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Bounds on the Non-convexity of Ranges of Vector Measures with Atoms

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Abstract. Upper bounds are given for the distance between the range, matrix range and partition range of a vector measure to the respective convex hulls of these ranges. The bounds are specified in terms of the maximum atom size, and generalize convexity results of Lyapounov (1940) and Dvoretzky, Wald and Wolfowitz (1951). Applications are given to the bisection problem, the "problem of the Nile", and fair division problems.

1. Introduction

Lyapounov's celebrated convexity theorem of 1940 (e.g. [**3, 10, 14, 15**]) asserts that the range of a finite-dimensional, atomless vector measure is convex and compact. A generalization of Lyapounov's theorem due to Dvoretzky, Wald and Wolfowitz [**6**] says that the same is true for the *matrix-k-range* and the *partition range* (see Definition 2.2 below).

If the vector measure has atoms, then convexity of all three ranges may fail in general, although atomlessness is not a necessary condition. Gouweleeuw [**9**] has given necessary and sufficient conditions for the range (or matrix-k-range) to be convex, as well as non-trivial sufficient conditions for the partition range to be convex.

A different approach was adopted by Elton and Hill [**7**], who proved a bound on how far from convex the range may be, as a function of the maximum atom size. The aim of this paper is to present such non-convexity inequalities for the three types of ranges mentioned above. Some of these are sharp, whereas in other cases the best possible bounds are not known to the author.

The first result is a slightly improved, but sharp, version of Elton and Hill's inequality. The proof presented here is very similar to that of Elton and Hill, with only a few minor adaptations. The original inequality is also included for the sake of comparison.

Next in line are two non-convexity inequalities for the matrix-k-range. These are proved using the improved inequality for the range, and a device of chaining

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together vector measures due to Blackwell. It is, however, the author's belief that these inequalities are not very sharp.

The last result is a sharp non-convexity bound for the partition range. Its proof (see [**1**]) is beyond the scope of this paper and is therefore omitted.

This paper is organized as follows. Section 2 lists the main results described above, accompanied by examples demonstrating their sharpness when applicable. Section 3 contains the necessary preparations for the proofs of the range and matrix-k-range inequalities, which then follow in Section 4. Section 5 gives applications of the main results to some well-known partitioning problems, including the bisection problem, the "problem of the Nile" and the problem of fair division. Section 6, finally, lists two open problems.

2. Non-convexity inequalities

Throughout this paper, $\mu, \mu_1, \ldots, \mu_n$ will always denote finite, non-negative, countably additive measures on a fixed measurable space $(\Omega, \mathcal{F})$. The *vector measure* $\vec{\mu} = (\mu_1, \ldots, \mu_n)$ is defined by

$$\vec{\mu}(A) := (\mu_1(A), \ldots, \mu_n(A)) \in \mathbb{R}^n, \quad A \in \mathcal{F}.$$

A set $E \in \mathcal{F}$ is called a (scalar) *atom* of μ if $\mu(E) > 0$ and for each $F \subset E, F \in \mathcal{F}$: $\mu(F) \in \{0, \mu(E)\}$. Similarly, E is a *vector atom* of $\vec{\mu} = (\mu_1, \ldots, \mu_n)$ if $\vec{\mu}(E) \neq \vec{0}$ and for each $F \subset E, F \in \mathcal{F} : \vec{\mu}(F) = \vec{\mu}(E)$ or $\vec{\mu}(F) = \vec{0}$. A (vector) measure is *atomless* if it does not have any atoms. A measure (resp. vector measure) is *purely atomic* if is assigns mass 0 (resp. $\vec{0}$) to the complement of the union of its atoms.

REMARK 2.1. From the definition of vector atom it can be seen that if E is a vector atom of $\vec{\mu}$, then

(i) E is a scalar atom of at least one μ_i;
(ii) for each $i \in \{1, \ldots, n\}$, either E is an atom of μ_i, or $\mu_i(E) = 0$.

Conversely, it follows from Lemma 2.4 (iii) in [**9**] that if E is a scalar atom of μ_i for some i, then E contains a vector atom F of $\vec{\mu}$ with $\vec{\mu}(F) = \vec{\mu}(E)$.

As a consequence, a vector measure is purely atomic if and only if all its component measures are.

A (measurable) *k-partition* is an ordered collection $(A_1, \ldots, A_k)$ of subsets of Ω such that $A_i \in \mathcal{F}$ $(i = 1, \ldots, k), A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^k A_i = \Omega$. Let Π_k denote the collection of all k-partitions of Ω.

In the following definition, $M_{n,k}(\mathbb{R})$ denotes the vector space of all $n \times k$ matrices with real entries.

