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OPTIMAL STOPPING RULES FOR CORRELATED RANDOM WALKS WITH A DISCOUNT

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Abstract

Optimal stopping rules are developed for the correlated random walk when future returns are discounted by a constant factor β per unit time. The optimal rule is shown to be of dual threshold form: one threshold for stopping after an up-step, and another for stopping after a down-step. Precise expressions for the thresholds are given both for the positively and the negatively correlated case. The optimal rule is illustrated by several numerical examples.

Keywords: Correlated random walk; stopping rule; optimality principle; discount factor; momentum

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1. Introduction

The main goal of this paper is to answer the following basic question. Suppose an investor owns a commodity whose price process follows a correlated random walk. If future returns are discounted by a constant factor β per unit time, when should the investor sell the commodity so as to maximize his expected return?

A *correlated random walk* (CRW), first introduced by Goldstein [9], is a process $S_n = S_0 + X_1 + \cdots + X_n$, where S_0 is any integer, and X_n, $n \geq 0$ is a two-state Markov chain with one-step transition matrix

$$\begin{array}{cc} & \begin{array}{cc} +1 & \quad -1 \end{array} \\ \begin{array}{c} +1 \\ -1 \end{array} & \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix} \end{array}$$

Thus, if the price goes up at stage n, it will take another step up at stage $n+1$ with probability p, or a step down with probability $1-p$. Likewise, if the price takes a step down at stage n, it will take another step down at the next stage with probability q, or a step up with probability $1-q$.

To avoid trivialities, we assume that $0 < p, q < 1$.

In terms of the above notation, the goal of this paper is to find an optimal stopping rule for the sequence $\beta^n S_n$, for $0 < \beta < 1$. For uncorrelated random walks, this problem was treated by Dubins and Teicher [6]. They showed that the optimal rule is to stop the first time the walk exceeds the threshold $G(\beta)/[1-G(\beta)]$, where G is the probability-generating function of the first-passage time of 1 when the walk starts at zero. Darling *et al.* [5] and Ferguson [7] also discuss various interesting optimal stopping problems for sums of i.i.d. random variables.

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Correlated random walks have been studied widely since Goldstein's paper. Gillis [8] developed a d-dimensional version, and conjectured it to be transient for all $d \geq 3$. Gillis' conjecture was proved first by Iossif [11], and later, for more general correlated random walks by Chen and Renshaw [3]. Results for the one-dimensional CRW took less time to develop. For example, Seth [19] obtained return probabilities and first-passage time distributions for the symmetric CRW, and Jain [12] generalized Seth's results to the non-symmetric case. Renshaw and Henderson [18] obtained occupation probabilities and a diffusion approximation for symmetric CRW. Mohan [15] and Mukherjea and Steele [16] considered a correlated gambler's ruin problem. Other works involving correlated random walks with one or two boundaries are due to Proudfoot and Lampard [17], Jain [13], Zhang [20] and, most recently, Böhm [2]. Whilst most of these authors use difference equations and generating functions, Böhm's approach is entirely combinatorial.

Papers concerning the correlated random walk have often been motivated by applied problems. For example, Goldstein [9] was interested in modeling certain physical diffusion processes, Henderson and Renshaw [10] used the CRW in two dimensions as a model for tree root growth, and Renshaw and Henderson [18] studied the behavior of a certain kind of pinball machine. Motivated by the study of ocean waves, Mauldin *et al.* [14] introduced a more general class of processes called directionally reinforced random walks, in which the distribution of each step depends on the entire run length in the most recent direction.

A comprehensive list of references for the correlated random walk, that also includes several other interesting applications, can be found in Chen and Renshaw [3].

Admittedly, the correlated random walk is an oversimplified model for something as intricate as a financial market. However, very little seems to be known on optimal stopping for processes exhibiting momentum. A recent paper by Allaart and Monticino [1] was a first attempt to analyze the effect of momentum on simple investment strategies. That paper studies finite-horizon single- and multiple-stopping problems for directionally reinforced random walks, though most of the explicit results are for the case of a correlated random walk. The present note aims to gain further insight.

It should also be noted that the issue of momentum in the stock market is a controversial one. Rather than taking a stand in the debate, this paper merely aims to demonstrate how a wary investor could take advantage of momentum when present, and to obtain some insight into the magnitude of this advantage.

The organization of this paper is as follows. Section 2 introduces the notation necessary to state the main theorems. Section 3 gives the optimal stopping rules. It will be shown that there are two thresholds L and U that bound from above the continuation zones when the walk takes a step down, respectively up. If the correlation between steps is positive then $L \leq U$, so one should always continue when the walk is below L, stop only after a downward step when the walk is between L and U, and always stop when the walk is at or above U. If the correlation is negative, then $L \geq U$ and the optimal rule can be described similarly. The thresholds are given by Theorems 3.1 and 3.2 below, and explicit expressions for the optimal stopping values are given in Theorem 3.3. Section 4 compares the return of the optimal rule with stopping rules that ignore the correlation between steps. Section 5 contains the proofs. Finally, Section 6 touches on extensions to more general settings.

