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OPTIMAL BUY/SELL RULES FOR CORRELATED RANDOM WALKS

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Abstract

Correlated random walks provide an elementary model for processes that exhibit directional reinforcement behavior. This paper develops optimal multiple stopping strategies - buy/sell rules - for correlated random walks. The work extends previous results given in Allaart and Monticino (2001) by considering random step sizes and allowing possibly negative reinforcement of the walk's current direction. The optimal strategies fall into two general classes - cases where conservative buy-and-hold type strategies are optimal and cases in which aggressive trading strategies of successively buying and selling the commodity depending on whether the price goes up or down are followed. Simulation examples are given based on an stock index fund to illustrate the variation in return possible using the theoretically optimal stop rules compared to simpler buy-and-hold strategies.

Keywords: Correlated random walk; multiple stopping; buy/sell strategies

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1. Introduction

This paper develops optimal multiple stopping strategies - buy/sell rules - for commodities whose prices follow a generalized class of correlated random walks. The work extends previous results given in Allaart and Monticino (2001) by considering

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random step sizes and allowing for possibly negative reinforcement of a walk's current direction. Interestingly, the optimal strategies fall into two general classes - cases where a conservative buy-and-hold strategy is optimal and cases in which a very aggressive trading strategy of successively buying and selling the commodity depending on whether the price goes up or down is followed.

Define a random walk with correlation (RWC), $\{S_n\}_{n\geq 0}$, by $S_0 \equiv s_0 \in \mathbb{R}$, and $S_n = S_0 + X_1 + \cdots + X_n$ for $n \geq 1$, where the increments $X_1, X_2, \ldots$ form a Markov sequence in the following way. Let Y_u and Y_d be real-valued random variables with finite expectations. For each $n \in \mathbb{N}$, $\mathcal{L}(X_{n+1}|X_n \geq 0) = \mathcal{L}(Y_u)$, and $\mathcal{L}(X_{n+1}|X_n < 0) = \mathcal{L}(Y_d)$, where $\mathcal{L}$ denotes probability law. Let $p = \mathrm{P}(Y_u \geq 0)$, and $q = \mathrm{P}(Y_d < 0)$. (Assume $0 < p, q < 1$ to avoid uninteresting cases.) So, if the price goes up at time n, it will go up (or stay equal) at time $n+1$ with probability p and will go down with probability $1-p$. The distribution of the price change at time $n+1$ after a price increase at time n is $\mathcal{L}(Y_u)$. Similarly, if the price goes down at n, it will go down at time $n+1$ with probability q and will go up or stay the same with probability $1-q$. The distribution of the price change at $n+1$ given a price decrease at time n is $\mathcal{L}(Y_d)$.

The objective of this paper is to determine a sequence of buying and selling times that maximize the investor's expected return from trading a commodity whose price follows the walk S_n, given an investment time horizon N and transaction cost $c \geq 0$. That is, the goal is to find stopping times

$$0 \leq \tau_1 < \tau_2 < \cdots < \tau_{2m} \leq N,$$

such that if the commodity is bought at times τ_{2j-1} and sold at times τ_{2j}, then the expected value of the total return

$$P(N) = \sum_{j=1}^{m} (S_{\tau_{2j}} - S_{\tau_{2j-1}}) - 2mc$$

is maximized, where m is possibly random. Note that the investor incurs a transaction cost for each trade whether it is a buy or sell. Since the investor is required to sell the commodity by time N if it is ever bought, each buy is paired with a sell.

Optimal trading strategies are developed for four basic cases of commodity price behavior. The first is trivial: the price process is a supermartingale and the obvious optimal strategy is never to buy. On the other hand, when the price process is a submartingale it is optimal to hold onto the commodity until the time horizon once the commodity is purchased. It is just a question of whether to buy given the investment time horizon. The third case considered is when the price is expected to decrease on the next step if the price went up on the previous stage, and the price is expected to increase if the price went down on the previous stage. The form of the optimal strategy depends on the transaction costs for this case. Either the commodity is bought and sold a single time according buy and sell signals; or, if costs are low enough, then it is optimal to successively buy and sell according to the familiar investment axiom of "buying on the dips and selling on the peaks." The final case considered reverses the commodity behavior and optimal strategies of the third case. Optimal strategies for all cases are given in the next section.

The random walk model considered here is an immediate extension of the correlated random walks introduced by Goldstein (1955). Basic properties of correlated random walks such as transition probabilities and first passage times have been examined in a number of papers. For instance, Seth (1963) gives return probabilities and first-passage time distributions for symmetric correlated random walks. Jain (1971) generalizes these results to the non-symmetric case. Renshaw and Henderson (1981) present occupation probabilities and a diffusion approximation. Gillis (1955) developed a d-dimensional version, and conjectured it to be transient for all $d \geq 3$. Gillis' conjecture was proved by Iossif (1986), and then for more general correlated random walks by Chen and Renshaw (1994). Other results on correlated random walks and boundary problems include Proudfoot and Lampard (1972), Jain (1973), Mukherjea and Steele (1987), Zhang (1992) and Böhm (2000).

