leq 1$. The case $p < q$ can be handled similarly. Finally, for $p = q$, the statement follows since

$$\frac{p}{1-p}\left(\frac{1}{2c} - 1\right) \leq \frac{p}{1-p}\left(\frac{1}{2p} - 1\right) = \frac{1-2p}{2(1-p)} \leq 1,$$

so (3.9) holds for every $K \geq 1$, and hence $f(K)$ is decreasing for $K \geq 1$.

Remark 3.3. For the case of a classical symmetric random walk ($p = q = 1/2$), Theorem 3.1 gives $\Delta^- = \lceil 1/(2c) - 1 \rceil$. Since there is no correlation between steps, it follows immediately that $\Delta^+ = \lceil 1/(2c) - 1 \rceil$ as well, though this seems difficult to verify from the expressions of Theorem 3.1. Ferguson and MacQueen [4] give the optimal threshold $\Delta^* = \lfloor 1/(2c) \rfloor$ for this case. Note that if $1/(2c)$ is non-integer, the values of Δ^- and Δ^* coincide. If $1/(2c)$ is an integer, the thresholds disagree, but the proof of Theorem 3.1 shows that it is immaterial whether $\sigma(\Delta^-)$ or $\sigma(\Delta^*)$ is used. For, if $K = 1/(2c) - 1$, then (3.9) above holds with equality, and hence $f_0^-(K+1) = f_0^-(K)$.

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