DEFINITION 2.2. For a vector measure $\vec{\mu} = (\mu_1, \ldots, \mu_n)$,

(i) $\mathcal{R}(\vec{\mu}) := \{\vec{\mu}(A) : A \in \mathcal{F}\} \subset \mathbb{R}^n$ is the *range* of $\vec{\mu}$.
(ii) $\mathcal{MR}_k(\vec{\mu}) := \{(\mu_i(A_j))_{i=1,j=1}^{n,k} : (A_1, \ldots, A_k) \in \Pi_k\} \subset M_{n,k}(\mathbb{R})$ is the *matrix-k-range* of $\vec{\mu}$.
(iii) $\mathcal{PR}(\vec{\mu}) := \{(\mu_1(A_1), \ldots, \mu_n(A_n)) : (A_1, \ldots, A_n) \in \Pi_n\} \subset \mathbb{R}^n$ is the *partition range* of $\vec{\mu}$.

PROPOSITION 2.3. [Lyapounov (1940)]. *$\mathcal{R}(\vec{\mu})$ is compact, and if $\vec{\mu}$ is atomless, then $\mathcal{R}(\vec{\mu})$ is convex.*

PROPOSITION 2.4. [Dvoretzky, Wald and Wolfowitz (1951)]. *If $\vec{\mu}$ is atomless, then $\mathcal{MR}_k(\vec{\mu})$ is convex and compact.*

Proposition 2.4 was later improved by Dubins and Spanier [**5**], who proved that $\mathcal{MR}_k(\vec{\mu})$ is always compact.

A direct consequence of Proposition 2.4 is the following:

PROPOSITION 2.5. *If $\vec{\mu}$ is atomless, then $\mathcal{PR}(\vec{\mu})$ is convex and compact.*

The main goal of this paper is to generalize the above convexity results to measures with atoms, as was first done by Elton and Hill (1987). In order to do so, the following notation is needed. Recall that for a vector $x \in \mathbb{R}^n$, the p-norm $\|x\|_p$ of x is defined by

$$\|x\|_p := \begin{cases} \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \max_{1 \le i \le n} |x_i| & \text{if } p = \infty. \end{cases}$$

Note that the norms $(\|.\|_p, p \in [1, \infty])$ are related via the sharp inequalities

$$\|x\|_q \le \|x\|_p \quad \text{and} \quad n^{-1/p}\|x\|_p \le n^{-1/q}\|x\|_q \quad \text{for } p \le q < \infty, \tag{2.1}$$

and

$$\|x\|_\infty \le \|x\|_p \le n^{1/p}\|x\|_\infty \quad \text{for } p < \infty. \tag{2.2}$$

(See, for example, Theorems 16 and 19 in [**11**]).

By identifying $M_{n,k}(\mathbb{R})$ with $\mathbb{R}^{nk}$, the norm $\|.\|_p$ can be naturally extended to $M_{n,k}(\mathbb{R})$ as follows:

$$\|(a_{i,j})_{i=1,j=1}^{n,k}\|_p := \begin{cases} \left(\sum_{i=1}^n \sum_{j=1}^k |a_{i,j}|^p\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \max_{1 \le i \le n, 1 \le j \le k} |a_{i,j}| & \text{if } p = \infty. \end{cases}$$

If x and y are points in $\mathbb{R}^n$, then $d_p(x, y) = \|x - y\|_p$ denotes the distance between x and y. For a set S in $\mathbb{R}^n$ and a point x in $\mathbb{R}^n$, $d_p(x, S) = \inf_{y \in S} d_p(x, y)$ is the distance from x to S, and $D_p(S)$ denotes the Hausdorff distance from S to its convex hull $\mathrm{co}(S)$:

$$D_p(S) := \sup_{x \in \mathrm{co}(S)} d_p(x, S).$$

For $M_{n,k}(\mathbb{R})$, the distances d_p and D_p are defined similarly.

DEFINITION 2.6. For $\alpha \ge 0$ and $p \in [1, \infty]$, $\mathcal{P}_{n,p}(\alpha)$ is the collection of all n-dimensional vector measures $\vec{\mu}$ for which $\|\vec{\mu}(E)\|_p \le \alpha$ for each atom E of $\vec{\mu}$.

The following theorem generalizes the convexity statement of Proposition 2.3. A proof is given in Section 4 below.

THEOREM 2.7. *Let $\vec{\mu}$ be a vector measure, and let $1 \le p \le 2$.*

(i) If $\vec{\mu} \in \mathcal{P}_{n,\infty}(\alpha)$, then $D_2(\mathcal{R}(\vec{\mu})) \le \alpha n/2$.

(ii) If $\vec{\mu} \in \mathcal{P}_{n,p}(\alpha)$, then $D_p(\mathcal{R}(\vec{\mu})) \le \frac{1}{2}\alpha n^{1/p}$.

The bound in (ii) is attained for all $p \in [1, 2]$. The bound in (i) is of the correct order of magnitude in n.

Theorem 2.7 (i) is the original generalization of Lyapounov's theorem by Elton and Hill. Note that (ii) implies (i), as follows easily by substituting $p = 2$ in (ii), and using (2.2). As a consequence, Elton and Hill's inequality holds under more general conditions, namely whenever $\|\vec{\mu}(E)\|_2 \leq \alpha\sqrt{n}$ for each vector atom E of $\vec{\mu}$. The following example shows that the bound in (ii) is attained for all $p \in [1,2]$.