2. Notation

The following notation is needed to state the main results of this paper. Let E_z^+ and E_z^- denote expectation operators given that $S_0 = z$ and $X_0 = 1$, respectively -1. Similarly, P^+ and P^- denote probability measures given that $S_0 = 0$ and $X_0 = 1$, respectively -1. For any real number x, let $\lceil x \rceil$ be the smallest integer greater than or equal to x.

As mentioned in the introduction, the optimal rule depends in a crucial way on whether the correlation between steps is positive or negative. Assuming that $\mathrm{P}(X_0 = 1) = 1/2$, Mohan [15] showed that the correlation coefficient between two consecutive steps X_n and X_{n+1} is given by

$$R = \frac{r}{\sqrt{1-(p-q)^2}},$$

where

$$r = p + q - 1.$$

(For different zeroth step probabilities, R is the limit of this correlation as $n \to \infty$.) Therefore, the cases $R > 0$ and $R < 0$ will be called respectively the positively and the negatively correlated case. Note that $R = 0$ is the case treated by Dubins and Teicher [6].

For any integer s, let $N(s)$ be the first time the walk is at s. That is,

$$N(s) = \min\{n \geq 0 : S_n = s\},$$

with the convention that the minimum of an empty set is $+\infty$. Define

$$\varrho_+ = \mathrm{E}_0^+ \, \beta^{N(1)}, \quad \text{and} \quad \varrho_- = \mathrm{E}_0^- \, \beta^{N(1)},$$

with the understanding that $\beta^{N(1)} = 0$ on the event $\{N(1) = \infty\}$. To maintain the continuity of this paper, the calculation of ϱ_+ and ϱ_- will be deferred until Section 5.

3. The optimal stopping rule

To motivate the results in this section, consider the following basic question, to be answered at the end of this section.

Example 3.1. *(A basic investment problem.)* Let $\beta = .95$, and $p = q = .70$, and suppose the price of stock X has just gone up to 6. If you are not currently holding the stock, should you buy now?

For any integer z, let $V^+(z)$ denote the optimal expected return when the walk starts at z on an up-run. That is,

$$V^+(z) = \sup_\tau \mathrm{E}_z^+[\beta^\tau S_\tau] = \sup_\tau \mathrm{E}_0^+[\beta^\tau (z + S_\tau)], \tag{3.1}$$

where the supremum is taken over all extended stopping rules τ for which $0 \leq \tau \leq \infty$ almost surely. Similarly, define

$$V^-(z) = \sup_\tau \mathrm{E}_z^-[\beta^\tau S_\tau] = \sup_\tau \mathrm{E}_0^-[\beta^\tau (z + S_\tau)].$$

The goal of this section is to determine the functions V^+ and V^-, and to find stopping rules τ^* that attain the suprema.

Since $\lim_{n\to\infty} \beta^n(z+S_n) = 0$ almost surely, the expression $\beta^\infty(z+S_\infty)$ will be interpreted as zero. Thus the reward for never stopping is zero. An immediate consequence of this convention is that an optimal rule exists and is given by the *principle of optimality*:

> Stop at the first stage for which the present return is greater than or equal to the optimal expected return if you continue.

For, $\mathrm{E}_z^{+/-}[\sup_n \beta^n S_n^+] \le \sup_n \beta^n(z+n) < \infty$, and hence the conditions of Theorem 4.5′ in [4] are satisfied.

It is now easy to establish the general form of the optimal stopping rule, given in (3.2) below.

Proposition 3.1. *There exist unique positive integers U and L such that*

(i) $V^+(z) > z$ if and only if $z < U$, and

(ii) $V^-(z) > z$ if and only if $z < L$.

Consequently, the optimal rule is of the form

$$\tau^* = \min\{n \ge 0 : (S_n \ge U \textit{ and } X_n = 1) \textit{ or } (S_n \ge L \textit{ and } X_n = -1)\}. \tag{3.2}$$

Proof. Let

$$\kappa := \sup_{\tau\ge 1} \frac{\mathrm{E}_0^+ \beta^\tau S_\tau}{1 - \mathrm{E}_0^+ \beta^\tau}, \tag{3.3}$$

where the supremum is over all stopping rules τ such that $P^+(\tau \ge 1) = 1$. Note that $\kappa \le \sup_n \beta^n n/(1-\beta) < \infty$. Furthermore, κ has the property that $z \ge \kappa$ if and only if $z \ge \mathrm{E}_0^+ \beta^\tau (z + S_\tau)$ for all τ. Thus $V^+(z) > z$ if and only if $z < \kappa$, so $U := \lceil \kappa \rceil$ satisfies condition (i). Clearly, this choice of U is unique, and U is positive since $\mathrm{E}_0^+ \beta^\tau S_\tau > 0$ when $\tau = N(1)$. The statement concerning L is proved similarly. Finally, the form of the optimal rule (3.2) follows immediately from (i) and (ii), and the principle of optimality.