Gambler ruin type problems for correlated random walks are examined by Mohan (1955) and Mukherjea and Steele (1986). Optimal buy/sell strategies are developed for a more general class of processes called directionally reinforced random walks in Allaart and Monticino (2001). Results given there are extended here by allowing random step sizes and accounting for negative reinforcement addressed by the "buy on the dips and sell on the peaks" strategies. Allaart (2003) examines the single stop problem of when to sell a commodity whose price follows a correlated random walk in order to maximize discounted return.

Other applications using correlated random walks include Goldstein (1951) where certain physical diffusion processes are modeled, and Henderson and Renshaw (1980) in which tree root growth is examined. Renshaw and Henderson (1981) studied the behavior of a certain kind of pinball machine. Mauldin *et al.* (1996) use the more general directionally reinforced random walks as an elementary model of ocean surface waves. A comprehensive list of references for correlated random walks is given in Chen and Renshaw (1994).

The problem explored here is motivated by the popular notion among proponents of stock market technical analysis that movements in security prices are not memoryless. In particular, price changes - up or down - from one day to the next affect succeeding day changes. Various assertions are given to justify this idea of price momentum, such as fundamental information about a company ripples out from insiders to investment professionals to individual investors. As this happens, the theory goes that the stock price is pushed ever higher if the information is favorable, or lower if the news is bad. Of course, it is far from agreed that the market consistently exhibits any behavior other than a random walk about an underlying trend (see, for instance, Malkiel (1999)). Regardless of one's belief in this postulated market phenomena, very little seems to be known about optimal stopping for processes exhibiting momentum. The intention here is not to take a stand on the existence of price momentum in the market, but to gain insight into how an investor might take advantage of momentum, if present, by examining a simple model of such processes. Section 3 illustrates the strategies developed through the behavior of a stock index fund. Using a model based on the price history of the fund, simulation results are given that indicate the variation in return possible using the theoretically optimal stop rules and compares this to simpler buy-and-hold strategies.

2. Optimal Buy/Sell Strategies

This section presents optimal buy/sell strategies. As mentioned above, there are four main cases that determine the form of the optimal strategy. Some necessary notation is developed first.

Let $E^+ := \mathrm{E}[Y_u]$ and $E^- := \mathrm{E}[Y_d]$. Define the *total correlation* by $r := p + q - 1$, and the *drift* by

$$\delta := \frac{(1-q)E^{+} + (1-p)E^{-}}{1-r}.$$

Recall that each purchase of the commodity is paired with a sell. So without loss of generality assume that a single transaction cost C representing total buying and selling costs is (only) assessed when the commodity is bought.

For $j \in \mathbb{N}$, let $V_H^+(j)$ denote the optimal expected additional net gain when there are j time periods remaining, the last increment was nonnegative, and after seeing that increment a decision was made resulting in the commodity now being held. Let $V_F^+(j)$ denote the same, except that after the last decision moment the commodity is *not* being held. Similarly, define $V_H^-(j)$ and $V_F^-(j)$, replacing "nonnegative" with "negative". For convenience, define $V_H^+(0) = V_H^-(0) = V_F^+(0) = V_F^-(0) = 0$. The following recursive relationships hold:

$$V_H^+(j+1) = E^+ + p\max\{V_H^+(j), V_F^+(j)\} + (1-p)\max\{V_H^-(j), V_F^-(j)\}, \tag{2.1}$$

$$V_F^+(j+1) = p\max\{V_H^+(j) - C, V_F^+(j)\} + (1-p)\max\{V_H^-(j) - C, V_F^-(j)\}, \tag{2.2}$$

$$V_H^-(j+1) = E^- + (1-q)\max\{V_H^+(j), V_F^+(j)\} + q\max\{V_H^-(j), V_F^-(j)\}, \tag{2.3}$$

$$V_F^-(j+1) = (1-q)\max\{V_H^+(j) - C, V_F^+(j)\} + q\max\{V_H^-(j) - C, V_F^-(j)\}. \tag{2.4}$$

The following lemma collects some elementary properties of the functions defined above. The straightforward proofs are omitted.

Lemma 1. *(a) $V_H^+(j)$, $V_F^+(j)$, $V_H^-(j)$ and $V_F^-(j)$ are all nondecreasing in j.*

(b) $V_F^+(j) \geq 0$ and $V_F^-(j) \geq 0$ for all j.

(c) If $E^+ \geq 0$, then $V_H^+(j) \geq V_F^+(j)$ for all j. If $E^- \geq 0$, then $V_H^-(j) \geq V_F^-(j)$ for all j.