EXAMPLE 2.8. Let $\mu_i = \alpha\delta_{\{i\}}$, $i = 1,\dots,n$, where δ denotes Dirac measure. Then $\mathcal{R}(\vec{\mu}) = \{0,\alpha\}^n$ and hence $\mathrm{co}(\mathcal{R}(\vec{\mu})) = [0,\alpha]^n$. In particular, $y = (\alpha/2,\dots,\alpha/2) \in \mathrm{co}(\mathcal{R}(\vec{\mu}))$, and for each $x \in \mathcal{R}(\vec{\mu})$, $\|x-y\|_p = \frac{1}{2}\alpha n^{1/p}$.

Elton and Hill give the following example to show that the bound in Theorem 2.7 (i) is of the correct order of magnitude in n.

EXAMPLE 2.9. Fix $n \in \mathbb{N}$, let $m = 2^k \leq n < 2^{k+1}$, and let $\{w_i\}_{i=1}^{m-1}$ be the $m-1$ mean-zero *Walsh functions* on m points (see [**18**]). Then $w_i \in \{-1,1\}^m$, $w_i \perp w_j$ for $i \neq j$, and $w_i \perp \vec{1}$ for each i, where $\vec{1} = (1,1,\dots,1)$. For example, when $n = 4$ (so $k = 2$ and $m = 4$),

$$w_1 = (1,1,-1,-1), \quad w_2 = (1,-1,1,-1), \quad \text{and} \quad w_3 = (1,-1,-1,1).$$

Let $\Omega = \{1,2,\dots,m-1\}$, and define $\vec{\mu}(\{j\}) = (w_i + \vec{1})/2$, $j = 1,\dots,m-1$. Let $y = \vec{\mu}(\Omega)/2 = \vec{\mu}(\emptyset)/2 + \vec{\mu}(\Omega)/2 \in \mathrm{co}(\mathcal{R}(\vec{\mu}))$. It can be shown (see [**7**]) that $d_2(x,y) \geq m/4$ for each $x \in \mathcal{R}(\vec{\mu})$.

Since $2m = 2^{k+1} > n$, it follows by rescaling that the best possible upper bound in Theorem 2.7 (i) is at least $\alpha n/8$ for general n, and at least $\alpha n/4$ if n is a power of 2.

The next example shows that the statement of Theorem 2.7 (ii) is false for $p > 2$ and large n. No non-trivial inequalities are known to the author for $p > 2$.

EXAMPLE 2.10. Let $m = 2^k \leq n < 2^{k+1}$, and let $\vec{\mu}$ be the same vector measure as in Example 2.9. Then $\|\vec{\mu}(\{j\})\|_p = \left(\frac{m}{2}\right)^{1/p}$ for each j, so $\vec{\mu} \in \mathcal{P}_{m,p}\left(\left(\frac{m}{2}\right)^{1/p}\right)$. From Example 2.9 it follows that $D_2(\mathcal{R}(\vec{\mu})) \geq m/4$, hence using (2.1) it follows that $D_p(\mathcal{R}(\vec{\mu})) \geq m^{1/p}m^{-1/2}m/4 = m^{1/p}m^{1/2}/4$. Since $1/p < 1/2$ it follows that $D_p(\mathcal{R}(\vec{\mu})) > \frac{1}{2}\left(\frac{m}{2}\right)^{1/p} m^{1/p}$ for sufficiently large m.

The following theorem gives upper bounds on the non-convexity of the matrix-k-range. Its proof is given in Section 4 below.

THEOREM 2.11. *Let $\vec{\mu}$ be a vector measure and let $k \in \mathbb{N}$.*
(i) If $\vec{\mu} \in \mathcal{P}_{n,\infty}(\alpha)$, then $D_2(\mathcal{MR}_k(\vec{\mu})) \leq \alpha n\sqrt{2k}$.
(ii) If $\vec{\mu} \in \mathcal{P}_{n,2}(\alpha)$, then $D_2(\mathcal{MR}_k(\vec{\mu})) \leq \alpha\sqrt{2nk}$.

The next theorem gives a sharp non-convexity bound for the partition range. Its proof can be found in [**1**].

THEOREM 2.12. *If $\vec{\mu} \in \mathcal{P}_{n,\infty}(\alpha)$, then*

$$D_\infty(\mathcal{PR}(\vec{\mu})) \leq \frac{n-1}{n}\alpha,$$

and this bound is attained.

EXAMPLE 2.13. (sharpness of Theorem 2.12) Let $\mu_i = \alpha\delta_{\{0\}}, i = 1,\dots,n$. Then $\mathcal{PR}(\vec{\mu}) = \{\alpha u_i : i = 1,\dots,n\}$, where u_i denotes the $i-$th unit vector in $\mathbb{R}^n$ with 1 in the $i-$th position and zeroes elsewhere. It follows that $\mathrm{co}(\mathcal{PR}(\vec{\mu})) = \{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = \alpha\}$. In particular, $y = (\alpha/n,\dots,\alpha/n) \in \mathrm{co}(\mathcal{PR}(\vec{\mu}))$, and for each $x \in \mathcal{PR}(\vec{\mu})$, $\|x-y\|_\infty = \alpha(n-1)/n$.