The relative ordering of the thresholds L and U depends on whether the correlation between steps is positive or negative. Thus, their evaluation is dealt with separately for the positively correlated case (Theorem 3.1) and for the negatively correlated case (Theorem 3.2). The expressions given by these theorems may seem tedious, but qualitatively the results can be summarized as follows. In the positively correlated case, $U \ge L$, and the calculation of U and L depends on which feature of the model has the more significant impact. Roughly speaking, if β is small, the discounting weighs in more heavily than the directional reinforcement between steps, and the thresholds for stopping after an up-step or down-step are quite close together. Moreover, the formula for U is the same as in the uncorrelated case. On the other hand, if β is sufficiently close to one, then the reinforcement takes charge, creating arbitrarily large gaps between the thresholds (as illustrated in Table 1 below). In that case, the threshold U is controlled by the one-stage look-ahead rule: stop after an up-step if and only if the present return

	$\beta = .60$	$\beta = .75$	$\beta = .90$	$\beta = .95$	$\beta = .99$	$\beta = .999$	$\beta = .9999$
$p = .50$	$(\mathbf{1}, \mathbf{1})$	$(\mathbf{1}, \mathbf{1})$	$(\mathbf{2}, \mathbf{2})$	$(\mathbf{3}, \mathbf{3})$	$(\mathbf{7}, \mathbf{7})$	$(\mathbf{22}, \mathbf{22})$	$(\mathbf{71}, \mathbf{71})$
$p = .55$	$(\mathbf{1}, \mathbf{1})$	$(\mathbf{1}, \mathbf{1})$	$(\mathbf{2}, \mathbf{2})$	$(\mathbf{3}, \mathbf{3})$	$(10, 7)$	$(100, 23)$	$(1000, 77)$
$p = .60$	$(\mathbf{1}, \mathbf{1})$	$(\mathbf{2}, \mathbf{1})$	$(\mathbf{3}, \mathbf{2})$	$(4, 3)$	$(20, 7)$	$(200, 26)$	$(2000, 85)$
$p = .70$	$(\mathbf{1}, \mathbf{1})$	$(\mathbf{2}, \mathbf{1})$	$(4, 2)$	$(8, 3)$	$(40, 8)$	$(400, 32)$	$(4000, 106)$
$p = .80$	$(\mathbf{1}, \mathbf{1})$	$(2, 1)$	$(6, 2)$	$(12, 3)$	$(60, 10)$	$(600, 41)$	$(6000, 137)$
$p = .90$	$(2, 1)$	$(3, 1)$	$(8, 1)$	$(16, 3)$	$(80, 13)$	$(800, 58)$	$(8000, 203)$

TABLE 1: The thresholds U and L in the symmetric, positively correlated case ($p = q \geq 1/2$). Values are tabulated as pairs (U, L). Bold face indicates that $\beta \leq \beta_0$.

is greater than or equal to the expected return if you take one more observation and then stop. The expression for L is more complicated.

In the negatively correlated case, $L \geq U$, and the situation is simpler: If the walk starts below U, the event $\{S_n \geq L, X_n = -1\}$ cannot occur before the event $\{S_n \geq U, X_n = 1\}$, so the optimal rule is simply $N(U)$. Thus U is given by the same formula as in the uncorrelated case. The expression for L is relevant only if the walk starts above U on a down-run.

Theorem 3.1. (Positively correlated steps.) *Assume $R \geq 0$. Then*

(i) $U \geq L$.

(ii) $U = \left\lceil \max\left\{ \dfrac{\varrho_+}{1-\varrho_+}, \dfrac{\beta(2p-1)}{1-\beta} \right\} \right\rceil$.

(iii) $\dfrac{\varrho_+}{1-\varrho_+} \geq \dfrac{\beta(2p-1)}{1-\beta}$ *if and only if* $\beta \leq \beta_0 := \dfrac{2}{1+\sqrt{1+8r(2p-1)}}$.

(iv) If $\beta \leq \beta_0$, then $L = \lceil \varrho_-/(1-\varrho_-) \rceil \geq U - 1$.

If $\beta \geq \beta_0$, then $U \geq 2$, and

$$L = \begin{cases} U-1 & \text{if } \; U < 2/(1-\varrho_+\varrho_-) \\ s^* & \text{otherwise,} \end{cases}$$

where s^* is the smallest integer s ($1 \leq s \leq U-2$) such that

$$s \geq \varrho_+\varrho_- \left[\beta(1-p) \sum_{i=1}^{U-s-2} (\beta p)^{i-1}(s+i) + (\beta p)^{U-s-2} U \right]. \tag{3.4}$$

Table 1 gives the thresholds L and U in the symmetric case, for various values of p and β. Note that if $\beta \leq 1/2$, then $U = L = 1$. To see this, observe that $\varrho_+ = \sum_{n=1}^{\infty} \beta^n \, \mathrm{P}^+(N(1) = n) \leq \beta \sum_{n=1}^{\infty} \mathrm{P}^+(N(1) = n) \leq \beta \leq 1/2$. Thus both quantities in the maximum in Theorem 3.1 (ii) are at most 1.

Remark 3.1. When $\beta \leq \beta_0$, Theorem 3.1 shows that $L \geq U - 1$. Thus the event $\{S_n \geq L, X_n = -1\}$ cannot occur before the event $\{S_n \geq U, X_n = 1\}$, unless it occurs at time zero. It follows that, if $S_0 < U - 1$ or $X_0 = 1$, the optimal rule is simply $N(U)$. Only if $S_0 = U - 1$ and $X_0 = -1$ does the exact value of L matter: The optimal rule is $\tau = N(U)$ if $L = U$, but $\tau \equiv 0$ if $L = U - 1$. Observe also that the case $\beta \leq \beta_0$ includes all uncorrelated random walks.