(d) If $E^+ \leq 0$, then $V_H^+(j) - C \leq V_F^+(j)$ for all j. If $E^- \leq 0$, then $V_H^-(j) - C \leq V_F^-(j)$ for all j.

(e) If $E^+ \geq E^-$, then $V_H^+(j) \geq V_H^-(j)$ and $V_F^+(j) \geq V_F^-(j)$ for all j. If $E^+ \leq E^-$, the reverse inequalities hold.

Next, define the differences

$$D^+(j) := V_H^+(j) - V_F^+(j), \qquad D^-(j) := V_H^-(j) - V_F^-(j),$$

$$D_H(j) := V_H^+(j) - V_H^-(j), \qquad D_F(j) := V_F^+(j) - V_F^-(j).$$

Observe that with j time periods remaining, it is optimal to buy after an up-step if and only if $D^+(j) \geq c$, and after a down-step if and only if $D^-(j) \geq c$.

Lemma 2. *(a) If $E^+ \geq E^-$, then $D^+(j) \geq D^-(j)$ for all $j \geq 0$.*

(b) If $E^+ \leq E^-$, then $D^+(j) \leq D^-(j)$ for all $j \geq 0$.

Proof. Assume $E^+ \geq E^-$. For $j \geq 0$, define

$$\begin{aligned}\Delta(j) = \max\{V_H^+(j), V_F^+(j)\} - \max\{V_H^-(j), V_F^-(j)\} \\ - \max\{V_H^+(j) - C, V_F^+(j)\} + \max\{V_H^-(j) - C, V_F^-(j)\}.\end{aligned}$$

It will be shown inductively that, for all $j \in \mathbb{N}$,

$$D^+(j) \geq D^-(j), \qquad \Delta(j-1) \geq 0. \tag{2.5}$$

Since $D^+(1) - D^-(1) = E^+ - E^-$ and $\Delta(0) = 0$, (2.5) holds for $j = 1$. Assume it holds for $j = k$. Then, by (2.1)-(2.4),

$$\begin{aligned}D^+(k+1) - D^-(k+1) &= D_H(k+1) - D_F(k+1) \\ &= E^+ - E^- + r\Delta(k).\end{aligned} \tag{2.6}$$

If $V_H^-(k) - C \geq V_F^-(k)$, then $V_H^+(k) - C \geq V_F^+(k)$ by the induction hypothesis, and $\Delta(k) = 0$. Likewise, if $V_H^+(k) \leq V_F^+(k)$, then $V_H^-(k) \leq V_F^-(k)$ and again $\Delta(k) = 0$. In both cases, (2.5) clearly holds for $j = k+1$. Assume, therefore, that

$$V_H^-(k) - C < V_F^-(k), \qquad \text{and} \qquad V_H^+(k) > V_F^+(k). \tag{2.7}$$

Then

$$\Delta(k) = \min\{D^+(k), C\} - \max\{D^-(k), 0\}. \tag{2.8}$$

This, along with the induction hypothesis and (2.7) implies that $\Delta(k) \geq 0$. If $r \geq 0$, this immediately yields $D^+(k+1) \geq D^-(k+1)$. If $r < 0$, then by (2.8) and the induction hypothesis,

$$\Delta(k) \leq D^+(k) - D^-(k) = E^+ - E^- + r\Delta(k-1) \leq E^+ - E^-. \tag{2.9}$$

Since $r \geq -1$, substituting (2.9) into (2.6) yields that $D^+(k+1) \geq D^-(k+1)$. This proves (a). The proof of (b) is analogous. □

Theorem 1. *If it is optimal to buy after an up-step (down-step) with j time periods remaining, then it is optimal to buy after an up-step (down-step) with $j+1$ time periods remaining.*

Proof. Suppose $D^+(j) \geq C$. By Lemma 1(d), this implies $E^+ \geq 0$. Thus, if $V_H^-(j) - C \geq V_F^-(j)$, then (2.1) and (2.2) immediately yield that $D^+(j+1) \geq C$. Suppose, therefore, that $V_H^-(j) - C < V_F^-(j)$. By Lemma 2, this implies that $E^+ \geq E^-$ and hence, by Lemma 1(e), $V_F^+(j) \geq V_F^-(j)$. Now $V_H^+(j+1) \geq V_H^+(j)$, and

$$\begin{aligned} V_F^+(j+1) &= p\{V_H^+(j) - C\} + (1-p)V_F^-(j) \\ &\leq p\{V_H^+(j) - C\} + (1-p)V_F^+(j), \end{aligned}$$

so that

$$D^+(j+1) \geq (1-p)\{V_H^+(j) - V_F^+(j)\} + pC \geq C.$$

This completes the proof for buying after an up-step. The proof for down-steps is similar. □

The optimal buy/sell strategies are as follows.