The following immediate consequence of Theorem 2.12 improves on an earlier result of Hill and Tong ([**12**], Theorem 3.2).

COROLLARY 2.14. *If* $\vec{\mu} \in \mathcal{P}_{n,\infty}(\alpha)$*, then*

$$D_2(\mathcal{PR}(\vec{\mu})) \leq \frac{n-1}{\sqrt{n}}\alpha.$$

EXAMPLE 2.15. The bound in Corollary 2.14 is of the correct order of magnitude in n: let $\mu_i = \alpha\delta_{\{i\}}, i = 1, \ldots, n$; then $\mathcal{PR}(\vec{\mu}) = \{0, \alpha\}^n$, hence $\mathrm{co}(\mathcal{PR}(\vec{\mu})) = [0, \alpha]^n$. In particular, $y = (\alpha/2, \ldots, \alpha/2) \in \mathrm{co}(\mathcal{PR}(\vec{\mu}))$, and for each $x \in \mathcal{PR}(\vec{\mu})$, $\|x - y\|_2 = \alpha\sqrt{n}/2$.

3. Preliminaries

The goal of this section and the next is to prove Theorems 2.7 and 2.11. For a proof of Theorem 2.12 the reader is referred to [**1**].

Most of the definitions and lemmas in this section are taken from Elton and Hill [**7**]. However, some of the statements are slightly more general than the corresponding statements in [**7**]. Most of the proofs are short, and are included here in order to make this paper more self-contained.

LEMMA 3.1. *For each* $\vec{\mu}$*, each* $\varepsilon > 0$ *and each* $q \in [1, \infty]$*, there exists a measurable partition* $\{B_i\}_{i=1}^N$ *of* Ω *satisfying*

$$\forall B \in \mathcal{F}, \exists J \subset \{1, \ldots, N\} : \|\vec{\mu}(B) - \vec{\mu}(\bigcup_{j \in J} B_j)\|_q < \varepsilon. \tag{3.1}$$

PROOF. Since $\mathcal{R}(\vec{\mu})$ is bounded, there is an ε-net $\{x^{(1)}, \ldots, x^{(m)}\}$ of $\mathcal{R}(\vec{\mu})$; that is $\{x^{(1)}, \ldots, x^{(m)}\} \subset \mathcal{R}(\vec{\mu})$, and for each $x \in \mathcal{R}(\vec{\mu})$ there is an $i \leq m$ such that $\|x - x^{(i)}\|_q < \varepsilon$. Let $\{A_i\}_{i=1}^m$ satisfy $\vec{\mu}(A_i) = x_i$, $i = 1, \ldots, m$, and let $\{B_i\}_{i=1}^N \subset \mathcal{F}$ be a measurable partition of Ω such that $\sigma(B_1, \ldots, B_N) = \sigma(A_1, \ldots, A_m)$. (Such a partition exists because $\sigma(A_1, \ldots, A_m)$ is finite.) It is easily seen that $\{B_i\}_{i=1}^N$ satisfies (3.1). □

The next lemma is stated and proved in [**7**] for $p = \infty$ only; the more general statement below requires a different proof. It will be used in the next section for $p = 2$.

LEMMA 3.2. *For each* $p \in [1, \infty)$*, each* $\vec{\mu} \in \mathcal{P}_{n,p}(\alpha)$ *and each* $B \in \mathcal{F}$ *there exists a measurable partition* $\{B_i\}_{i=1}^k$ *of* B *such that* $\|\vec{\mu}(B_i)\|_p \leq \alpha$ *for all* $i \leq k$.

PROOF. Let $B \in \mathcal{F}$. By Rényi [**17**], p.83, each μ_i has at most countably many atoms, hence $\vec{\mu}$ has at most countably many vector atoms. Let A be the union of all the vector atoms of $\vec{\mu}$. Then $A \in \mathcal{F}$. Since μ_1 is atomless on $B \backslash A$, there is a measurable partition $(C_j)_{j=1}^l$ of $B \backslash A$ such that $\mu_1(C_j) \leq \alpha n^{-1/p}$ for all $j \leq l$ (where $1/\infty = 0$). Repeating this argument for μ_2 and each C_j, then for μ_3, etc., yields a partition $(D_j)_{j=1}^L$ of $B \backslash A$ such that $\mu_i(D_j) \leq \alpha n^{-1/p}$ for all $i \leq n$ and $j \leq L$, which implies $\|\vec{\mu}(D_j)\|_p \leq \alpha$ for all $j \leq L$.