Remark 3.2. The inequality (3.4) can of course be solved experimentally, but a more systematic way is to expand the summation in (3.4) using the identity

$$\sum_{i=1}^{k}(s+i)x^{i-1} = \frac{s(1-x^k)}{1-x} + \frac{1-x^k\{1+k(1-x)\}}{(1-x)^2},$$

where the second term follows by differentiating the finite geometric sum $\sum_{i=1}^{k} x^i$. Thus, after some simplifications, (3.4) can be written as

$$s \geq \varrho_- \varrho_+ \left[\beta(1-p)\left\{\frac{s-(U-2)(\beta p)^{U-s-2}}{1-\beta p} + \frac{1-(\beta p)^{U-s-2}}{(1-\beta p)^2}\right\} + (\beta p)^{U-s-2}U\right]. \tag{3.5}$$

This inequality can be solved easily with the help of a computer. However, a surprisingly accurate estimate of s^* may be obtained by ignoring the exponential terms in (3.5). This gives the simpler inequality

$$s \geq \varrho_- \varrho_+ \beta(1-p)\left[\frac{s}{1-\beta p} + \frac{1}{(1-\beta p)^2}\right]. \tag{3.6}$$

Using the identity

$$\varrho_- \varrho_+ \beta(1-p) = \varrho_+ - \beta p$$

(to be derived in Section 5), if follows that (3.6) holds if and only if

$$s \geq \hat{L} := \left\lceil \frac{\varrho_+ - \beta p}{(1-\varrho_+)(1-\beta p)} \right\rceil.$$

The estimate $\hat{L}$ was found to give the correct prediction for the value of L in many of the cases of Table 1, except where U is very small, in which case the exponential terms in (3.5) contribute significantly.

Theorem 3.2. (Negatively correlated steps.) *Assume $R \leq 0$. Then*

(i) $U \leq L$.

(ii) $U = \left\lceil \dfrac{\varrho_+}{1-\varrho_+} \right\rceil$, *and*

(iii) L *is the smallest integer* $s \geq U$ *such that*

$$s \geq \beta(1-q) \sum_{i=1}^{s-U+1} (s-i+2)(\beta q)^{i-1} + \varrho_-(\beta q)^{s-U+1}U.$$

Theorem 3.3. *Let U and L be given by Theorem 3.1 or Theorem 3.2, as appropriate. Then*

$$V^+(z) = \begin{cases} \varrho_+^{L-z+1}\left[\beta(1-p)\sum_{i=1}^{U-L-1}(\beta p)^{i-1}(L+i-1) + (\beta p)^{U-L-1}U\right], & z \leq L < U \\ \beta(1-p)\sum_{i=1}^{U-z}(\beta p)^{i-1}(z+i-2) + (\beta p)^{U-z}U, & L < z < U \\ \varrho_+^{U-z}U, & z < U \leq L \\ z, & z \geq U, \end{cases}$$

and

$$V^-(z) = \begin{cases} \varrho_- \varrho_+^{L-z} \left[\beta(1-p) \sum_{i=1}^{U-L-1} (\beta p)^{i-1}(L+i-1) + (\beta p)^{U-L-1} U \right], & z < L < U \\ \beta(1-q) \sum_{i=1}^{z-U+1} (\beta q)^{i-1}(z-i+2) + \varrho_-(\beta q)^{z-U+1} U, & U \le z < L \\ \varrho_- \varrho_+^{U-z-1} U, & z < U \le L \\ z, & z \ge L. \end{cases}$$

Example 3.2. *(A basic investment problem, ctd.)* We can now answer the basic investment question posed in Example 3.1. For $\beta = .95$ and $p = q = .70$, Table 1 shows that $U = 8$ and $L = 3$. So if you buy at a price of 6 on an up-run, the expected return will be $V^+(6) \approx 6.100$. If the expected gain of .100 beats transaction costs you should buy, and subsequently sell as soon as the price either goes down or reaches 8, whichever happens first.

4. Comparison with best "uncorrelated" stopping rules.

It is interesting to compare the performance of the optimal rule to that of stopping rules that ignore the correlation between steps. To this end, we will consider two investors. One investor acknowledges the correlation and is able to accurately assess the probabilities p and q; the other assumes that there is no correlation, and selects a stopping rule based on that belief. How much better can the first investor do?

If the walk is symmetric (that is, $p = q$), it is easy to compare the two investors' returns, for in this case the second investor will likely assume that $p = 1/2$, and hence use the rule that is optimal with respect to the value $p = 1/2$, regardless the true value of p. For simplicity, let us assume that the walk starts at zero, having reached zero on an up-step. The return of the first investor is simply $V := V^+(0)$. The return of the second investor is obtained as follows: calculate the Dubins-Teicher threshold $G(\beta)/(1-G(\beta))$ that corresponds to the value $p = 1/2$. Call this threshold (rounded up to the nearest integer) K. Then the return of the second investor is $V' := \varrho_+^K K$, where ϱ_+ is calculated using the true value of p.

Table 2 shows the values of V and V' for several choices of p and β. Also given is the first investor's relative advantage $\alpha := ((V - V')/V') \times 100\%$. As might be expected, the advantage of acknowledging the correlation increases dramatically as p increases and β is held fixed. For realistic values of p, however, the advantage is quite small.