Case I. $E^+ \leq 0$, $E^- \leq 0$.

The price process, S_n, is a supermartingale in this case. Thus, the optimal strategy is not to buy the commodity at all.

Case II. $E^+ > 0$, $E^- > 0$.

In this case, the price process is a submartingale. Hence, once the commodity is bought, it is optimal to hold it until the time horizon. The question is whether to buy the commodity. Intuitively, the commodity should be bought only if there is enough time between purchase and the time horizon for the positive drift of the walk to generate an expected gain greater than the transaction cost.

In this case, (2.1) and (2.3) simplify to

$$V_H^+(j+1) = E^+ + pV_H^+(j) + (1-p)V_H^-(j), \tag{2.10}$$

$$V_H^-(j+1) = E^- + (1-q)V_H^+(j) + qV_H^-(j). \tag{2.11}$$

Hence,

$$D_H(j+1) = rD_H(j) + (E^+ - E^-), \quad j \geq 0.$$

Since $D_H(0) = 0$, it follows that

$$D_H(j) = \frac{E^+ - E^-}{1-r}(1 - r^j), \qquad j \geq 0.$$

Substituting $V_H^-(j) = V_H^+(j) - D_H(j)$ into (2.10) and iterating gives

$$V_H^+(j) = \sum_{k=1}^{j} \{E^+ - (1-p)D_H(k-1)\}, \qquad j \in \mathbb{N}.$$

Thus, after some manipulations,

$$V_H^+(j) = \delta j + \frac{(1-p)(E^+ - E^-)}{(1-r)^2}(1 - r^j), \qquad j \in \mathbb{N}, \tag{2.12}$$

and

$$V_H^-(j) = \delta j - \frac{(1-q)(E^+ - E^-)}{(1-r)^2}(1 - r^j), \qquad j \in \mathbb{N}. \tag{2.13}$$

There are two subcases.

Case II (a). $E^+ \geq E^-$.

Let j^+ be the smallest j for which $D^+(j) \geq C$, and let j^- be the smallest j for which $D^-(j) \geq C$. Then $j^+ \leq j^-$ by Lemma 2, and an easy induction argument shows that $V_F^+(j) = V_F^-(j) = 0$ for all $j \leq j^+$. Thus, j^+ is the smallest j such that $V_H^+(j) \geq C$. Since $\delta > 0$, such a j exists by (2.12).

If $V_H^-(j^+) \geq C$, then $j^- = j^+$. Otherwise, let $j^+ \leq j < j^-$. Then

$$V_F^-(j+1) = (1-q)\{V_H^+(j) - C\} + qV_F^-(j),$$

so, by (2.3),

$$D^-(j+1) = qD^-(j) + E^- + (1-q)C.$$

Since

$$D^-(j^+) = V_H^-(j^+) - V_F^-(j^+) = V_H^-(j^+),$$

it follows that for all $j^+ < j \leq j^-$,

$$D^-(j) = \{V_H^-(j^+) - b^-\}q^{j-j^+} + b^-, \qquad b^- := \frac{E^-}{1-q} + C. \tag{2.14}$$

Hence, j^- is the smallest $j \geq j^+$ such that $\{V_H^-(j^+) - b^-\}q^{j-j^+} + b^- \geq C$. Since $b^- > C$, j^- is finite.

In summary, to determine whether to buy the commodity, first find j^+, the smallest j such that $V_H^+(j) \geq C$, using (2.12). Compute $V_H^-(j^+)$ using (2.13). Then use (2.14) to find j^-, the smallest $j \geq j^+$ such that $D^-(j) \geq C$. Now, with j time periods remaining, it is optimal to buy after an up-step if and only if $j \geq j^+$, and after a down-step if and only if $j \geq j^-$. Again, the commodity is held until the time horizon if it is purchased.

Case II (b). $E^+ < E^-$.

Here the order of j^+ and j^- is reversed. First, find j^-, the smallest j such that $V_H^-(j) \geq C$, using (2.13). Compute $V_H^+(j^-)$ using (2.12). Then use the formula

$$D^+(j) = \{V_H^+(j^-) - b^+\}p^{j-j^-} + b^+, \qquad b^+ := \frac{E^+}{1-p} + C.$$

to find j^+, the smallest $j \geq j^-$ such that $D^+(j) \geq C$. Finally, use j^+ and j^- as in Case II (a).

Case III. $E^+ \leq 0$, $E^- > 0$.

This case and the next are the most interesting as the optimal strategies may involve multiple trades in and out of the commodity. Some additional definitions are needed before specifying the optimal strategies.