The argument for $B \cap A$ is slightly different. If the number of vector atoms of $\vec{\mu}$ is finite, then there is nothing left to prove. Otherwise, let the atoms of $\vec{\mu}$ be

$E_1, E_2, \ldots$ Since

$$\sum_{j=1}^{\infty} \|\vec{\mu}(E_j)\|_p \leq \sum_{j=1}^{\infty} \|\vec{\mu}(E_j)\|_1 = \sum_{j=1}^{\infty}\sum_{i=1}^{n} \mu_i(E_j) = \sum_{i=1}^{n}\sum_{j=1}^{\infty} \mu_i(E_j) = \sum_{i=1}^{n} \mu_i(A) < \infty,$$

there is $j_0 \in \mathbb{N}$ such that

$$\left\| \vec{\mu}\left(\bigcup_{j=j_0+1}^{\infty} E_j \right) \right\|_p \leq \sum_{j=j_0+1}^{\infty} \|\vec{\mu}(E_j)\|_p \leq \alpha.$$

Taking intersections of the sets $E_1, \ldots, E_{j_0}$ and $\bigcup_{j=j_0+1}^{\infty} E_j$ with B completes the proof. □

LEMMA 3.3. *For all $p, q \in [1, \infty], \varepsilon > 0$ and $\vec{\mu} \in \mathcal{P}_{n,p}(\alpha)$, there is a purely atomic vector measure $\vec{\mu}_0 \in \mathcal{P}_{n,p}(\alpha)$ with finitely many atoms, such that*

$$D_q(\mathcal{R}(\vec{\mu})) \leq D_q(\mathcal{R}(\vec{\mu}_0)) + \varepsilon.$$

The idea of the proof of Lemma 3.3 is that Lemma 3.1 and a repeated application of Lemma 3.2 yield a partition $\{B_i\}_{i=1}^{N}$ of Ω satisfying both (3.1) and $\|\vec{\mu}(B_i)\|_p \leq \alpha$ for all $i \leq N$. The restriction $\vec{\mu}_0$ of $\vec{\mu}$ to $\sigma(B_1, \ldots, B_N)$ then has the desired property. (See [**7**], §3 for the details).

Lemma 3.3 says that it is in fact sufficient to prove Theorem 2.7 for purely atomic measures with a finite number of atoms. Since the range of such a vector measure is a finite set, this reduction turns the problem into one of finite geometry.

For the remainder of this section, V is a finite set of (not necessarily distinct) points in $\mathbb{R}_+^n = \{(r_1, ..., r_n) : r_i \in \mathbb{R}, r_i \geq 0 \text{ for all } i \leq n\}$, and $|V|$ denotes the cardinality of V.

DEFINITION 3.4.

$$\Sigma(V) = \left\{ \sum_{x_i \in V} \delta_i x_i : \delta_i \in \{0, 1\} \right\}; \qquad C(V) = \left\{ \sum_{x_i \in V} t_i x_i : t_i \in [0, 1] \right\}.$$

LEMMA 3.5. $\mathrm{co}(\Sigma(V)) = C(V)$.

The next lemma states that $C(V)$ can be expressed as the union of translates of subsets of the form $C(\hat{V})$ where $|\hat{V}| \leq n$. Let the vector sum $V_1 \oplus V_2$ of two sets V_1 and V_2 be defined by $V_1 \oplus V_2 = \{v_1 + v_2 : v_1 \in V_1, v_2 \in V_2\}$.

LEMMA 3.6. $C(V) = \bigcup \{\Sigma(V \backslash \hat{V}) \oplus C(\hat{V}) : \hat{V} \subset V, |\hat{V}| \leq n\}$.

A direct algebraic proof of Lemma 3.6 can be found in [**7**]. Here a different proof is given, based on the Shapley-Folkman lemma from convex geometry.

LEMMA 3.7. (Shapley and Folkman - see [**2**], §5) *Let $V_1, \ldots, V_k$ be nonempty subsets of $\mathbb{R}^n$. Then for each $y \in \mathrm{co}(V_1) \oplus \cdots \oplus \mathrm{co}(V_k)$ there exists a representation $y = x_1 + \cdots + x_k$, with $x_i \in \mathrm{co}(V_i)$ for all i, but $x_i \notin V_i$ for at most n indices i.*

LEMMA 3.8. $\mathrm{co}(\oplus_{i=1}^{k} V_i) = \oplus_{i=1}^{k} \mathrm{co}(V_i)$ *for all $V_1, \ldots, V_k$.*

PROOF. Straightforward. □

PROOF OF LEMMA 3.6. Clearly $C(V) \supset \bigcup\{\Sigma(V\backslash\hat{V}) \oplus C(\hat{V}) : \hat{V} \subset V, |\hat{V}| \leq n\}$. Conversely, let $x_1, \dots, x_k$ denote the elements of V (counting multiplicities), and let $V_i := \{0, x_i\}$. Then $\Sigma(V) = \oplus_{i=1}^k V_i$, and (using Lemma 3.8) $C(V) = \operatorname{co}(\Sigma(V)) = \oplus_{i=1}^k \operatorname{co}(V_i)$. Now fix $y \in C(V)$; then by Lemma 3.7 there is a representation $y = \sum_{i=1}^k y_i$, where $y_i \in \operatorname{co}(V_i)$ for all i, and $y_i \notin V_i$ for at most n indices i. Let $I := \{i : y_i \notin V_i\}$, and define $\hat{V} := \{x_i\}_{i \in I}$. It is easily checked that $y \in \Sigma(V\backslash\hat{V}) \oplus C(\hat{V})$. □

The next lemma is critical for the proof of Theorem 2.7. Its proof is taken from [**7**].