5. Proofs of the main theorems

The proofs of Theorems 3.1 and 3.2 will follow after a series of preliminary results. The first lemma shows how the ordering of the thresholds U and L depends on the correlation R between steps.

Lemma 5.1. *If $R \ge 0$, then $U \ge L$. Otherwise, $U \le L$.*

Proof. Let $W^+(z) = \sup_{\tau \ge 1} E_z^+ \beta^\tau S_\tau$, where the supremum is over all stopping rules

		$\beta = .60$	$\beta = .75$	$\beta = .90$	$\beta = .95$	$\beta = .99$	$\beta = .999$	$\beta = .9999$
$p = .55$	V	.3598	.4808	.8539	1.245	2.869	9.149	28.83
	V'	.3598	.4808	.8539	1.245	2.846	9.031	28.63
	α	0	0	0	0	.80%	1.30%	.71%
$p = .60$	V	.3858	.5192	.9388	1.382	3.260	10.23	32.03
	V'	.3858	.5095	.9219	1.352	3.105	9.848	31.27
	α	0	1.90%	1.84%	2.21%	4.99%	3.86%	2.43%
$p = .70$	V	.4374	.6401	1.181	1.821	4.284	13.01	40.21
	V'	.4374	.5657	1.060	1.572	3.642	11.55	36.79
	α	0	13.1%	11.4%	15.8%	17.6%	12.7%	9.29%
$p = .80$	V	.4892	.7746	1.578	2.538	5.981	17.48	53.11
	V'	.4892	.6223	1.208	1.812	4.241	13.44	42.97
	α	0	24.5%	30.6%	40.0%	41.0%	30.0%	23.6%
$p = .90$	V	.5892	.9771	2.239	3.849	9.779	27.43	80.93
	V'	.5428	.6821	1.380	2.103	4.991	15.84	50.80
	α	8.55%	43.2%	62.3%	83.0%	95.9%	73.2%	59.3%

TABLE 2: Comparison of the expected returns V and V'.

τ such that $1 \le \tau \le \infty$ almost surely. Define $W^-(z)$ analogously. Then

$$W^+(z) = \beta[pV^+(z+1) + (1-p)V^-(z-1)],$$

and

$$W^-(z) = \beta[(1-q)V^+(z+1) + qV^-(z-1)].$$

Suppose first that $p + q \ge 1$. Then by the monotinicity of $V^+(z)$ and $V^-(z)$,

$$\begin{aligned} W^+(z) - W^-(z) &= \beta r[V^+(z+1) - V^-(z-1)] \\ &\ge \beta r[V^+(z) - V^-(z)], \end{aligned} \tag{5.1}$$

where $r = p + q - 1$. We claim that $V^+(z) \ge V^-(z)$. This is trivial if $V^-(z) = z$, so assume that $V^-(z) > z$. Then $V^+(z) - V^-(z) \ge W^+(z) - W^-(z) \ge \beta r[V^+(z) - V^-(z)]$, and since $\beta r < 1$, this implies that $V^+(z) - V^-(z) \ge 0$, establishing the claim.

If $R \le 0$, the inequality sign in (5.1) flips, and argueing similarly we conclude that $V^+(z) \le V^-(z)$. Using the definitions of L and U from Proposition 3.1, this completes the proof.

In what follows, say that at stage n the walk is in state $(s, +)$ if $S_n = s$ and $X_n = 1$; and in state $(s, -)$ if $S_n = s$ and $X_n = -1$. (In fact, the process (S_n, X_n) is a bivariate Markov chain, and so the states $(s, +)$ and $(s, -)$ have a natural interpretation as states of a Markov chain.)

Proposition 5.1. *If $R \ge 0$, then*

$$U = \left\lceil \max\left\{ \frac{\varrho_+}{1-\varrho_+}, \frac{\beta(2p-1)}{1-\beta} \right\} \right\rceil. \tag{5.2}$$

Proof. Recall that $U = \lceil \kappa \rceil$, where κ is defined by (3.3). Choosing $\tau = N(1)$ in (3.3) gives that $U \ge \varrho_+/(1-\varrho_+)$. Choosing $\tau \equiv 1$ gives $U \ge \beta(2p-1)/(1-\beta)$. Hence U is at least as large as the right side of (5.2).

For the converse inequality, suppose the walk is in state $(U-1, +)$, and consider two cases.

Case 1. $L \geq U - 1$. It is then optimal to continue until the walk reaches U. Hence $U - 1 < \mathrm{E}^{+}_{U-1}\, \beta^{N(U)} S_{N(U)} = \mathrm{E}^{+}_{0}\, \beta^{N(1)} U = \varrho_+ U$, so $U - 1 < \varrho_+/(1 - \varrho_+)$.

Case 2. $L < U - 1$. It is then optimal to continue until the walk either reaches U, or takes a step to the left. Since one or the other must clearly occur at the next stage, the rule $\tau \equiv 1$ is optimal. Hence $U - 1 < \mathrm{E}^{+}_{U-1}\, \beta S_1 = \beta(U - 1 + 2p - 1)$, so $U - 1 < \beta(2p - 1)/(1 - \beta)$.