Define

$$j_s := \inf\left\{j \geq 1 : 1 - q^j \geq \frac{(1-q)|E^+|}{(1-p)E^-}\right\}, \tag{2.15}$$

where the infimum of an empty set is taken to be ∞. Let

$$f^-(j) := \begin{cases} \dfrac{1-q^j}{1-q}E^-, & j \leq j_s + 1, \\ (j - j_s - 1)\delta + r(q^{j_s}E^- - \delta)\dfrac{1 - r^{j-j_s-1}}{1-r} + \dfrac{1-q^{j_s+1}}{1-q}E^-, & j > j_s + 1, \end{cases}$$

and define

$$j_b := \inf\{j \geq 1 : f^-(j) \geq C\}. \tag{2.16}$$

It is not difficult to see that j_s is finite if and only if $\delta > 0$. Similarly, j_b is finite if and only if $\delta > 0$ or $E^- > C(1-q)$.

Theorem 2. *If $E^+ \le 0$, $E^- > 0$, then the optimal strategy is*

(i) Never sell after a down-step;

(ii) Never buy after an up-step;

(iii) Buy after a down-step if and only if there are at least j_b time periods remaining;

(iv) Sell after an up-step if and only if $|E^+| \ge C(1-p)$ or the number of remaining time periods is at most $\min(j_b, j_s)$.

Proof. Parts (i) and (ii) of the optimal strategy follow directly from (c) and (d) of Lemma 1, respectively. To prove parts (iii) and (iv), let j^* be the smallest value of j (possibly infinite) such that $D^-(j) \ge C$. Since $V_F^+(j) = V_F^-(j) = 0$ for $j \le j^*$, it follows that j^* is in fact the smallest j such that $V_H^-(j) \ge C$. It will now be shown that $j^* = j_b$. There are again two subcases:

Case III (a). $|E^+| \ge C(1-p)$.

In this case,

$$V_H^+(j) \le V_F^+(j) \qquad \text{for all } j. \tag{2.17}$$

This follows by a routine induction argument. In particular, $V_H^+(1) = E^+ \le 0 = V_F^+(1)$. If $V_H^+(j) \le V_F^+(j)$ for some j, then, since $V_H^-(j) \ge V_F^-(j)$,

$$\begin{aligned} V_H^+(j+1) &= pV_F^+(j) + (1-p)V_H^-(j) + E^+, \\ V_F^+(j+1) &\ge pV_F^+(j) + (1-p)\{V_H^-(j) - C\}, \end{aligned} \tag{2.18}$$

so that

$$V_H^+(j+1) - V_F^+(j+1) \le E^+ + (1-p)C \le 0.$$

By (2.17), it is always optimal to sell after an up-step. Hence part (iv) of the optimal strategy follows for the case $|E^+| \ge C(1-p)$.

Next, (2.17) implies that for $k < j^*$,

$$V_H^-(k+1) = E^- + (1-q)V_F^+(k) + qV_H^-(k) = E^- + qV_H^-(k). \tag{2.19}$$

Hence,

$$V_H^-(j) = \frac{1-q^j}{1-q}E^-, \qquad 1 \le j \le j^*, \tag{2.20}$$

and it follows that

$$j^* = \inf\left\{ j : \frac{1-q^j}{1-q}E^- \ge C \right\}. \tag{2.21}$$

Since $|E^+| \geq C(1-p)$, (2.21) and (2.15) imply that $j^* \leq j_s$. Hence, by (2.16) and the definition of $f^-(j)$, $j^* = j_b$. This establishes part (iii) of the optimal strategy in the case $|E^+| \geq C(1-p)$.

Case III (b). $|E^+| < C(1-p)$.

If $j \geq j^*$, equality holds in (2.18), so that

$$V_H^+(j+1) - V_F^+(j+1) \geq E^+ + C(1-p) > 0. \tag{2.22}$$

Thus, if the commodity is being held after an up-step with more than j^* time periods remaining, it is optimal to hold it for at least one more time period.

Next, let $j < j^*$ be fixed, and suppose that $V_H^+(k) \leq 0$ for $k = 1, \ldots, j$. Then

$$V_H^+(j+1) = E^+ + (1-p)V_H^-(j).$$

Since $j < j^*$, (2.19) holds for $k = 0, 1, \ldots, j$. Hence, $V_H^-(j)$ is given by (2.20), so the following equivalences hold:

$$V_H^+(j+1) \geq 0 \quad \iff \quad V_H^-(j) \geq |E^+|/(1-p) \quad \iff \quad j \geq j_s. \tag{2.23}$$