LEMMA 3.9. *For all $x, y \in \mathbb{R}^n$ and all $t \in [0,1]$,*

$$\min\{\|x + (1-t)y\|_2^2, \|x - ty\|_2^2\} \leq \|x\|_2^2 + \|y\|_2^2/4.$$

PROOF. Let $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ denote the standard inner product on $\mathbb{R}^n$. First consider the case where $2\langle x, y\rangle \geq (t^2 - (1-t)^2)\|y\|_2^2$. Then

$$\begin{aligned}\|x - ty\|_2^2 &= \|x\|_2^2 + t^2\|y\|_2^2 - 2t\langle x, y\rangle \\ &\leq \|x\|_2^2 + t^2\|y\|_2^2 - t(t^2 - (1-t)^2)\|y\|_2^2 \\ &= \|x\|_2^2 + t(1-t)\|y\|_2^2 \leq \|x\|_2^2 + \|y\|_2^2/4.\end{aligned}$$

The case where $2\langle x, y\rangle \leq (t^2 - (1-t)^2)\|y\|_2^2$ is similar, yielding $\|x + (1-t)y\|_2^2 \leq \|x\|_2^2 + \|y\|_2^2/4$. □

LEMMA 3.10. *Let $|V| = n$. If $\|y\|_2 \leq 1$ for all $y \in V$, then $D_2(\Sigma(V)) \leq \sqrt{n}/2$.*

PROOF. Let $V = \{x_1, ..., x_n\}$ and fix $x = \sum_{i=1}^n t_i x_i \in C(V)$. Applying Lemma 3.9 n times implies the existence of $(\delta_i)_{i=1}^n \in \{0,1\}^n$ satisfying

$$\left\|\sum_{i=1}^n \delta_i x_i - x\right\|_2^2 = \left\|\sum_{i=1}^n (\delta_i - t_i) x_i\right\|_2^2 \leq n \cdot \max\{\|y\|_2^2 : y \in V\}/4 \leq n/4.$$

□

4. Proofs of the range and matrix range inequalities

PROOF OF THEOREM 2.7. Since (ii) implies (i), it is enough to prove (ii). Note first that if (ii) holds for $p = 2$, then it holds for each $p \in [1,2]$. For, suppose that $\vec{\mu} \in \mathcal{P}_{n,p}(\alpha)$; then also $\vec{\mu} \in \mathcal{P}_{n,2}(\alpha)$ in view of (2.1), so $D_2(\mathcal{R}(\vec{\mu})) \leq \frac{1}{2}\alpha n^{1/2}$, and a second application of (2.1) yields $D_p(\mathcal{R}(\vec{\mu})) \leq \frac{1}{2}\alpha n^{1/p}$.

It is therefore sufficient to prove (ii) for $p = 2$. Let $\varepsilon > 0$ and $\vec{\mu} \in \mathcal{P}_{n,2}(\alpha)$. By Lemma 3.3 there exists a purely atomic vector measure $\vec{\mu}_0 \in \mathcal{P}_{n,2}(\alpha)$ with finitely many atoms such that

$$D_2(\mathcal{R}(\vec{\mu})) \leq D_2(\mathcal{R}(\vec{\mu}_0) + \varepsilon. \tag{4.1}$$

Now $\mathcal{R}(\vec{\mu}_0) = \Sigma(V)$, where $V = \{\vec{\mu}_0(E) : E$ is a vector atom of $\vec{\mu}_0\}$, so applying Lemma 3.6, Lemma 3.10 and rescaling yields

$$D_2(\mathcal{R}(\vec{\mu}_0) \leq \alpha\sqrt{n}/2,$$

which together with (4.1) completes the proof of (ii), since ε was arbitrary. □

PROOF OF THEOREM 2.11. Since (ii) implies (i), it suffices to prove (ii). Fix $(A_1, \ldots, A_k)$ and $(B_1, \ldots, B_k)$ in Π_k and $t \in [0,1]$. Following Dubins and Spanier [**5**], define the $2nk$-dimensional vector measure $\vec{m}$ by

$$\vec{m}(S) = (\mu_i(S \cap A_j), \mu_i(S \cap B_j))_{i=1,j=1}^{n,k}.$$

Note that $\vec{m}$ and $\vec{\mu}$ have the same vector atoms. If E is a vector atom of $\vec{m}$, then, since distinct vector atoms are essentially disjoint, it can be assumed that $E \subset A_{j_1}$ and $E \subset B_{j_2}$ for some j_1 and j_2. Hence $E \cap A_j = \emptyset$ for all $j \neq j_1$ and $E \cap B_j = \emptyset$ for all $j \neq j_2$, so