Since U is an integer, it follows in both cases that U is no larger than the right side of (5.2). This establishes the proposition.

The next lemma is important mainly for computational purposes. The explicit expressions given below for ϱ_+ and ϱ_- have little impact on the theoretical arguments in the proofs that follow.

Lemma 5.2. *Let $r = p + q - 1$. Then*

$$\varrho_+ = [1 + r\beta^2 - \{(1 + r\beta^2)^2 - 4pq\beta^2\}^{1/2}]/2\beta q, \tag{5.3}$$

and

$$\varrho_- = [1 - r\beta^2 - \{(1 + r\beta^2)^2 - 4pq\beta^2\}^{1/2}]/2\beta(1 - p). \tag{5.4}$$

Moreover, for $s \geq 2$,

$$\mathrm{E}^{+}_{0}\, \beta^{N(s)} = \varrho_+^s, \tag{5.5}$$

and

$$\mathrm{E}^{-}_{0}\, \beta^{N(s)} = \varrho_- \varrho_+^{s-1}. \tag{5.6}$$

Proof. Let $G^{(s)}_+(x) = \mathrm{E}^{+}_{0}\, x^{N(s)}$ and $G^{(s)}_-(x) = \mathrm{E}^{-}_{0}\, x^{N(s)}$ be the probability-generating functions of $N(s)$ given that $S_0 = 0$ and $X_0 = 1$, respectively -1. Notice that $\varrho_+ = G^{(1)}_+(\beta)$ and $\varrho_- = G^{(1)}_-(\beta)$. The functions $G^{(s)}_+$ and $G^{(s)}_-$ satisfy (see [13])

$$G^{(1)}_+(x) = px + (1 - p)xG^{(1)}_-(x)G^{(1)}_+(x), \tag{5.7}$$

$$G^{(1)}_-(x) = (1 - q)x + qxG^{(1)}_-(x)G^{(1)}_+(x), \tag{5.8}$$

$$G^{(s)}_+(x) = [G^{(1)}_+(x)]^s, \qquad s \geq 2, \tag{5.9}$$

and

$$G^{(s)}_-(x) = G^{(1)}_-(x)[G^{(1)}_+(x)]^{s-1}, \qquad s \geq 2. \tag{5.10}$$

Simultaneously solving the equations (5.7) and (5.8), and substituting $x = \beta$ gives (5.3) and (5.4). The remaining identities are immediate consequences of (5.9) and (5.10).

Lemma 5.3. *The following inequalities are equivalent.*

(i) $\dfrac{\varrho_+}{1 - \varrho_+} \geq \dfrac{\beta(2p - 1)}{1 - \beta}$.

(ii) $\beta \leq \beta_0 := \dfrac{2}{1 + \sqrt{1 + 8r(2p - 1)}}$.

(iii) $\dfrac{\varrho_+}{1 - \varrho_+} \leq \dfrac{\varrho_-}{1 - \varrho_-} + 1$.

Proof. That (i) and (ii) are equivalent follows (eventually) from equation (5.3). That (i) and (iii) are equivalent follows from the identity

$$\varrho_- \varrho_+ = \frac{\varrho_+ - \beta p}{\beta(1-p)},$$

which is in turn a consequence of (5.7).

Note that in the symmetric case ($p = q$), (ii) becomes

$$\beta \leq \frac{2}{1 + \sqrt{1 + 8(2p-1)^2}}.$$

This may be solved for p in terms of β, yielding

$$p \leq \frac{1}{2}\left(1 + \sqrt{\frac{1-\beta}{2\beta^2}}\right).$$

This expresses in an alternative way that if $\beta \leq \beta_0$, the reinforcement between steps is relatively weak compared to the effect of the discounting.

Proof of Theorem 3.1. The first three statements follow directly from Lemma 5.1, Proposition 5.1, and Lemma 5.3, respectively. It remains only to calculate L. Suppose first that $\beta \leq \beta_0$, so $U = \lceil \varrho_+/(1-\varrho_+) \rceil$. A reasoning analogous to that in the first part of the proof of Proposition 5.1 shows that $L \geq \lceil \varrho_-/(1-\varrho_-) \rceil$. Since (iii) of Lemma 5.3 holds, it follows that $L \geq U - 1$.

To establish the converse inequality, $L \leq \lceil \varrho_-/(1-\varrho_-) \rceil$, suppose the walk is in state $(U-1, -)$, and consider two cases.

Case 1. $U < (1-\varrho_-)^{-1}$. Then for the stopping rule $\tau = N(U)$,

$$\mathrm{E}_{U-1}^{-} \beta^\tau S_\tau = \mathrm{E}_0^{-} \beta^{N(1)} U = \varrho_- U > U - 1,$$

so it is optimal to continue. Hence $L = U < (1-\varrho_-)^{-1}$.

Case 2. $U \geq (1-\varrho_-)^{-1}$. Then it is optimal to stop, since otherwise the rule τ from Case 1 would be optimal. We conclude that $L \leq U - 1 < \varrho_+/(1-\varrho_+) \leq (1-\varrho_-)^{-1}$ in view of Lemma 5.3 (iii).

In both cases it follows that $L < \varrho_-/(1-\varrho_-) + 1$, and therefore $L \leq \lceil \varrho_-/(1-\varrho_-) \rceil$.