If $j^* \leq j_s$, then (2.23) implies that $V_H^+(j) < 0$ for all $j \leq j^*$, which means that (2.20) holds for all $j \leq j^*$. Hence,

$$j^* = \inf\left\{ j : \frac{1-q^j}{1-q} E^- \geq C \right\} = j_b.$$

If $j^* > j_s$, then (2.20) holds for $1 \leq j \leq j_s + 1$. However, (2.23) shows that with $j_s + 1$ or more time periods remaining it is no longer optimal to sell after an up-step, so that the calculation of $V_H^-(j)$ is different when $j > j_s + 1$. In fact, for $j_s < j < j^*$, $V_H^+(j)$ and $V_H^-(j)$ satisfy the difference equations (2.10) and (2.11), but with different initial conditions. Specifically,

$$V_H^-(j_s+1) = \frac{1-q^{j_s+1}}{1-q} E^- \qquad \text{(by (2.20))},$$

and

$$\begin{aligned} V_H^+(j_s+1) &= E^+ + pV_F^+(j_s) + (1-p)V_H^-(j_s) \\ &= E^+ + (1-p)\frac{1-q^{j_s}}{1-q} E^- \end{aligned}$$

(again by (2.20) and the fact that $V_F^+(j_s) = 0$, since $j_s < j^*$). Define

$$D(k) := D_H^+(k + j_s + 1), \qquad k = 0, 1, 2, \ldots.$$

Then $D(k) = rD(k-1) + (E^+ - E^-)$ for $k = 1, 2, \ldots, j^* - j_s - 1$, and hence,

$$D(k) = r^k D(0) + (E^+ - E^-)\frac{1 - r^k}{1 - r}, \qquad k = 1, 2, \ldots, j^* - j_s - 1, \tag{2.24}$$

where

$$\begin{aligned} D(0) &= V_H^+(j_s + 1) - V_H^-(j_s + 1) \\ &= E^+ + (1-p)\frac{1 - q^{j_s}}{1 - q}E^- - \frac{1 - q^{j_s+1}}{1 - q}E^- \\ &= E^+ + \frac{rq^{j_s} - p}{1 - q}E^-. \end{aligned} \tag{2.25}$$

Now substituting $V_H^+(j) = V_H^-(j) + D(j - j_s - 1)$ into (2.11) gives

$$V_H^-(j+1) = V_H^-(j) + (1-q)D(j - j_s - 1) + E^-, \qquad j = j_s + 1, \ldots, j^* - 1.$$

Iterating this recursive relationship yields, for $j = j_s + 2, \ldots, j^*$,

$$V_H^-(j) = V_H^-(j_s + 1) + \sum_{k=1}^{j - j_s - 1} \{E^- + (1-q)D(k-1)\}. \tag{2.26}$$

After straightforward calculations using (2.24), (2.25) and the definitions of r and δ, (2.26) reduces to

$$V_H^-(j) = (j - j_s - 1)\delta + r(q^{j_s}E^- - \delta)\frac{1 - r^{j - j_s - 1}}{1 - r} + \frac{1 - q^{j_s+1}}{1 - q}E^-.$$

Thus, for all $j \le j^*$, $V_H^-(j) = f^-(j)$, and it follows that $j^* = j_b$, which proves part (iii) of the optimal strategy. Part (iv) follows from the comment following (2.22) and the equivalences (2.23) (which hold for $j < j^*$). $\square$

Case IV. $E^+ > 0$, $E^- \le 0$.

This case is a mirror image of Case III, and can be treated in the same way. The optimal strategy is stated formally in the next theorem.

Analogous to Case III, define

$$j_s := \inf\left\{ j \ge 1 : 1 - p^j \ge \frac{(1-p)|E^-|}{(1-q)E^+} \right\}.$$

Let

$$f^+(j) := \begin{cases} \dfrac{1-p^j}{1-p} E^-, & j \le j_s + 1, \\ (j - j_s - 1)\delta + r(p^{j_s} E^+ - \delta) \dfrac{1 - r^{j-j_s-1}}{1-r} + \dfrac{1-p^{j_s+1}}{1-p} E^+, & j > j_s + 1, \end{cases}$$

and define

$$j_b := \inf\{j \ge 1 : f^+(j) \ge C\}.$$

Theorem 3. *If $E^+ > 0$, $E^- \le 0$, then the optimal strategy is*

(i) Never sell after an up-step;

(ii) Never buy after a down-step;

(iii) Buy after an up-step if and only if there are at least j_b time periods remaining;

(iv) Sell after a down-step if and only if $|E^-| \ge C(1-q)$ or the number of remaining time periods is at most $\min(j_b, j_s)$.

3. Example of Return Variance

This section presents simulation results to illustrate the variance in return possible when applying optimal trading strategies. Some questions for further study are also mentioned after the simulation results. Simulations are based on data from a mutual fund that tracks the common stock performance of the 1,000 largest publicly traded U.S. companies - the Schwab 1000 Index fund (SNXFX). The intention is not to advocate that this fund or any commodity necessarily obeys a random walk with correlation. Rather, the example provides a guide to what sort of return variance could be expected if the trading strategies were applied under the RWC assumption. Since the exact distribution on return is difficult to compute even for simple distributions of Y_u and Y_d, simulation is used in the examples given here.