$$\|\vec{m}(E)\|_2^2 = 2\|\vec{\mu}(E)\|_2^2 \leq 2\alpha^2.$$

Applying Theorem 2.7 (ii) to $\vec{m}$ now yields $D_2(\mathcal{R}(\vec{m}) \leq \alpha\sqrt{2} \cdot \sqrt{2nk}/2 = \alpha\sqrt{nk}$, so there exists a set $S \in \mathcal{F}$ with $\|\vec{m}(S) - t\vec{m}(\Omega)\|_2 \leq \alpha\sqrt{nk}$, that is,

$$\sum_{i=1}^{n}\sum_{j=1}^{k}(\mu_i(S \cap A_j) - t\mu_i(A_j))^2 + \sum_{i=1}^{n}\sum_{j=1}^{k}(\mu_i(S \cap B_j) - t\mu_i(B_j))^2 \leq \alpha^2 nk. \tag{4.2}$$

Since $\mid \mu_i(S \cap B_j) - t\mu_i(B_j) \mid = \mid \mu_i(B_j \backslash S) - (1-t)\mu_i(B_j) \mid$, it follows from (4.2) that

$$\sum_{i=1}^{n}\sum_{j=1}^{k}(\mu_i(S \cap A_j) - t\mu_i(A_j))^2 + \sum_{i=1}^{n}\sum_{j=1}^{k}(\mu_i(B_j \backslash S) - (1-t)\mu_i(B_j))^2 \leq \alpha^2 nk. \tag{4.3}$$

Letting $C_j = (A_j \cap S) \cup (B_j \backslash S)$ it follows from (4.3) that

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{j=1}^{k}(\mu_i(C_j) - t\mu_i(A_j) - (1-t)\mu_i(B_j))^2 \\ &\quad = \sum_{i=1}^{n}\sum_{j=1}^{k}(\mu_i(A_j \cap S) + \mu_i(B_j \backslash S) - t\mu_i(A_j) - (1-t)\mu_i(B_j))^2 \\ &\qquad = \|M + N - P - Q\|_2^2 \leq 2(\|M - P\|_2^2 + \|N - Q\|_2^2) \leq 2\alpha^2 nk, \end{aligned} \tag{4.4}$$

where the matrices M, N, P and Q are defined in the obvious manner, i.e. $M = (\mu_i(A_j \cap S))_{i=1,j=1}^{n,k}$, etc. Taking square roots on both sides of (4.4) completes the proof of (ii). □

REMARK 4.1. The device of chaining together vector measures was introduced by Blackwell [**4**]. It was used by Dubins and Spanier [**5**] to derive convexity of the matrix range from Lyapounov's theorem in the atomless case, and by Hill and Tong [**12**] to obtain a non-convexity bound for the partition range from Theorem 2.7 (i). However, the inequality presented in Theorem 2.12 is much stronger than Hill and Tong's result. It is therefore the author's belief that sharper inequalities than those presented in Theorem 2.11 can be found.

5. Applications

Lyapounov's theorem has been applied in a number of areas including optimal stopping theory, control theory and statistical decision theory. In principle, any application of Propositions 2.3, 2.4 or 2.5 can be generalized to measures with atoms using the corresponding generalization from Section 2. The aim of this section is to illustrate this by a few examples, including the bisection problem, the "problem of the Nile" and the problem of fair division.

1. The objective in the *bisection problem* is to find a set $A \in \mathcal{F}$ such that $\mu_i(A) = \frac{1}{2}\mu_i(\Omega)$ for all i. If $\mu_1, \ldots, \mu_n$ are atomless, then the existence of such a set is a direct consequence of Proposition 2.3. The following theorem generalizes this result to measures with atoms:

THEOREM 5.1. *If $\|\vec{\mu}(E)\|_2 \leq \alpha$ for every vector atom E of $\vec{\mu}$, then there exists a set $A \in \mathcal{F}$ satisfying*

$$| \mu_i(A) - \frac{1}{2}\mu_i(\Omega) | \leq \alpha\sqrt{n}/2 \quad \textit{for all } i \leq n.$$

PROOF. Since $(0, \ldots, 0) = \vec{\mu}(\emptyset) \in \mathcal{R}(\vec{\mu})$ and $(\mu_1(\Omega), \ldots, \mu_n(\Omega)) = \vec{\mu}(\Omega) \in \mathcal{R}(\vec{\mu})$, it follows that $\mathrm{co}(\mathcal{R}(\vec{\mu}))$ contains the vector $(\frac{1}{2}\mu_1(\Omega), \ldots, \frac{1}{2}\mu_n(\Omega))$. Apply Theorem 2.7 (ii). □

An approximate bisection result based on Theorem 2.7 (i) can, of course, be stated and proved analogously.

For an application of Theorem 2.7 to the bang-bang principle of control theory, see Elton and Hill [**7**].

2. Fisher's *"problem of the Nile"* (see Dubins and Spanier [**5**]). "Each year the Nile would flood, thereby irrigating or perhaps devastating parts of the agricultural land of a predynastic Egyptian village. The value of different portions of the land would depend upon the height of the flood. In question was the possibility of giving to each of the k residents a piece of land whose value would be $1/k$ of the total land value no matter what the height of the flood."