Turn next to the case $\beta \geq \beta_0$. By Lemma 5.3, $U \geq \varrho_+/(1-\varrho_+) \geq (1-\varrho_-)^{-1} > 1$, so we conclude that $U \geq 2$. Furthermore, as in Case 2 above, it follows that $L \leq U - 1$.

If $U < 2/(1 - \varrho_-\varrho_+)$, then

$$U - 2 < \varrho_-\varrho_+ U = \mathrm{E}_{U-2}^{-} \beta^{N(U)} S_{N(U)},$$

so in state $(U-2, -)$ it is strictly optimal to continue. Hence $L > U - 2$, and it must be the case that $L = U - 1$.

Assume for the remainder of the proof that $U \geq 2/(1 - \varrho_-\varrho_+)$. If $S_0 < L$, the optimal rule can be written as

$$\tau^* = \min\{n \geq N(L+1) : S_n = U \text{ or } X_n = -1\}.$$

Thus, one approach would be to calculate the expected return from the rule τ^* and maximize over L. However, this seems to lead to tedious algebra. For an alternative method, define, for all $1 \leq s \leq U-2$, the stopping rule

$$\tau(s) = \min\{n \geq N(s+2) : S_n = U \text{ or } X_n = -1\}.$$

Assume the walk begins in state $(s, -)$. Recall that L is the smallest integer s such that for all stopping rules τ,

$$s \geq \mathrm{E}_s^- \, \beta^\tau S_\tau.$$

Since $\tau(L-1) = \tau^*$, the inequality

$$s \geq \mathrm{E}_s^- \, \beta^{\tau(s)} S_{\tau(s)}, \tag{5.11}$$

holds if $s = L$, but does not hold if $s = L-1$. To show that L is the *smallest* s for which (5.11) holds, it suffices to show that $f(s) := s - \mathrm{E}_s^- \, \beta^{\tau(s)} S_{\tau(s)}$ is increasing. To compute the expectation in (5.11), observe that we can write

$$\tau = N(s+2) + \tau'(s),$$

where $\tau'(s)$ is independent of $N(s+2)$, and the distribution of $\tau'(s)$ is that of $\sigma := \min\{n \geq 0 : S_n = U \text{ or } X_n = -1\}$, conditional on the event $\{S_0 = s+2, X_0 = 1\}$. Hence

$$\mathrm{E}_s^- \, \beta^{\tau(s)} S_{\tau(s)} = \mathrm{E}_s^- \, \beta^{N(s+2)} \, \mathrm{E}_{s+2}^+ \, \beta^\sigma S_\sigma.$$

By (5.6), $\mathrm{E}_s^- \, \beta^{N(s+2)} = \mathrm{E}_0^- \, \beta^{N(2)} = \varrho_- \varrho_+$. To compute the other expectation, let $T^- = \min\{n \geq 1 : X_n = -1\}$. Note that, given that $S_0 = s+2$, $\sigma = \min\{T^-, U-s-2\}$. Thus, since T^- has a geometric distribution conditional on the event $X_0 = 1$,

$$\begin{aligned}
\mathrm{E}_{s+2}^+ \, \beta^\sigma S_\sigma = \sum_{i=1}^{U-s-2} & \mathrm{E}_{s+2}^+ [\beta^\sigma S_\sigma | T^- = i] \, \mathrm{P}^+(T^- = i) \\
& + \mathrm{E}_{s+2}^+ [\beta^\sigma S_\sigma | T^- > U-s-2] \, \mathrm{P}^+(T^- > U-s-2) \\
= \beta(1-p) & \sum_{i=1}^{U-s-2} (\beta p)^{i-1}(s+i) + (\beta p)^{U-s-2} U.
\end{aligned}$$

Now, by conveniently regrouping terms, if follows that

$$
\begin{aligned}
f(s)-f(s-1) &= 1-\varrho_-\varrho_+[\beta(1-p)\sum_{i=1}^{U-s-2}(\beta p)^{i-1}+2\beta(1-p)(\beta p)^{U-s-2} \\
&\qquad\qquad\qquad\qquad + (1-\beta)(\beta p)^{U-s-2}U] \\
&\geq 1-\varrho_-\varrho_+[\beta(1-p)\sum_{i=1}^{U-s-2}(\beta p)^{i-1}+2\beta(1-p)(\beta p)^{U-s-2} \\
&\qquad\qquad\qquad\qquad + \{1-2\beta(1-p)\}(\beta p)^{U-s-2}] \\
&\geq 1-[\beta(1-p)\sum_{i=1}^{U-s-2}(\beta p)^{i-1}+(\beta p)^{U-s-2}] \\
&\geq 1-[\sum_{i=1}^{U-s-2}(\beta p)^{i-1}(1-\beta p)+(\beta p)^{U-s-2}] \\
&= 0,
\end{aligned}
$$

where the first inequality follows since $U=\lceil \beta(2p-1)/(1-\beta)\rceil \leq 1+\beta(2p-1)/(1-\beta)$, so $(1-\beta)U \leq 1-2\beta(1-p)$. Thus L is the smallest s for which (3.4) holds, provided such an s exists. But since $U \geq 2/(1-\varrho_-\varrho_+)$, (3.4) holds when $s=U-2$, and so the number s^* in the statement of Theorem 3.1 is well-defined.