The RWC model used in the simulations is based on the daily closing price of the SNXFX fund from June 1, 2004 to May 31, 2005 (Figure 1). To gain a sense of whether a random walk with correlation model is at all a reasonable representation of the data, a standard statistical test (see Anderson and Goodman (1957)) was performed to evaluate the hypothesis that the price movement of the fund - up or down - from one day to the next is independent of the previous day's movement. For the 252 trading days in the data set, the (chi-square) test had a .0387 level of significance, indicating

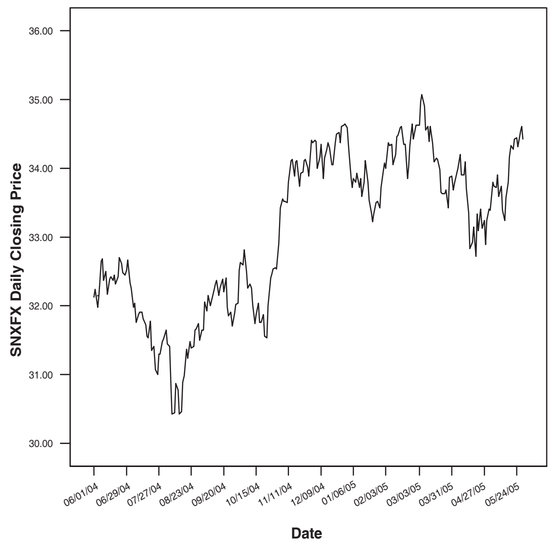

FIGURE 1: Daily closing price of Schwab 1000 Index fund from June 1, 2004 through May 31, 2005.

that the independence model could be rejected. Of course, the test does not indicate whether models other than a RWC model may provide better explanations for the data. Figures 2 and 3 show the histograms on the price change of the fund given the previous day's price went up or down, respectively. The figures also show a fitted normal density. A standard t-test yields that the hypothesis that the means of the two distributions are equal can be rejected at a (two-tail) significance level of .00035. In the simulation, it was assumed that for any trade – buying or selling – 1000 shares were traded. So, motivated by the data, an RWC model with $Y_u \sim \mathcal{N}(-17, 220)$, $p = .469$, $Y_d \sim \mathcal{N}(45, 230)$, and $q = .422$ was used.

The optimal strategy given by Theorem 2 was applied over a time horizon of $N = 65$, corresponding to the sixty-five NYSE trading days June 1, 2005 through August 31, 2005. Figure 4 shows the distribution on return for the strategy of buying 1000 shares of the fund on the first day and then holding it until it is sold at the time horizon, with no transaction costs. The distribution can be shifted to the left by C to account for transaction costs. For the assumed distributions on Y_u and Y_d, the optimal strategy will buy after a down step and sell after an up step whenever the transaction costs,

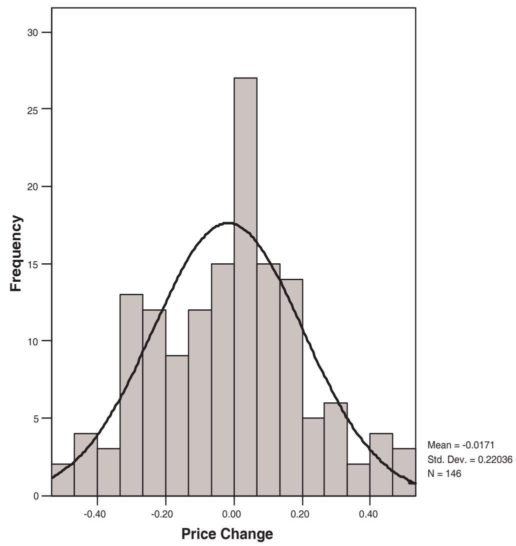

FIGURE 2: Histogram of price changes given price went up previous day for Schwab 1000 Index fund (June 1, 2004 - May 31, 2005).

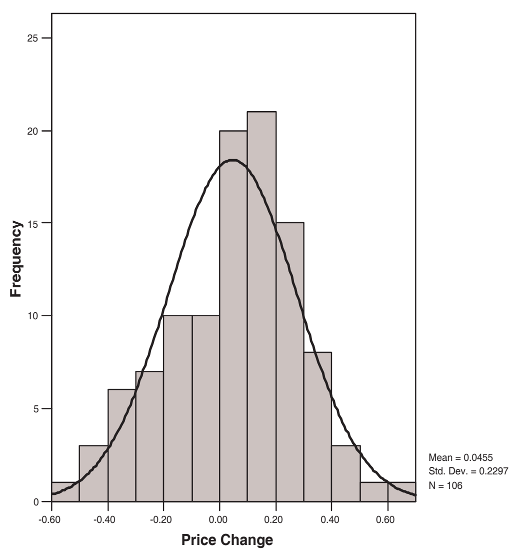

FIGURE 3: Histogram of price changes given price went down previous day for Schwab 1000 Index fund (June 1, 2-4 - May 31, 2005).