Neyman [**16**] proved that the problem has a solution under the assumption that there are only a finite number, say n, of possible flood heights. The following theorem generalizes Neyman's result.

THEOREM 5.2. *If $\mu_1, \ldots, \mu_n$ are probability measures and $\|\vec{\mu}(E)\|_2 \leq \alpha$ for every vector atom E of $\vec{\mu} = (\mu_1, \ldots, \mu_n)$, then there exists a k-partition $(A_1, \ldots, A_k)$ of Ω such that*

$$| \mu_i(A_j) - \frac{1}{k} | \leq \alpha\sqrt{2nk} \quad \textit{for all } i \leq n \textit{ and } j \leq k.$$

PROOF. For $r = 1, \ldots, k$, let $(A_j^{(r)})_{j=1}^k$ be the partition with $A_r^{(r)} = \Omega$ and $A_j^{(r)} = \emptyset$ if $j \neq r$. Then

$$M_r := (\mu_i(A_j^{(r)}))_{i=1,j=1}^{n,k} \in \mathcal{MR}_k(\vec{\mu})$$

for $r = 1, \ldots, k$. Note that M_r is the matrix with only 1's in the r-th column and 0's elsewhere. It follows that $\frac{1}{k}M_1 + \ldots \frac{1}{k}M_k \in \mathrm{co}(\mathcal{MR}_k(\vec{\mu}))$. Applying Theorem 2.11 (ii) gives the desired result. □

3. In the classical *fair division problem* the objective is to find a partition $(A_1, \ldots, A_n)$ of Ω such that

$$\mu_i(A_i) \geq \frac{1}{n} \quad \text{for all } i \leq n, \tag{5.1}$$

where $\mu_1, \ldots, \mu_n$ are probability measures. If $\mu_1, \ldots, \mu_n$ are atomless, then Proposition 2.5 guarantees the existence of such a partition, even with equality in (5.1). If, in addition, $\mu_i \neq \mu_j$ for some $i \neq j$, then there is a partition for which (5.1) holds with *strict* inequality (see for example Dubins and Spanier [**5**]).

If the measures have atoms then in general a partition satisfying (5.1) need not exist. However, if the measures are not all equal and the atoms are sufficiently small, then the differentiation between the measures may compensate for the non-divisibility of the atoms, making fair division still possible. To make this more precise, let

$$M := \sup\{\sum_{i=1}^{n} \mu_i(B_i) \mid (B_i)_{i=1}^{n} \text{ is a partition of } \Omega\}.$$

The following theorem generalizes a result of Elton, Hill and Kertz [**8**].

THEOREM 5.3. *If $\|\vec{\mu}(E)\|_\infty \leq \alpha$ for every vector atom E of $\vec{\mu}$, then there exists a partition $(A_1, \ldots, A_n)$ of Ω such that*

$$\mu_i(A_i) \geq (n - M + 1)^{-1} - \frac{n-1}{n}\,\alpha,\ i = 1, \ldots, n.$$

PROOF. As in Legut [**13**], applying Theorem 2.12 at the place where [**13**] uses convexity of the partition range. □

COROLLARY 5.4. *If $\|\vec{\mu}(E)\|_\infty \leq (M-1)(n-1)^{-1}(n-M+1)^{-1}$ for every vector atom E of $\vec{\mu}$, then there is a partition $(A_1, \ldots, A_n)$ of Ω satisfying (5.1).*

EXAMPLE 5.5. Let $n = 3$ and suppose that $M = 2$ (note that 2 is a realistic value in this case, since M can assume any value between 1 and 3). If $\|\vec{\mu}(E)\|_\infty \leq \frac{1}{4}$ for every vector atom E of $\vec{\mu}$, then Corollary 5.4 implies the existence of a fair division in the sense of (5.1).

6. Open Problems

PROBLEM 1. Find a non-convexity inequality analogous to Theorem 2.7 (ii) for $p = \infty$; in other words, find the best possible (or at least a good) constant $K(n, \alpha)$ such that if $\vec{\mu} \in \mathcal{P}_{n,\infty}(\alpha)$, then $D_\infty(\mathcal{R}(\vec{\mu})) \leq K(n, \alpha)$. An example of the significance of such a sharp bound is that it would yield the best possible upper bound in Theorem 5.1. Note that by Example 2.9, the order of magnitude of $K(n, \alpha)$ must be at least $\sqrt{n}$.

PROBLEM 2. Find a sharp non-convexity inequality for the matrix-k-range. The inequalities given in Theorem 2.11 are probably far from sharp, as was already pointed out in Remark 4.1. In fact, no examples are known to the author of vector measures $\vec{\mu}$ for which $D_2(\mathcal{MR}_k(\vec{\mu}))$ is unbounded in k. Is there an upper bound for the non-convexity of $\mathcal{MR}_k(\vec{\mu})$ that does not depend on k? What about $D_\infty(\mathcal{MR}_k(\vec{\mu}))$?

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