Proof of Theorem 3.2. Statement (i) follows from Lemma 5.1, and (ii) follows since Case 1 in the proof of Proposition 5.1 applies. To prove (iii), suppose the walk begins in state $(z,-)$, where $U \leq z < L$. Then the optimal rule is to stop as soon as the walk takes a step to the right, unless prior to that the walk reaches level $U-1$, in which case one should continue until the walk returns to U. Define the stopping times $\tau^*=\min\{n: S_n \geq U \text{ and } X_n=1\}$, and $T^+=\min\{n: X_n=1\}$. Observe that on $\{T^+ > z-U+1\}$, $S_{\tau^*}=U$. Furthermore, $\mathrm{E}_z^-[\beta^{\tau^*}|T^+ > z-U+1] = \beta^{z-U+1}\,\mathrm{E}_{U-1}^-\,\beta^{N(U)} = \beta^{z-U+1}\,\mathrm{E}_0^-\,\beta^{N(1)} = \beta^{z-U+1}\varrho_-$. Thus,

$$
\begin{aligned}
V^-(z) &= \mathrm{E}_z^-\,\beta^{\tau^*}S_{\tau^*} \\
&= \sum_{i=1}^{z-U+1}\mathrm{E}_z^-[\beta^{\tau^*}S_{\tau^*}|T^+=i]\,\mathrm{P}^-(T^+=i) \\
&\qquad + \mathrm{E}_z^-[\beta^{\tau^*}S_{\tau^*}|T^+ > z-U+1]\,\mathrm{P}^-(T^+ > z-U+1) \\
&= \beta(1-q)\sum_{i=1}^{z-U+1}(z-i+2)(\beta q)^{i-1}+\varrho_-(\beta q)^{z-U+1}U.
\end{aligned}
$$

Since $V^-(z) > z$ if and only if $z < L$, (iii) follows.

Proof of Theorem 3.3. For $z \geq \min\{U,L\}$, the calculation of V^+ and V^- is shown essentially in the proofs of Theorems 3.1 and 3.2. If $z < \min\{U,L\}$, the appropriate expression from (5.5) or (5.6) gets pre-multiplied, since the walk must first pass through U or L before the optimal rule can stop.

Arbitrary first-step probabilities. So far we have assumed that the walk has already started some time before stage 1, and the distribution of the first step is determined by the direction of the zeroth step. This is of course no real restriction. If it is desirable to assume that the walk actually begins with step 1, the direction of the first step for instance being determined by the flip of a coin, then the optimal rule and expected return follow easily from the previous results using the optimality equation. Let $0 \le \alpha \le 1$ be a parameter such that $\mathrm{P}(X_1 = 1) = \alpha = 1 - \mathrm{P}(X_1 = -1)$, and define $V(z) = \sup_\tau \mathrm{E}_z\, \beta^\tau S_\tau$. Then

$$V(z) = \max\{z, \beta[\alpha V^+(z+1) + (1-\alpha)V^-(z-1)]\},$$

and at least one observation should be taken if the second term in the maximum is greater than the first.

6. Extensions

The qualitative results of this paper can be extended without difficulty to a variety of more general settings. The key point in each of the examples given below is that the proof of Proposition 3.1 does not depend on the assumption of unit steps, but applies in more general situations as well.

1) Let $X_n, n \ge 0$ be the Markov chain defined in Section 1, and let $Y, Y_1, Y_2, \ldots$ and $Z, Z_1, Z_2, \ldots$ be i.i.d. sequences of positive random variables, independent of the sequence $\{X_n\}$. Define $S_0 = 0$, and for $n \ge 1$,

$$S_n = \sum_{i=1}^{n} [Y_i \mathrm{I}(X_i = 1) - Z_i \mathrm{I}(X_i = -1)],$$

where $\mathrm{I}(A)$ denotes the indicator random variable of the event A. Thus, steps to the right are distributed as Y, whereas steps to the left are distributed as Z. It is easy to see that the conclusions of Proposition 3.1 and Lemma 5.1 hold, and hence the optimal stopping rule is of the same general form as in the special case considered above. Of course, the thresholds may be difficult to calculate in general.

2) Let $X_n, n \ge 0$ be a *general* Markov chain with state space $I \subset \mathbf{Z}$. Define $S_0 = 0$, and $S_n = \sum_{j=1}^{n} X_j$ for $n \ge 1$. To analyze the structure of the optimal rule, introduce

$$\begin{aligned} V^i(z) &:= \sup_\tau \mathrm{E}_z^i\, \beta^\tau S_\tau \\ &:= \sup_\tau \mathrm{E}[\beta^\tau (z + S_\tau) | S_0 = z, X_0 = i], \qquad i \in I. \end{aligned}$$

Imitating the proof of Proposition 3.1 for each V^i, we see that there is a threshold u_i for each state $i \in I$, such that it is optimal to stop at the first time n at which $S_n \ge u_{X_n}$. If the state space I is bounded from above, then each of the thresholds is finite, and so are the optimal values $V^i(z)$. It is interesting to ask for natural conditions on the Markov chain that guarantee a finite optimal value when the state space is unbounded.

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