C, are below 32. For $C < 32$, $j_b = j_s = 1$. Simulations (100,000 Monte Carlo points and the walk assumed to have taken a down step at time 0) were performed for several values of $C < 32$. For $C = 0$, the average return for the buy-and-hold strategy was 854 with a standard deviation of 1634 while the average return for the optimal strategy was 1382 with a standard deviation of 1186. Moreover, for this case, the optimal strategy outperformed buy-and-hold more than 65% of the time. While the optimal strategy significantly improves upon the average return observed for the buy-and-hold strategy, both strategies have fairly large deviations in return. For $C = 10$, the average return for the optimal strategy was 1203 with a standard deviation of 1176; for $C = 20$, the average return was 1021 with a standard deviation of 1167; and, for $C = 30$, the average return was 845 with a standard deviation of 1154. The examples indicate that, as transaction costs increase, the average performances of the buy-and-hold and optimal strategies become similar. Moreover, the buy-and-hold strategy becomes nearly as likely to outperform the optimal strategy as the optimal strategy is to outperform the buy-and-hold strategy – the 50^{th} percentile of the difference between optimal strategy and buy-and-hold return is approximately 56 when $C = 30$ (Figure 5). It is a bit surprising that the standard deviation in return was lower for the more aggressive optimal strategies than for the buy-and-hold strategies, for all the cases considered. It is also interesting to note the return of the optimal strategy when applied to the actual performance of the fund SNXFX. For $C = 0$, the return from the optimal strategy would have been 1510 compared to 860 for the buy-and-hold strategy – actual returns for both strategies are relatively close to the average returns in the simulations. For $C = 10$, the return from the optimal strategy would have been 1320 (fund was bought and sold nineteen times), again close to the average simulation return. Similarly, for $C = 20$ and $C = 30$, the return from the optimal strategy would have been 1130 and 940, respectively.

There are several natural further questions to pursue:

- Many commodity funds place a limit on the number of trades that can occur within a given time. What are optimal strategies given a bound on the number of trades?
- How does a (tax) penalty for rapid buying and selling affect the form of the optimal trading strategy?
- What is the optimal strategy if there is a (dividend) reward for holding the

commodity at certain times?

- The simulations suggest that the variance in return of the optimal strategy is smaller than for buy-and-hold strategies. Is this true in general?
- The optimal strategies given here assume that the same number of units of the commodity are bought and sold each time. What is the optimal strategy when the number of units traded is allowed to vary?

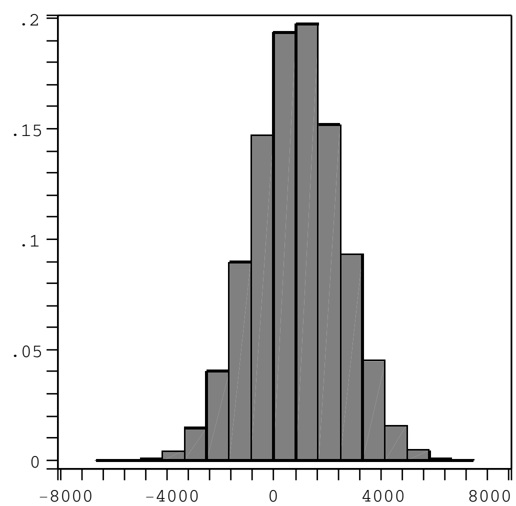

FIGURE 4: Distribution of return for the buy-and-hold strategy for C = 0 and N = 65 (100,000 Monte Carlo points). Mean return was 854 with standard deviation of 1634.

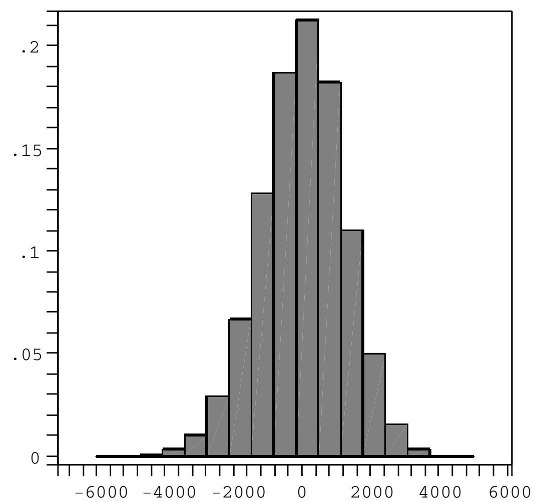

FIGURE 5: Distribution of the difference in return from following optimal strategy and following buy-and-hold strategy, for C = 30 and N = 65 (10,000 Monte Carlo points). Median is 56 and mean is 22.

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